UNIVERSIDAD DE SONORA

División de Ciencias Exactas y Naturales

Programa de Posgrado en Matemáticas

American option pricing as a free-boundary problem

T E S I S

Que para obtener el grado académico de:

Maestro en Ciencias
(Matemáticas)

Presenta:

Adriana Ocejo Monge

Director de Tesis: Dr. Oscar Vega Amaya

Hermosillo, Sonora, México, 26 de junio de 2009.

SINODALES

Dr. Oscar Vega Amaya
Universidad de Sonora, Hermosillo, México.

Dr. Carlos Gabriel Pacheco González
Centro de Investigación y de Estudios Avanzados del I. P. N., Ciudad de México.

Dr. Adolfo Minjárez Sosa
Universidad de Sonora, Hermosillo, México.

Dr. Agustín Brau Rojas
Universidad de Sonora, Hermosillo, México.

UNIVERSIDAD DE SONORA

Acta de Examen de Grado

1942

EL SABER DE MIS HIJOS HARA MI GRANDEZA

En la ciudad de Hermosillo, Sonora, México, siendo las 18:00 horas del día 26 de Junio del 2009, se reunieron en:

El Auditorio del Departamento de Matemáticas

de la Universidad de Sonora, los señores:

DR. Oscar Vega Amaya

Dr. Carlos Gabriel Pacheco González

Dr. Adolfo Minjárez Sosa

bajo la Presidencia del primero y fungiendo como Secretario el último, para efectuar el Examen de Grado a:

ADRIANA OCEJO MONGE

de: Maestro en Ciencias (Matemáticas)

quien presentó el trabajo de tesis

"Valuación de la opción Americana como Problema de Frontera Libre"

Acta No. 9

Foja 9

Libro 1

Exp. No. 203201196

Los sinodales replicaron a la sustentante y después de debatir entre sí reservada y libremente, la declararon:

Aprobada por unanimidad

Acto continuo el Presidente del Jurado le hizo saber el resultado de su examen y para constancia se levanta la presente, firmando los que intervinieron.

Firma del sustentante

Oscar Vega Amaya

PRESIDENTE

Adolfo Minjárez

SECRETARIO

VOCAL

Dr. Fernando Luque Vásquez Coordinador del Programa de: Posgrado en Matemáticas de la Universidad de Sonora, hace constar que las firmas que anteceden corresponden a los sinodales que intervinieron en el examen que contiene la presente acta.

Hermosillo, Sonora, a 26 de Junio del 2009.

Dr. Fernando Luque Vásquez
Coordinador del Programa

DIVISIÓN DE CIENCIAS EXACTAS Y NATURALES
COORDINACIÓN
POSGRADO EN MATEMÁTICAS

Ing. Enrique Gurrola Mac, Secretario de Unidad en funciones de Vicerrector de la Unidad Regional Centro de la Universidad de Sonora, hace constar que: el Dr. Fernando Luque Vásquez es a la fecha de expedición de este documento, Coordinador del Programa arriba citado y suya la firma que aparece al calce del mismo.

Hermosillo, Sonora, a 26 de Junio del 2009.

EL SABER DE MIS HIJOS HARA MI GRANDEZA
UNIDAD REGIONAL CENTRO
VICERRECTORIA

Ing. Enrique Gurrola Mac
Secretario de Unidad en funciones de
VICERRECTOR de la
UNIDAD REGIONAL CENTRO

To my lovely parents.

Acknowledgements

My first thanks go to Jesus Christ, who is my inspiration. Also, I am so greatful to my parents, Adriana and Humberto, because of their love and encouragement in my career.

I wish to thank my thesis advisers Dr. Oscar Vega and Dr. Carlos Gabriel Pacheco. Their supervision have substantially enriched the final product and working with them could not be more friendly.

Also, thanks to CONACYT for support my studies financially.

Contents

Introduction

An American option is a financial contract between two parties; in the case of an American put, one pays to the other for the right to sell certain asset at any time in the future before an expiration date; or in the case of the American call, one pay for the right to buy the asset. There are two important questions to answer here. The first one is,

When is the right moment to exercise such an option? or equivalently, when is the optimal time to stop in order to maximize revenue (at least in average)?

The second question is

What should be the right price of such a contract?

In this thesis we deal with both questions, and provide satisfactory answers for each one. Both questions are important to solve in financial markets; the first one is of interest to the buyer of the option, and second to the seller. Indeed, for the first question one looks for a policy to exercise efficiently the contract, that is to say we try to find an *optimal stopping rule* that solves an *optimal stopping problem.* For the second question, one is willing to determine the "fair" price of the contract.

It turns out that in order to provide good answers to these questions, one relies heavily on strong mathematical tools. However, the solutions of the problems are somewhat elegant and quite straightforward to understand, as well as to carry out in practical implementations. We shall see how both issues are intertwined, we shall be able to understand the insights the problem.

We will see that for the American call the problem is simplified when assuming assets with no dividends, and the main problem becomes the American put, still assuming non-dividend paying stocks. Thus, the main part of the thesis is about American put. The starting point is the *Risk Neutral framework*, based on the celebrated works of Black and Scholes. Indeed, one uses a geometric Brownian motion for modeling, and the so-called *Risk Neutral measure* (also called *martingale measure*) for pricing. Let us then give some historical background on the American put problem.

The pricing of American options has enjoyed a fast development the last decades. The flexibility of the contract to early exercise suggest the formulation of the option's price as an optimal stopping problem. McKean [22] showed in 1969 that this optimal stopping problem could be transformed into a free-boundary problem. Despite McKean's formula provides an explicit representation of the option's price in terms of the unknown optimal stopping boundary function, it is quite difficult to implement analytical or numerical examination. Thereafter, important properties of the optimal stopping boundary were studied by van Moerbeke [38] in 1976.

Earlier in 1973, Black and Scholes [3] obtained the price of a European option using a free-arbitrage assumption, which means that the market does not allow opportunities to make money without risk. The same year, Merton [23] proved that the American call option on non-dividend paying stocks is the same as the European call option price. Also, he pointed out that such a methodology does not apply to American put options, but that McKean's results could be adapted to this end.

The use of financial arguments using replicating portfolios emerged as an efficient technique for the pricing of options, by means of the so-called equivalent martingale measure of the market. The pioneering works in this line are due to Cox, Ross, and Rubinstein [5] in 1979; Harrison and Kreps [12] in 1979; and Harrison and Pliska [13] in 1981. The application of these arguments to the pricing of the American put option is due to Bensoussan [1] and Karatzas [17].

Later in 1991, Jacka studied the free-boundary problem formulation arising from the optimal stopping problem. He showed that the optimal strategy is given by a boundary function that satisfies an integral equation called *the free-boundary equation.* However, the uniqueness of the solution to this integral equation in the class of continuous increasing functions was proved until 2005 by Peskir [29].

Now we present how the thesis is organized.

In Chapter 1 we state the background and several results used in this work.

In Chapter 2 we discuss the Arbitrage-Free Pricing Theory to price contingent claims (such as the payoff of the European option), and it is determined the existence and uniqueness of the martingale measure. The existence of this measure is intimately related to the abscense of arbitrage, while the uniqueness is related to the completeness of the market (due to Harrison and Pliska [13]) in the sense that every contingent claim can be replicated. In this context, the fair price of a contingent claim is the expected value of the discounted claim under the unique martingale measure. After that, we apply this results to the *Binomial model* for the price process, which is the simplest option pricing approach. At the end of Chapter 2, we arrived to the *Black and Scholes pricing formula* for the European option and give the price of the American option as the solution to an optimal stopping problem.

In Chapter 3, the theory of optimal stopping problems for time homogeneous strong Markov processes is established. Here it is showed that there exists a solution to the optimal stopping problem under certain conditions. Also, it is given a characterization of the value function as the smallest superharmonic function which dominates the gain function.

In Chapter 4 we transfer the optimal stopping problem, representing the price of the American put, to a free-boundary problem. This step is mainly due to the Markovian structure of the price process by means of the infinitesimal generator. It is proved that the optimal strategy to exercise the option is determined by a function of the time known as *the optimal stopping boundary.* The idea is that the holder will optimally exercise the option the first time that the price process falls below a barrier, that is to say, the optimal stopping boundary.

In summary, thanks to the free-boundary formulation, it is derived the optimal stopping rule by the *first passage time* of the geometric Brownian motion to a barrier determined by the *free-boundary equation* (an integral equation). And the fair price is given by the present value of the expected value of the profit made at the first passage time. This is exposed in Chapter 5.

Chapter 1

Preliminaries

1.1 Markov processes and stopping times

Definition 1.1. (Filtrations)

(i) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{T}$ an index set. A *filtration* $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ is an increasing family of sub-σ-algebras of $\mathcal{F}$, that is, $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $s \leq t$. Thus, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathcal{T}}, P)$ is called a *filtered probability space.*

(ii) A stochastic process $X = \{X_t\}_{t\in\mathcal{T}}$ defined on $(\Omega, \mathcal{F}, P)$ is said to be *adapted* to the filtration $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ if X_t is $\mathcal{F}_t$-measurable, for each $t \in \mathcal{T}$. In this case, we say that X is defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathcal{T}}, P)$.

Definition 1.2. (Stopping times)

(i) Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathcal{T}}, P)$ be a filtered probability space. A random variable $\tau : \Omega \to \mathcal{T}$ is a *stopping time* if the event $[\tau \leq t]$ is $\mathcal{F}_t$-measurable for each $t \in \mathcal{T}$.

(ii) Let τ be a stopping time. An event $A \in \mathcal{F}$ is said to be *prior* to τ if

$$A \cap [\tau \leq t] \in \mathcal{F}_t, \quad \forall\, t \in \mathcal{T}.$$

Denote by $\mathcal{F}_\tau$ the family of all the prior events, that is,

$$\mathcal{F}_\tau := \{A \in \mathcal{F}_\infty : A \cap [\tau \leq t] \in \mathcal{F}_t, \forall\, t \in \mathcal{T}\}. \tag{1.1}$$

where $\mathcal{F}_\infty = \sigma(\mathcal{F}_t : t \geq 0)$. It can be readily verified that $\mathcal{F}_\tau$ is a σ-algebra.

In the sequel the following notation is used. Given a topological space E the Borel σ-algebra is denoted by $\mathcal{B}(E)$.

Definition 1.3. (Markov process)

(i) A stochastic process $X = \{X_t\}_{t\geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ and taking values on a measurable space $(E, \mathcal{B}(E))$ is a *Markov process* if it satisfies the so-called *Markov property*, that is, for each $s \leq t$ and $B \in \mathcal{B}(E)$ it follows that

$$P(X_t \in B \mid \mathcal{F}_s) = P(X_t \in B \mid X_s). \tag{1.2}$$

The measure

$$\pi(B) := P(X_0 \in B),$$

is called the *initial distribution measure.*

(ii) Let $X = \{X_t\}_{t\geq 0}$ be a Markov process. We say that X is a *strong Markov process* if it satisfies the *strong Markov property* , that is, for each a.s. finite stopping time τ and $B \in \mathcal{B}(E)$ we have

$$P(X_{t+\tau} \in B \mid \mathcal{F}_\tau) = P(X_{t+\tau} \in B \mid X_\tau). \tag{1.3}$$

The probabilistic structure of a Markov process X can also be determined by a non-negative function called *transition probability.*

Definition 1.4. (Transition probability function)
A function $P : \mathcal{B}(E)\times\mathbb{R}_+\times E\times\mathbb{R}_+ \to [0, 1]$ is a *transition probability* (t.p.) function if for all $s \leq t \leq r$ it satisfies the following properties:

(i) $P(\cdot, t, x, s)$ is a probability measure on $\mathcal{B}(E)$ for each $x \in E$.

(ii) $P(B, t, \cdot, s)$ is $\mathcal{B}(E)$-measurable for each $B \in \mathcal{B}(E)$.

(iii) P satisfies the *Chapman-Kolmogorov equation*, that is, for all $x \in E$ and $B \in \mathcal{B}(E)$ we have

$$P(B, r, x, s) = \int_E P(B, r, y, t)P(dy, t, x, s) \tag{1.4}$$

for all $B \in \mathcal{B}(E)$, $x \in E$, and non-negative numbers $s \leq t \leq r$.

A t.p. function P is the t.p. of a Markov process X if for each $B \in \mathcal{B}(E)$ and $s \leq t$ we have that

$$P(X_t \in B \mid X_s) = P(B, t, X_s, s). \tag{1.5}$$

The Markov process X is said to be *time-homogeneous* if its t.p. is such that for all $s, t \geq 0$ and $B \in \mathcal{B}(E)$,

$$P(B, t, x, 0) = P(B, s + t, x, s). \tag{1.6}$$

In such case, we will write $P(B, t, x) := P(B, t, x, 0)$.

Definition 1.5. (Markov family)
Let $(\Omega, \mathcal{F})$ and $(E, \mathcal{B}(E))$ be measurable spaces, and $\{\mathcal{F}_t\}_{t\geq 0}$ a filtration. A *Markov family* is a family of probability spaces

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{P_x : x \in E\}) \tag{1.7}$$

and an E-valued process $X = \{X_t\}_{t\geq 0}$ defined on $(\Omega, \mathcal{F})$ and adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, satisfying the following conditions

(i) $(t, x) \mapsto P_{t,x}(A)$ is $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(E)$-measurable for each $A \in \mathcal{F}$.

(ii) $P_{t,x}(X_t = x) = 1$ for each $x \in E$ and $t \geq 0$.

(iii) For each $x \in E$, $s \geq t$, $u \geq 0$, and $B \in \mathcal{B}(E)$, the next property holds:

$$P_{t,x}(X_{s+u} \in B \mid X_s) = P_{s,X_s}(X_u \in B), \quad P_{t,x}-\text{ a.s.} \tag{1.8}$$

The shift operator

Assume that $(\Omega, \mathcal{F})$ is the *canonical space* , that is,

$$\Omega = E^{\mathbb{R}_+}, \quad \mathcal{F} = \mathcal{B}(E^{\mathbb{R}_+}).$$

Then, identify each $\omega \in \Omega$ with a sample path $t \mapsto X_t(\omega)$ of the E-valued process X. In this sense, we have that $X_t(\omega) = \omega(t)$. Let $\mathcal{F}_t := \sigma(X_s, s \leq t)$. For each $t \geq 0$, define the *shift operator* $\theta_t : \Omega \to \Omega$ by

$$\theta_t(\omega)(s) := \omega(t+s), \quad \forall\, s \geq 0,\, \omega \in \Omega. \tag{1.9}$$

Also, given a finite stopping time τ, define the operator

$$\theta_\tau(\omega) := \theta_t(\omega), \quad \text{if } \tau(\omega) = t. \tag{1.10}$$

For all $s, t \geq 0$ and finite stopping times σ and τ, the following equalities can be verified:

$$X_s \circ \theta_t = X_{s+t} \tag{1.11}$$

$$X_s \circ \theta_\tau = X_{s+\tau} \tag{1.12}$$

$$X_\sigma \circ \theta_t = X_{\sigma\circ\theta_t + t} \tag{1.13}$$

$$X_\sigma \circ \theta_\tau = X_{\sigma\circ\theta_\tau + \tau}. \tag{1.14}$$

For instance, we will verify (1.13). Let $\omega \in \Omega$, then

$$\begin{aligned}(X_\sigma \circ \theta_t)(\omega) &= X_\sigma(\theta_t(\omega)) \\ &= X_{\sigma(\theta_t(\omega))}(\theta_t(\omega)) \\ &= X_{\sigma\circ\theta_t + t}(\omega).\end{aligned}$$

Also, if $A \in \mathcal{F}$ then it is easy to check that

$$[X_{s+t} \in A] = \theta_t^{-1}[X_s \in A]. \tag{1.15}$$

Using the shift operator, the Markov property (1.8) and the strong Markov property (1.3) can be written as

$$P_{t,x}(X_u \circ \theta_s \in B \mid X_s) = P_{s,X_s}(X_u \in B), \quad P_{t,x} - \text{ a.s.}$$

$$P_{t,x}(X_u \circ \theta_\tau \in B \mid X_\tau) = P_{\tau,X_\tau}(X_u \in B), \quad P_{t,x} - \text{ a.s.}$$

for all $s \geq t \geq 0$, $x \in E$, $B \in \mathcal{B}(E)$, and finite stopping times τ.

1.2 Stochastic Differential Equations

Let $\{B_t\}_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. The *Itô integral* will be denoted by

$$\int_S^T g(t,\omega)dB_t, \tag{1.16}$$

where $0 \leq S < T$, defined for functions in the class $\mathcal{V}(S,T)$ introduced next. A function $g(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$ belongs to the class $\mathcal{V}(S,T)$ if

(i) $g(t,\omega)$ is $\mathcal{B}(\mathbb{R}_+)\times\mathcal{F}$-measurable.

(ii) $g(t,\omega)$ is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$.

(iii) $P(\int_S^T g^2(t,\omega)<\infty)=1$.

Denote by $\mathcal{L}_2(S,T)$ the subclass of $\mathcal{V}(S,T)$ such that

(iii)' $g\in L^2(dt\times dP)$, that is, $E\left(\int_S^T g(t,\omega)^2dt\right)<\infty$.

Proposition 1.1. *Let $g(t,\omega)\in\mathcal{L}_2(0,T)$ for all T. Then the Itô integral*

$$M_t(\omega)=\int_0^t g(s,\omega)dB_s, \tag{1.17}$$

is a martingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$.

See Øksendal [27, Corollary 3.2.6] for a proof.

Definition 1.6. An *Itô process* is a stochastic process $X=\{X_t\}_{0\leq t\leq T}$ of the form

$$X_t=X_0+\int_0^t u(s,\omega)ds+\int_0^t v(s,\omega)dB_s, \tag{1.18}$$

where $u(t,\omega)$ and $v(t,\omega)$ are adapted, measurable processes such that $v\in\mathcal{V}(0,T)$ and u satisfies the condition

$$P\left(\int_0^T |u(s,\omega)|ds<\infty\right)=1.$$

Theorem 1.2. (Martingale Representation) *Let $X=\{X_t\}_{t\geq 0}$ be a martingale with respect to P and suppose that $X_t\in L^2(dP)$ for all $t\geq 0$. Then, there exists a unique stochastic process $g(s,\omega)$ such that $g\in\mathcal{V}(0,t)$ and*

$$X_t(\omega)=E(X_0)+\int_0^t g(s,\omega)\,dB_s, \qquad a.s. \tag{1.19}$$

for all $t\geq 0$.

See Øksendal [27, p. 53] for a proof.

Theorem 1.3. (Itô's formula) *Let X be an Itô process satisfying (1.18) and $g(t,x)\in C^{1,2}([0,\infty)\times\mathbb{R})$. Then*

$$Y_t=g(t,X_t) \tag{1.20}$$

is an Itô process, and setting $u=u(t,X_t), v=v(t,X_t), g=g(t,X_t)$ we have

$$dY_t=[g_t+u\,g_x+\frac{v^2}{2}\,g_{xx}]dt+v\,g_x\,dB_t. \tag{1.21}$$

See Øksendal [27] p. 44 for a proof.

Diffusions

We are interested in processes satisfying the stochastic differential equation (SDE)

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t, \tag{1.22}$$

with initial condition $X_0 = Z$, where Z is a random variable.

The conditions (C1)-(C2) below guarantee the existence and uniqueness of solutions to SDE's, they are called the *Itô conditions* :

(C1) $|\mu(t,x)| + |\sigma(t,x)| \le K(1+|x|)$,

(C2) $|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le K|x-y|$,

where $x, y \in \mathbb{R}$ and K is a positive constant. Note that condition (C2) implies that μ and σ are Lipshitz continuous.

Theorem 1.4. (Existence and uniqueness of solution) *Consider the stochastic differential equation*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t, \qquad X_0 = Z. \tag{1.23}$$

where $Z \in L^2(dP)$ and Z is independent of Brownian motion B_t for all $t \ge 0$. If $\mu(t,x)$ and $\sigma(t,x)$ satisfy the Itô conditions, then there exists a unique t-continuous solution $X = \{X_t(0,Z)\}_{0\le t\le T}$ to (1.23), adapted to the filtration $\mathcal{F}_t^Z = \sigma(Z, B_s : s \le t)$ and bounded in $L^2(dP)$.

See Øksendal [27, p. 68] for a proof.

A solution to the SDE (1.23) is called an *Itô diffusion* and the functions $\mu(t,x)$ and $\sigma(t,x)$ are called the *drift* and *diffusion* coefficients, respectively.

Proposition 1.5. *Let $X = \{X_t(0,Z)\}_{0\le t\le T}$ be an Itô diffusion. Also, suppose that $\sigma(t,x)$ is continuous and that*

$$E\left(\int_0^T \sigma^2(t, X_t)\, dt\right) < \infty. \tag{1.24}$$

Then, the process X is a martingale if and only if the drift is zero, that is, $\mu(t,x) = 0$.

Proof. If X is a martingale then Theorem 1.2 implies that $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$ can also be written as

$$dX_t = \hat{\sigma}(t, X_t)dB_t,$$

so that $0 = \mu(t, X_t)dt + (\sigma(t, X_t) - \hat{\sigma}(t, X_t))dB_t$. Define

$$M_s := \int_0^s \mu(t, X_t)dt = -\int_0^s (\sigma(t, X_t) - \hat{\sigma}(t, X_t))dB_t.$$

Since $\int_0^s \mu(t, X_t)dt$ is of bounded variation and $\int_0^s (\sigma(t, X_t) - \hat{\sigma}(t, X_t))dB_t$ is a martingale, it follows that $M = \{M_s\}_{s\ge 0}$ is constant in time [1]. That is, $M_s = M_0$ and thus $\sigma - \hat{\sigma} \equiv 0$ implying $\mu \equiv 0$.

[1] See a proof of this in [24, Result 9.b.1 page 74]

On the other hand, if $\mu(t,x)=0$ then

$$dX_t = \sigma(t, X_t)\, dB_t$$

is an Itô integral, and so is a martingale by Proposition 1.1. $\square$

Proposition 1.6. (Markov property) *Let $X=\{X_t(0,y)\}_{t\geq 0}$ be an Itô diffusion. Then X is a Markov process with t.p. given by*

$$P(B,t,x,s) = P(X_t(s,x)\in B), \tag{1.25}$$

where $\{X_t(s,x)\}_{t\geq s}$, is the unique solution to the SDE (1.22) with initial condition $X_s = x$.

See Friedman [10] or Øksendal [27, p. 115] for a proof.

From Proposition 1.6 the diffusion process $X=\{X_t\}_{t\geq 0}$ solving (1.22) defines the Markov family (see Definition 1.5)

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{P_{s,x} : 0\leq s\leq T, x\in\mathbb{R}\}), \tag{1.26}$$

where $P_{s,x}$ is the distribution law of the unique solution $X=\{X_t(s,x)\}_{t\geq s}$ to the SDE starting at $X_s = x$. In particular $P_{s,x}(X_t\in B) = P(X_t(s,x)\in B)$. Thus (1.5) and (1.25) yield

$$P_{s,X_s}(X_t\in B) = P(B,t,X_s,s) = P(X_t\in B \mid X_s), \tag{1.27}$$

for all $s\geq t$ and $B\in\mathcal{B}(\mathbb{R})$.

Remark 1.1. If the functions μ and σ are time-independent, the solution $\{X_t(s,x)\}_{t\geq s}$ is a time-homogeneous Markov process (see [27, p. 114]). Note also that for $t\geq s$,

$$P_{s,x}(X_t\in B) = P(B,t,x,s) = P(B,t-s,x,0) = P_{0,x}(X_{t-s}\in B).$$

If this is the case, the Markov family is

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{P_x, x\in\mathbb{R}\}), \tag{1.28}$$

where P_x is the distribution law of the unique solution $X=\{X_t(0,x)\}_{t\geq 0}$ to the SDE starting at $X_0 = x$.

Proposition 1.7. (Strong Markov property) *Let $X=\{X_t(s,x)\}_{t\geq s}$ be a homogeneous Itô diffusion and τ a P_x-finite stopping time. Then X is a strong Markov process, that is, the following condition is satisfied*

$$P_{s,x}(X_{t+\tau}\in B\mid \mathcal{F}_\tau) = P_{s,x}(X_{t+\tau}\in B\mid X_\tau) = P_{\tau,X_\tau}(X_t\in B), \tag{1.29}$$

for all $t\geq s$ and $B\in\mathcal{B}(\mathbb{R})$.

See Friedman [10] or Øksendal [27, p. 117] for a proof.

Definition 1.7. The *infinitesimal generator* $\mathcal{A}$ of a Markov process X is defined by

$$\mathcal{A}f(t,x) := \lim_{s\downarrow 0} \frac{E_{t,x}f(s+t, X_{s+t}) - f(t,x)}{s}, \tag{1.30}$$

where $E_{t,x}$ is the expectation with respect to $P_{t,x}$. Denote by $\mathcal{D}_{\mathcal{A}}$ the set of functions $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ for which the limit (1.30) exists for all $s \geq 0$ and $x \in E$.

Denote by $\mathcal{C}_0^{1,2}$ the class of functions $f \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ with compact support. It turns out that $\mathcal{C}_0^{1,2} \subset \mathcal{D}_{\mathcal{A}}$.

Proposition 1.8. *Let $X = \{X_t\}_{t\geq 0}$ be an Itô diffusion solving*

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dB_t, \tag{1.31}$$

where μ and σ are continuous. If $f \in \mathcal{C}_0^{1,2}$ then

$$\mathcal{A}f = \mu\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial^2 x} + \frac{\partial f}{\partial t}. \tag{1.32}$$

See Øksendal [27, pp. 123 and 220] for a proof.

It is convenient to define the differential operator $\mathbb{L}_X$ associated to the Itô difussion X by

$$\mathbb{L}_X := \mu\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial^2 x}. \tag{1.33}$$

Then, equation (1.32) becomes

$$\mathcal{A}f = \mathbb{L}_X f + \frac{\partial f}{\partial t}. \tag{1.34}$$

Proposition 1.9. (Dynkin's formula) *Let $X = \{X_t(s,x)\}_{t\geq s}$ be an Itô diffusion and $f \in \mathcal{C}_0^{1,2}$. Suppose that τ is a stopping time with $\tau \geq s$ such that $E_{s,x}\tau < \infty$. Then*

$$E_{s,x}f(\tau, X_\tau) = f(s,x) + E_{s,x}\int_s^\tau \mathcal{A}f(r, X_r)dr, \tag{1.35}$$

where $\mathcal{A}$ is the infinitesimal generator of X.

See Klebaner [21, Section 6.1] or Øksendal [27, p. 124] for a proof.

Theorem 1.10. (Girsanov) *Let $X = \{X_t\}_{t\geq 0}$ be an Itô process of the form*

$$dX_t = u(t,\omega)\, dt + dB_t, \qquad 0 \leq t \leq T, \quad X_0 = 0 \tag{1.36}$$

and define the process

$$M_t := \exp\left(-\int_0^t u(s,\omega)\, dB_s - \frac{1}{2}\int_0^t u^2(s,\omega)\, ds\right). \tag{1.37}$$

If the Novikov's condition is satisfied, namely,

$$E\left(\exp\left\{\frac{1}{2}\int_0^T u^2(s,\omega)\, ds\right\}\right) < \infty, \tag{1.38}$$

then

(a) *M_t is a martingale;*

(b) *X is a Brownian motion with respect to the measure*

$$Q(A) = E(I_A M_T), \qquad A \in \mathcal{F}_T. \tag{1.39}$$

See Øksendal [27, p. 162] for a proof.

Chapter 2

Arbitrage-free price of contingent claims

A *security* is a financial instrument such as a stock, a bond, or other asset. A *derivative security* is a contract whose value depends on the value of other security known as the *underlying*. A common type of derivative is the *European option*. Under this contract, the owner agrees to sell (put option) or buy (call option) certain asset at a specified future date (maduration) and at a specified price (strike price). If the owner is allowed to exercise the option *before* maduration, the contract is an *American option*.

Given the possibility of making a non-zero profit without risk by simultaneously sell and buy contracts, it sounds reasonable that such contracts must be worth a positive value to offset the inherent risk of the asset, which is *fair* for both parties. Otherwise, there is *arbitrage* in the market, that is, any party may trade and make a profit without risk.

In this chapter we develope the so-called Arbitrage-Free Pricing Theory and show that the absence of arbitrage is equivalent to the existence of a measure, called risk-neutral or martingale measure, which makes the discounted value of the underlying to be a martingale. This important result is applied later to the pricing of the European and American options under the discrete and continuous security models. In the continuous time setting, the problem is solved for the European option and its price is refered as the *Black-Sholes pricing formula*. For the American option, due to the flexibility of exercise it at any time until maduration, the determination of its price raises as the solution to an optimal stopping problem.

2.1 Discrete time

2.1.1 Arbitrage-Free Pricing Theory

Set $\mathcal{T} = \{0, 1, \ldots, T\}$ with $T < \infty$ and consider a financial market with $n+1$ securities

$$\mathbf{S}_t = (\beta_t, S_t^1, \ldots, S_t^n), \qquad t \in \mathcal{T}, \tag{2.1}$$

where S_t^i is the value of the risky security i at time t, and the riskless security value β_t corresponds to the cost at time t for lending one unit of money at the initial time $t = 0$ at a constant interest rate $r > 0$, so that

$$\beta_t = e^{rt}. \tag{2.2}$$

Assume that the market $\{\mathbf{S}_t\}_{t\in\mathcal{T}}$ is defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathcal{T}}, P)$, and $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ is the natural filtration for S_t.

A *portfolio process* is a random vector

$$\theta_t = (\theta_t^0, \theta_t^1, \dots, \theta_t^n), \qquad t = 1, 2, \dots, T, \tag{2.3}$$

where θ_t^i is the number of units of security i to be bought (if positive) or sold (if negative) from time $t-1$ to t. Since the choice of θ_t depends only on the information of the prices up to time $t-1$, then the process $\{\theta_t\}_{t\in\mathcal{T}}$ is *predictable*, that is, θ_t is $\mathcal{F}_{t-1}$-measurable.

The *initial portfolio value* is given by

$$V_0 = \theta_1^0\beta_0 + \sum_{i=1}^n \theta_1^i S_0^i, \tag{2.4}$$

while the *portfolio value* at time t before any transaction are made at this same time is

$$V_t = \theta_t^0\beta_t + \sum_{i=1}^n \theta_t^i S_t^i, \qquad t = 1, 2, \dots, T. \tag{2.5}$$

Since the process $\{\theta_t\}_{t\in\mathcal{T}}$ is adapted to $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$, so is $\{V_t\}_{t\in\mathcal{T}}$.

Definition 2.1. A portfolio process $\{\theta_t\}_{t\in\mathcal{T}}$ is *self-financing* if

$$V_t = \theta_{t+1}^0\beta_t + \sum_{i=1}^n \theta_{t+1}^i S_t^i, \qquad t = 1, \dots, T-1. \tag{2.6}$$

Using equations (2.5) and (2.6), the change in the portfolio value $\Delta V_t := V_{t+1} - V_t$ is given by

$$\Delta V_t = \theta_{t+1}^0\Delta\beta_t + \sum_{i=1}^n \theta_{t+1}^i \Delta S_t^i, \qquad t = 1, \dots, T-1, \tag{2.7}$$

where $\Delta S_t^i := S_{t+1}^i - S_t^i$ and $\Delta\beta_t := \beta_{t+1} - \beta_t$. Thus, the self-financing feature ensures that any change of the portfolio value at any time is only due to the changes of the security prices. Moreover, note that

$$V_t = V_0 + \sum_{u=0}^{t-1} \Delta V_u, \qquad t = 1, 2, \dots, T,$$

which provides another representation for the portfolio value (2.5), namely,

$$V_t = V_0 + \sum_{u=0}^{t-1} \theta_{u+1}^0 \Delta\beta_u + \sum_{u=0}^{t-1}\sum_{i=1}^n \theta_{u+1}^i \Delta S_u^i, \qquad t = 1, \dots, T. \tag{2.8}$$

Consider two portfolios P_1 and P_2 with value processes $\{V_t^1\}_{t\in\mathcal{T}}$ and $\{V_t^2\}_{t\in\mathcal{T}}$, respectively, and assume that

$$V_T^1 = V_T^2, \tag{2.9}$$

Then,

$$V_t^1 = V_t^2, \qquad t \in \mathcal{T}, \tag{2.10}$$

must hold in order to rule out a riskless profit. If this two portfolios did not have the same value at any time $0 \leq t < T$, say $V_t^1 < V_t^2$, then, at this time, sell the portfolio P_2 with value V_t^2 and buy the portfolio P_1 with value V_t^1. This immediately produces a riskless profit $V_t^2 - V_t^1 > 0$ over the period time from t to T since the portfolios have the same value at time T so that any obligation of P_2 is covered with P_1.

Thus, if V_0^1 is known and V_0^2 is unknown, the above discussion yields to the intuitive conclusion that the initial price of P_2 is given by $V_0^2 = V_0^1$. Then, V_0^2 is the *free-arbitrage price* of P_2.

An *arbitrage opportunity* is the existence of a portfolio which makes a riskless profit. Formaly, we have the next definition.

Definition 2.2. There is an arbitrage opportunity in the market $\{\mathbf{S}_t\}_{t\in\mathcal{T}}$ if there exists a self-financing portfolio $\{\theta_t\}_{t\in\mathcal{T}}$ such that

- $V_0 = 0$,
- $V_T \geq 0$ a.s., and
- $E(V_T) > 0$.

The absence of arbitrage opportunities in the market is a fundamental concept for determining the fair price of a derivative.

Definition 2.3. A *contingent claim* X is a $\mathcal{F}_T$-measurable function, representing the payoff of a derivative at time T.

In the case of the European call and put option, the contingent claim is the payoff function $X = \{S_T - K\}_+$ and $X = \{K - S_T\}$, respectively, where S_T is value of the underlying at time T.

The problem is to determine the price of a contingent claim X at the current time $t = 0$. To do this, suppose that we can find a self-financing portfolio $\{\theta_t\}_{t\in\mathcal{T}}$ such that its value at time T *attains* X, that is

$$V_T = X.$$

It is called a *replicating portfolio* and we say that $\{\theta_t\}_{t\in\mathcal{T}}$ *replicates* the contingent claim X, or that X is *attainable* by $\{\theta_t\}$.

Suppose that $\{\theta_t\}_{t\in\mathcal{T}}$ replicates the claim X. The portfolio value at time t is the amount of wealth needed at that time in order to *hedge* (offset) the accompanying risk when selling the claim X. This is why $\{\theta_t\}_{t\in\mathcal{T}}$ is also called *hedging portfolio* . At the time T, the writer will pay the obligation X which coincide with the value of the portfolio V_T. Thus, V_0 should be the fair price for the contingent claim X to avoid arbitrage opportunities. The next theorem states in a nutshell what we have just discussed.

Theorem 2.1. (Arbitrage-Free Pricing) *Let X be a contingent claim and suppose that there exists a replicating portfolio $\{\theta_t\}_{t\in\mathcal{T}}$ that attains X, whose initial value is V_0. If there are no arbitrage opportunities in the market, then V_0 is the fair price for X.*

The Fundamental Asset Pricing Theorem

According to Theorem 2.1, in order to find the price of a contingent claim X, it is required to find a replicating portfolio and then calculate its initial value V_0. In general, it is difficult to construct a replicating portfolio; even worse, this portfolio not need to exist. The markets free of this ambiguity have a particular name.

Definition 2.4. The market $\{\mathbf{S}_t\}_{t\in\mathcal{T}}$ is *complete* if every contingent claim X is attainable

Suppose that there exists a self-financing portfolio $\{\theta(t)\}_{t\in\mathcal{T}}$ that replicates the contingent claim X. There is an alternative method for calculating the initial portfolio value V_0 (fair price of X) using martingale methods without an explicit determination of the portfolio process. To explain how this can be done consider the representation for the portfolio value in (2.8) at time T:

$$X = V_0 + \sum_{u=0}^{T-1} \theta^0_{u+1}\Delta\beta_u + \sum_{u=0}^{T-1}\sum_{i=1}^{n} \theta^i_{u+1}\Delta S^i_u \tag{2.11}$$

and introduce the variables $I_0 := \sum_{u=0}^{T-1} \theta^0_{u+1}\Delta\beta_u$ and

$$I_i := \sum_{u=0}^{T-1} \theta^i_{u+1}\Delta S^i_u, \qquad i = 1, 2, \ldots, n. \tag{2.12}$$

Thus, equation (2.11) reads

$$X = V_0 + \sum_{i=0}^{n} I_i. \tag{2.13}$$

Now define the discounted price process of the security $i = 1, 2, \ldots, n$ is defined by

$$D^i_t := S^i_t/\beta_t, \qquad t \in \mathcal{T}. \tag{2.14}$$

Next, take $\tilde{V}_t = V_t/\beta_t$ so that from (2.5) we see that

$$\tilde{V}_t = \sum_{i=1}^{n} \theta^i_t D^i_t. \tag{2.15}$$

Apply the same reasoning used in (2.11)-(2.13) to obtain

$$\frac{X}{\beta_T} = V_0 + \sum_{i=1}^{n} \tilde{I}_i, \tag{2.16}$$

where $\tilde{I}_i$ is given as

$$\tilde{I}_i = \sum_{u=0}^{T-1} \theta^i_{u+1}\Delta D^i_u, \qquad i = 1, 2, \ldots, n, \tag{2.17}$$

and $\Delta D^i_u = D^i_{u+1} - D^i_u$.

Suppose that a new probability measure Q can be found so that the price process $\{D^i_t\}_{t\in\mathcal{T}}$ is a martingale for all $i = 1, 2, \dots, n$, that is,

$$E_Q(\Delta D^i_t \mid \mathcal{F}_t) = E_Q(D^i_{t+1} - D^i_t \mid \mathcal{F}_t) = 0, \quad \forall\, t = 0, 1, \dots, T-1.$$

Thus, by taking expectation in (2.17) get

$$\begin{aligned} E_Q(\tilde{I}_i) = &= E_Q\left(\sum_{u=0}^{T-1} \theta^i_{u+1} \Delta D^i_u \mid \mathcal{F}_0\right) \\ &= \sum_{u=0}^{T-1} E_Q[\, E_Q(\theta^i_{u+1} \Delta D^i_u \mid \mathcal{F}_u) \mid \mathcal{F}_0] \\ &= \sum_{u=0}^{T-1} E_Q[\, \theta^i_{u+1}\, E_Q(\Delta D^i_u \mid \mathcal{F}_u) \mid \mathcal{F}_0] \\ &= 0. \end{aligned}$$

Finally, putting all togheter in (2.16), after taking expectation E_Q, obtain

$$V_0 = E_Q(X/\beta_T) = e^{-rT} E_Q(X). \tag{2.18}$$

The measure Q is called *martingale measure* or *risk-neutral measure* for the market $\{\mathbf{S}_t\}_{t\in\mathcal{T}}$.

Remark 2.1. The martingale measure vanishes all the terms of the sum on the right-hand side of (2.16) after taking the expectation E_Q. This does not happen for any other measure for the first sumand

$$I_0 = \sum_{u=0}^{T-1} \theta^0_{u+1} \Delta\beta_u,$$

in (2.13), since the deterministic process $\{\beta_t\}_{t\in\mathcal{T}}$ is not a martingale under any measure.

It seems natural to ask the martingale measure Q to be equivalent to the original measure P. Thus, we have the next definition.

Definition 2.5. Let $\mathbf{S}_t = (\beta_t, S^1_t, \dots, S^n_t)$ be a financial market defined on $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ is the natural filtration. A probability measure Q is said to be a *martingale measure* or *risk-neutral measure* if the following conditions are satisfied:

i) $Q \sim P$, that is, $P(A) = 0$ if and only if $Q(A) = 0$; and

ii) $\{D^i_t\}_{t\in\mathcal{T}}$ is a martingale with respect to Q, for all $i = 1, 2, \dots, n$.

The pricing formula (2.18) is very appealing, since all what is needed to do is to calculate the conditional expectation of a known random variable. However, to obtain the formula (2.18) we made an important assumption, namely, the existence of a martingale measure Q for the discounted price process $\{D_t\}_{t\in\mathcal{T}}$.

A number of authors have proved the existence of a martingale measure in different ways; see Dalang, Morton and Willinger [6], Schachermayer [35], Rogers [33, Theorem 1], Pliska [32, p. 243], and Musiela [25, Theorem 2.6.1] for a proof. This result is called the *First Fundamental Theorem of Asset Pricing*, and we state it below.

Theorem 2.2. (First Fundamental Theorem of Asset Pricing)
Suppose that the number of trading periods is $T < \infty$. The following statements are equivalent:

(i) *The market is arbitrage-free, that is, there does not exist any arbitrage opportunity.*

(ii) *There exists a martingale measure for the market.*

In any of these cases the fair price of a replicable contingent claim X is

$$V_0 = e^{-rT} E_Q(X), \tag{2.19}$$

where V_0 is the initial value of any replicating portfolio.

A natural question arises: under which conditions the martingale measure, if it exists, is unique? Of course, there could be more than one. For instance, if the discounted price process $\{D_i(t)\}$ is constant, then any measure Q makes such a process a martingale. The feature of a market to have several martingale measures is closely related to incomplete markets. The next theorem characterise complete markets as those with a unique martingale measure. A proof of this result can be found in Musiela [25, Theorem 2.6.2].

Theorem 2.3. (Second Fundamental Theorem of Asset Pricing)
Suppose that the number of trading periods is $T < \infty$. An arbitrage-free market is complete if and only if there is only one martingale measure.

Completeness is the property that gives to the market a unique way of pricing contingent claims.

We have seen that the portfolio value process $\{V_t\}$ gives the fair price of the contingent claim X at any time under the assumption of a free-arbitrage market. In general, we can figure out the price of X at any time $t < T$ with the same reasoning. To see this, consider the representation (2.8), so that

$$X = V_t + \sum_{u=t}^{T-1} \theta_{u+1}^0 \Delta\beta_u + \sum_{u=t}^{T-1} \sum_{i=1}^{n} \theta_{u+1}^i \Delta S_u^i. \tag{2.20}$$

Then, the analogue of (2.16) is

$$\frac{X}{\beta_T} = \frac{V_t}{\beta_t} + \sum_{i=1}^{n} \tilde{I}_i, \tag{2.21}$$

where

$$\tilde{I}_i = \sum_{u=t}^{T-1} \theta_{u+1}^i \Delta D_u^i. \tag{2.22}$$

If Q is a martingale measure, after taking conditional expectation with respect to $\mathcal{F}_t$ under Q, we found that

$$\frac{V_t}{\beta_t} = E_Q(X/\beta_T \mid \mathcal{F}_t). \tag{2.23}$$

We conclude this section with the next result.

Proposition 2.4. *Consider an arbitrage-free market $\{\mathbf{S}_t\}_{t\in\mathcal{T}}$ and suppose that a contingent claim X is attainable. Then, the fair price of X at an arbitrary time $t \in \mathcal{T}$ is given by*

$$V_t = e^{-r(T-t)} E_Q(X \mid \mathcal{F}_t), \tag{2.24}$$

for each martingale measure Q.

The European option

A European option is an agreement between two parties to purchase or sell a certain security (or asset) by a fixed price K called the *strike price*, in a future time T called the *maturity*. A European *call* gives the holder the right to purchase the asset, while a European *put* gives the owner the right to sell the asset.

Example 2.6. The payoff of European call option is $X = \{S_T - K\}_+$. Thus, the price of the call option is

$$C_{Eur} = e^{-rT} E_Q(\{S_T - K\}_+).$$

Similarly, the price of the put option is

$$P_{Eur} = e^{-rT} E_Q(\{K - S_T\}_+).$$

2.1.2 American option pricing

The material developed in this part is close to the approach of Elliott and Kopp [9].

The European options can only be exercised *at* maturity time T. In this subsection we introduce the *American option*, which can be exercised at *any time* before maturity. Specifically, an American option is an agreement between two parties to purchase or sell certain asset at a given strike price K, *on or before* maturity T. An American call gives its holder the right to buy the asset, while an American put gives the right to sell it.

If the American option is exercised at time t, the claim is the payoff $X_t = \{S_t - K\}_+$ for a call and $X_t = \{K - S_t\}_+$ for a put. Let $A(t)$ be the expected gain if the option is exercised at the time t. First,

$$A(T) = X_T.$$

Now argue backwards in time. At time $T-1$, the holder has two options: either exercise the put to obtain X_{T-1}, or wait until the next period to obtain X_T. The latter option is equivalent to a European claim running from $T-1$ to T. Since the holder wants to earn the biggest profit, we conclude that

$$A(T-1) = \max\{\, X_{T-1},\, e^{-r}\, E_Q(A(T) \mid \mathcal{F}_{T-1})\,\}, \tag{2.25}$$

where Q is the martingale measure of the market. Next, argue inductively to obtain

$$A(t) = \max\{\, X_t,\, e^{-r}\, E_Q(A(t+1) \mid \mathcal{F}_t)\,\}, \qquad \forall\, 0 \le t \le T-1. \tag{2.26}$$

Thus, going backwards until time $t = 0$ the holder obtains the maximum profit $A(0)$.

A closed form for the American option price

We want to stop at the time that X_t, the payoff of the American option, becomes maximum on the entire period from 0 to T. The random variable X_t depends on the state of the price S_t. Thus, X_t is maximised at different times.

It is natural to think in the *fair price* A of the American option as the discounted payoff when the holder follows an optimal stopping rule. That is,

$$A = \sup_{0 \le \tau \le T} e^{-r\,\tau}\, E_Q(X_\tau), \tag{2.27}$$

where the supremum is taken over all stopping times τ taking values in $\{0, 1, 2, \ldots, T\}$. The article of Merton [23] discusses why this quantity must be the *fair price* of an American option, in the absence of arbitrage.

In what follows, we shall prove that our value for $A(0)$ in (2.26), which was found by iteration, actually coincides with A above. To do this, recall the recursive formula

$$A(t) = \max\{\, X_t, \quad e^{-r}\, E_Q(A(t+1) \mid \mathcal{F}_t)\,\}, \qquad \forall\, 0 \le t \le T-1.$$

If we discount both the payoff and the price processes, we obtain

$$Y_t := e^{-rt}\, X_t, \quad \tilde{A}(t) = e^{-rt}\, A(t); \tag{2.28}$$

then, formula (2.26) becomes

$$\tilde{A}(t) = \max\{\, Y_t, \quad E_Q(\tilde{A}(t+1) \mid \mathcal{F}_t)\,\}, \qquad \forall\, 0 \le t \le T-1. \tag{2.29}$$

Notice that $\tilde{A}(0) = A(0)$. We may identify the process $\{\tilde{A}(t)\}$ with the one described in equations (B.2)-(B.3) with respect to the discounted payoff $\{Y_t\}_{t=0}^{T}$ (see Appendix B). Therefore, Theorem B.5 implies that the stopping time

$$\tau^* = \inf\{k \le T :\, \tilde{A}(k) = Y_k\}, \tag{2.30}$$

is optimal, that is,

$$E_Q(Y_{\tau^*}) = \sup_{\tau \le T} E_Q(Y_\tau). \tag{2.31}$$

Moreover, the second part of Theorem B.5 implies

$$\tilde{A}(0) = E_Q(Y_{\tau^*}).$$

Since $\tilde{A}(0) = A(0)$, we finally obtain

$$A(0) = \sup_{\tau \le T} E_Q(e^{-r\,\tau} X_\tau). \tag{2.32}$$

We have proved the next theorem.

Theorem 2.5. (American option price in discrete time) *Assume that there exists a martingale measure Q and consider the process $\{\tilde{A}(t)\}_{t=0}^{T}$ defined by (2.29) with respect to the discounted payoff $\{Y_t\}_{t=0}^{T}$ with $Y_t = e^{-rt} X_t$. Then, the value of the American option*

$$A = \sup_{\tau \le T} e^{-r\,\tau} E_Q(X_\tau),$$

is equal to $A = \tilde{A}(0)$ and the optimal exercise time is

$$\tau^* = \inf\{k \le T : \tilde{A}(k) = Y_k\}.$$

No-early exercise for American call

Proposition 2.6. *The value of the American call option on a non-dividend paying asset is the same as the value of the European call option.*

Proof. We will show that for the American call option it is optimal to wait until maturity T.

Set $X_t = \{S_t - K\}_+$, for each $t = 0, 1, \ldots, T$. Recall the time t price of the European call option

$$V_t = e^{-r(T-t)} E_Q(X_T \mid \mathcal{F}_t).$$

The mapping $x \mapsto \{x - K\}_+$ is convex, then Jensen's inequality for conditional expectations implies

$$\{E_Q(S_T \mid \mathcal{F}_t) - K\}_+ \le E_Q(\{S_T - K\}_+ \mid \mathcal{F}_t),$$

and after multiplication by $e^{-r(T-t)}$ and using that $e^{-rt} S_t$ is a martingale under Q, we obtain

$$\{S_t - e^{-r(T-t)} K\}_+ \le e^{rt} E_Q(e^{-rT}\{S_T - K\}_+ \mid \mathcal{F}_t).$$

Since $e^{-r(T-t)} \le 1$, the last inequality becomes

$$e^{-rt}\{S_t - K\}_+ \le E_Q(e^{-rT}\{S_T - K\}_+ \mid \mathcal{F}_t).$$

Thus, we conclude that the process

$$Y_t := e^{-rt}\{S_t - K\}_+ = e^{-rt} X_t,$$

is a submartingale.

Let τ be a stopping time of $\{Y_s\}_{s=0}^{T}$ taking values in $\{t, \ldots, T\}$. Then, the Optional Sampling Theorem implies

$$Y_\tau \le E_Q(Y_T \mid \mathcal{F}_\tau),$$

which in turn yields, after taking conditional expectation, to

$$E_Q(e^{-r(\tau-t)} X_\tau \mid \mathcal{F}_t) \le e^{-r(T-t)} E_Q(X_T \mid \mathcal{F}_t), \quad \text{for all } t = 0, 1, \ldots, T. \tag{2.33}$$

The right-hand side is the value of the European call option at time t, while the left-hand side is the expected payoff of the American call when following the stopping rule τ. Thus, we conclude that the value of the European call cannot be less than the value of the American call, whichever stopping rule τ we select. Therefore, if one is the holder of an American call, it is optimal to wait until maturity T to exercise the option and thus the price of the American put is the same as the European counterpart. □

Remark 2.2. If the asset does not pay dividend, the American call should be worth it more than the European call.

Remark 2.3. We used the fact that the payoff function for the call is convex, which is not the case for the payoff of the put. Actually, the put case is rather difficult, and the main objective of the present work is to determine the optimal stopping rule.

2.2 The binomial model

Consider a market consisting of a riskless security with time t price $\beta_t = e^{rt}$, and a single risky security whose time t price S_t is determined by the so-called *binomial model*, which we derive in detail later.

In this section we will explicitly calculate the pricing formula for a given contingent claim X, by means of Proposition 2.4. To do this, we will figure out the martingale measure Q. First, we tackle the problem in a one-period framework and then pass to the T-period case.

Binomial model for price processes

Probably, the simplest model for the price of a single security evolving in discrete time is the binomial model. It offers a good approximation for the price behaviour and, despite its simplicity, we will see later that this model turns to be the basis for the most common and useful models in continuous time.

Let $Z_1, Z_2, \ldots, Z_T$ be independent Bernoulli random variables, that is,

$$P(Z_i = 1) = 1 - P(Z_i = 0) = p, \qquad \forall\, i = 1, 2, \ldots, T. \tag{2.34}$$

Then, the partial sum

$$W_n = \sum_{i=1}^{n} Z_i, \tag{2.35}$$

follows the *binomial distribution* given by

$$P(W_n = k) = C_k^n\, p^k (1-p)^{n-k}. \tag{2.36}$$

The process $\{W_n\}_{n=1}^{T}$ is called a *p-random walk.*

Suppose that the initial price is $S_0 = S > 0$, and at the next times, the price either goes up by a factor u with probability p or goes down by a factor d with probability $1-p$, with $d < u$.

It is clear that the price of the security at time t takes the form

$$S_t = S\, u^k\, d^{t-k}, \tag{2.37}$$

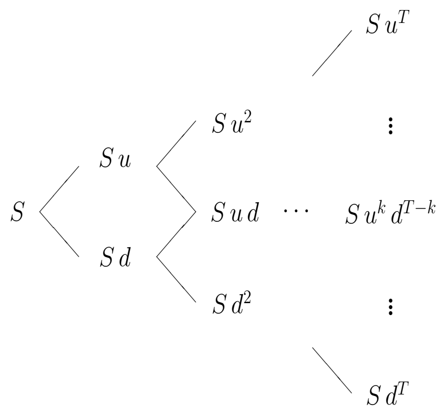

Figure 2.1: Binomial tree model

for any $0 \le k \le t$, see the Figure 2.1. Moreover, it is easily seen that

$$P(S_t = S\,u^k\,d^{t-k}) = P(W_t = k) = C_k^t\,p^k\,(1-p)^{t-k}, \tag{2.38}$$

and thus, we can write down the process S_t as

$$S_t = S\,u^{W_t}\,d^{t-W_t}, \qquad t = 1, 2, \ldots, T. \tag{2.39}$$

This is called the *binomial model* .

The martingale measure

Suppose that $t = 1$ so that $\beta_0 = 1, \beta_1 = e^r$, and say that $S_0 = S$. According to Definition 2.5, we look for a probability measure Q with

$$Q(S_1 = S\,u) = q = 1 - Q(S_0 = S\,d), \tag{2.40}$$

satisfying the condition

$$E_Q(S_1/\beta_1) = S_0. \tag{2.41}$$

Equation (2.41) can be rewritten as

$$e^{-r}\,E_Q(S_1) = S. \tag{2.42}$$

In other words, we look for $q > 0$ so that

$$e^{-r}(S\,u\,q + S\,d\,(1-q)) = S \tag{2.43}$$

Solving the last equation for q we get

$$q = \frac{e^r - d}{u - d}. \tag{2.44}$$

Note 2.4. Since Q is a probability measure, the parameters $d, u,$ and r have some restrictions. To begin $d < u$ in order to the denominator in (2.44) make sense. Moreover, both $u > e^r$ and $d < e^r$ must hold, otherwise $q \ge 1$ or $q \le 0$, respectively.

We conclude that Q is the *unique* martingale measure in the market if and only if

$$d < e^r < u. \tag{2.45}$$

One-period case

We find the fair price for the claim X in two different ways. The first method consists in the construction of a replicating portfolio and then to calculate its initial value. This method works because we already showed that there exists a martingale measure, namely (2.44) above. Then, Theorem 2.2 implies that the initial value of the replicating portfolio is the fair price of X.

Note that the payoff of X has two possible values, say X_u or X_d, depending on the value of the risky security if it goes up or down. Recall that the value of the portfolio at $t = 1$ is

$$V_1 = \theta_1^0 \beta_1 + \theta_1^1 S_1. \tag{2.46}$$

Thus, from (2.46), we obtain the system:

$$\theta_1^0 e^r + \theta_1^1 \, Su = X_u, \tag{2.47}$$

$$\theta_1^0 e^r + \theta_1^1 \, Sd = X_d, \tag{2.48}$$

which always has a solution under the assumption in (2.45). In fact, it is readily seen that

$$\theta_1^0 = \frac{u\, X_d - d\, X_u}{e^r(u-d)}, \qquad \theta_1^1 = \frac{X_u - X_d}{S(u-d)}. \tag{2.49}$$

Therefore, the initial value (recall (2.4)) is given by

$$V_0 = \theta_1^0 + \theta_1^1 S. \tag{2.50}$$

For the second method, we can directly calculate the price of X by means of the martingale measure q in (2.44) using the formula

$$V_0 = e^{-r} E_Q(X) = e^{-r}[X_u\, q + X_d(1-q)]. \tag{2.51}$$

It can be easily verified that equations (2.50) and (2.51) coincide.

T-period case

It turns out that the martingale measure Q' which makes $e^{-rt} S_t$ to be a martingale coincides with Q in (2.40), that is, the quantity q in (2.44) does not depend on the number of periods. To this end, fix t and note that the binomial model (2.39) implies that

$$S_{t+1} = u^{Z_{t+1}}\, d^{1-Z_{t+1}} S_t, \tag{2.52}$$

where Z_{t+1} is a Bernoulli variable so that

$$Q'(Z_{t+1} = 1) = 1 - Q'(Z_{t+1} = 0) = q'. \tag{2.53}$$

In particular,

$$E_{Q'}(u^{Z_{t+1}}\, d^{1-Z_{t+1}}) = u\, q' + d(1-q'). \tag{2.54}$$

We require that

$$E_{Q'}\left(\frac{S_{t+1}}{e^{r(t+1)}} \mid \mathcal{F}_t \right) = \frac{S_t}{e^{rt}}, \tag{2.55}$$

and we will see that $q' = q$. Combining (2.55) with the $\mathcal{F}_t$ measurability of S_t, together with the fact that Z_{t+1} is independent of $\mathcal{F}_t$, we obtain

$$\begin{aligned} 0 &= E_{Q'}(e^{-r(t+1)}\, S_{t+1} - e^{-rt} S_t \mid \mathcal{F}_t) && (2.56)\\ &= E_{Q'}(e^{-r}\, u^{Z_{t+1}}\, d^{1-Z_{t+1}}\, S_t - S_t \mid \mathcal{F}_t) && (2.57)\\ &= E_{Q'}(e^{-r}\, u^{Z_{t+1}}\, d^{1-Z_{t+1}} - 1 \mid \mathcal{F}_t) && (2.58)\\ &= e^{-r}\, E_{Q'}(u^{Z_{t+1}}\, d^{1-Z_{t+1}}) - 1. && (2.59) \end{aligned}$$

By means of (2.54) we finally obtain the martingale measure

$$q' = \frac{1 - d\,e^{-r}}{e^{-r}(u-d)} = \frac{e^r - d}{u - d} = q, \tag{2.60}$$

as claimed.

Valuation algorithm

To find the fair price of the contingent claim X, we use a procedure that relies on the formula (2.24) which can be rewritten as follows: $V(T) = X$ and

$$V_t = e^{-r} E_Q(V_{t+1} \mid \mathcal{F}_t), \qquad \forall\, 0 \le t \le T-1, \tag{2.61}$$

where V_t is the value of a replicating portfolio.

For each t, with $0 \le t \le T$, denote by $V(t,k)$ the value of the replicating portfolio at time t when the price of the security has gone up k times, with $0 \le k \le T$. We can calculate the initial portfolio value V_0 by backwards induction using the martingale measure q and formula (2.51) for a single-period. Let X_k be the value of the contingent claim if the price of the risky security goes up k times and

$$V(T,k) = X_k, \qquad \forall\, 0 \le k \le T. \tag{2.62}$$

Using formula (2.51) and the martingale measure (2.60), the value of the contingent claim at time $T-1$ is

$$V(T-1,k) = e^{-r}[q\, V(T,k+1) + (1-q)\, V(T,k)]. \tag{2.63}$$

Proceeding inductively for each t with $0 \le t \le T-1$, we have

$$V(t,k) = e^{-r}[q\, V(t,k+1) + (1-q)\, V(t,k)], \tag{2.64}$$

until we get to the value V_0, which must be the price of the claim X. See Figure 2.2

2.2.1 Binomial pricing formula for the European option

The approach of this subsection rely on the work of Cox, Ross, and Rubinstein [5].

In what follows we compute the fair price for the European option by direct calculation of the expectation $E_Q(X)$, where X is the payoff function. This is done under the assumption that the price of the underlying asset S_T follows the binomial model

$$S_T = u^{W_T}\, d^{T-W_T}\, S,$$

where $\{W_t\}$ is a q-random walk, and q is the martingale measure in (2.44).

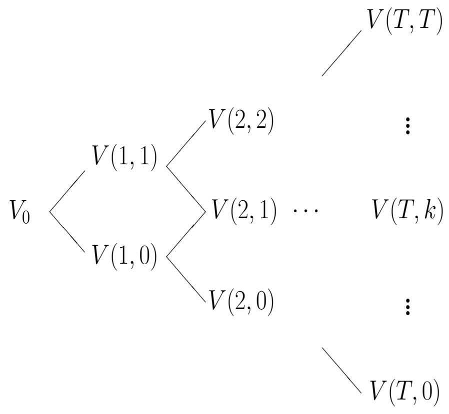

Figure 2.2: Value of contigent claim in time

Lemma 2.7. *Let* $\{W_n\}_{n=1}^T$ *be a* q*-random walk under* Q*, where* $W_n = \sum_{i=1}^n Z_i$ *and* $Q(Z_i = 1) = q$ *for all* $0 \le i \le T$*. Define*

$$Y_T := \prod_{i=1}^{T} e^{\alpha\, Z_i} E_Q^{-1}(e^{\alpha\, Z_i}), \tag{2.65}$$

for some $\alpha \in \mathbb{R}$*. Then,*

$$Q'(A) = E_Q(I_A\, Y_T), \tag{2.66}$$

defines a probability measure. Moreover, $\{W_n\}_{n=1}^T$ *is a* q'*-random walk under* Q'*, i.e.* $Q'(Z_i = 1) = q'$*, where*

$$q' = \frac{q\, e^{\alpha}}{q\, e^{\alpha} + (1-q)}. \tag{2.67}$$

Proof. It is straightforward that Q' is a probability measure. To prove the second part, we calculate the distribution of the variables Z_n under Q'. For each n, we have

$$a := E_Q(e^{\alpha\, Z_n}) = q\, e^{\alpha} + (1-q). \tag{2.68}$$

Fix $n \le T$. Then, using the independence of the variables Z_n we get

$$\begin{aligned}
Q'(Z_n = 1) &= E_Q(\, I_{[Z_n=1]}\, Y_T) && (2.69)\\
&= a^{-T}\, E_Q(\, I_{[Z_n=1]}\, e^{\alpha\, Z_n} \prod_{i \ne n} e^{\alpha\, Z_i}) && (2.70)\\
&= a^{-T}\, E_Q(\, I_{[Z_n=1]} e^{\alpha\, Z_n}) \prod_{i \ne n} E_Q(e^{\alpha\, Z_i}) && (2.71)\\
&= a^{-1}\, E_Q(\, I_{[Z_n=1]} e^{\alpha\, Z_n}) && (2.72)\\
&= a^{-1}\, q\, e^{\alpha}, && (2.73)
\end{aligned}$$

as required. $\square$

Theorem 2.8. (Binomial pricing formula for European option)
Let $X = \{S_T - K\}_+$ *be the payoff of the European call option, and let* q *be the risk-neutral measure that makes* $\{e^{-rt} S_t\}$ *a martingale. Then, the binomial price of the call option is*

$$C_{Eur} = e^{-rT}[S\,B_k(T, q') - K\,B_k(T, q)], \tag{2.74}$$

where $k = \min\{m \in \mathbb{N} : m \geq \log(K/Sd^T)/\log(u/d)\}$, *and*

$$q = \frac{e^r - d}{u - d}, \quad q' = \frac{q\,u}{e^r},$$

and $B_k(T, p)$ *is defined by*

$$B_k(T, p) := \sum_{i=k}^{T} C_i^T\, p^i (1-p)^{T-i}.$$

Proof. Consider the event $A := [S_t \geq K]$. Thus, according to Theorem 2.2, the fair price of the European call option is given by

$$C_{Eur} = e^{-rT}\, E_Q(\{S_T - K\}_+) = e^{-rT}[\,E_Q(S_T\, I_A) - K E_Q(I_A)\,]. \tag{2.75}$$

First we calculate $E_Q(I_A)$. Note that

$$\begin{aligned} E_Q(I_A) &= Q(S_T \geq K) \\ &= Q(S\,u^{W_T}\, d^{T-W_T} \geq K) \\ &= Q(u^{W_T}\, d^{-W_T} \geq K/S\,d^T). \end{aligned}$$

Now take $c := K/Sd^T$ and assume that m satisfies the condition

$$u^m\, d^{-m} \geq c. \tag{2.76}$$

Then, we have that

$$m \geq \frac{\log(c)}{\log(u/d)}, \tag{2.77}$$

and this is so because

$$\begin{aligned} \log(u^m\, d^{-m}) &= \log(u^m/d^m) \\ &= \log(u/d)^m \\ &= m\, \log(u/d). \end{aligned}$$

Let k be the minimum of the m's satisfying (2.77), that is,

$$k = \min\{m \in \mathbb{N} : m \geq \log(K/Sd^T)/\log(u/d)\}. \tag{2.78}$$

Thus,

$$E_Q(I_A) = \sum_{i=k}^{T} Q(W_T = i) = \sum_{i=k}^{T} C_i^T\, q^i (1-q)^{T-i}. \tag{2.79}$$

It remains to figure out $E_Q(S_T\, I_A)$. To this end, set $\alpha = \log(u/d)$ in Lemma 2.7 so that q' in (2.67) transforms into

$$q' = \frac{q(u/d)}{q(u/d) + (1-q)} = \frac{q\,u}{q\,u + (1-q)d} = \frac{q\,u}{e^r}, \tag{2.80}$$

where we use the fact that q is the martingale measure (2.60). Also, note that Y_T becomes

$$Y_T = \frac{1}{a^T} \prod_{i=1}^{T} \left(\frac{u}{d}\right)^{Z_i} = \frac{1}{a^T} \left(\frac{u}{d}\right)^{W_T} = \frac{1}{a^T} u^{W_T} d^{-W_T}, \tag{2.81}$$

where a is given in (2.68). Thus

$$S_T = S\, u^{W_T}\, d^{T-W_T} = S\, a^T d^T Y_T = S\, Y_T, \tag{2.82}$$

since

$$a\, d = [\, q(u/d) + (1-q)\,]d = q\, u + (1-q)d = 1.$$

Hence, putting equations (2.80)-(2.82) together into $E_Q(S_T\, I_A)$, and using the change of measure (2.66) in Lemma 2.7, we obtain

$$\begin{aligned} E_Q(S_T\, I_A) &= S E_Q(I_A\, Y_T) && (2.83)\\ &= S\, Q'(A) && (2.84)\\ &= S\, E_{Q'}(I_A) && (2.85) \end{aligned}$$

The same ideas preceding (2.79) for calculating the expectation $E_Q(I_A)$ can be applied in this case for $E_{Q'}(I_A)$, obtaining

$$E_{Q'}(I_A) = \sum_{i=k}^{T} Q'(W_T = i) = \sum_{i=k}^{T} C_i^T\, (q')^i (1-q')^{T-i}, \tag{2.86}$$

where k is given by (2.78). □

Note 2.5. Along with the put-call parity relation (see [30] and [37, p. 155]), the price P_{Eur} of a *European put option* is expressed in terms of the call price C_{Eur} in (2.74). Specifically, the put-call parity says that

$$P_{Eur} = C_{Eur} - S - K\, e^{-rT}. \tag{2.87}$$

2.2.2 Binomial algorithm for the American option

Suppose that the risky asset price follows the binomial model $S_t = S\, u^{W_t}\, d^{t-W_t}$. Consider an American put, that is, set $X_t = \{K - S_t\}_+$ for all $t = 0, 1, \ldots, T$. In analogy with the procedure (2.64), denote by $A(t,k)$ the price of the American put at time t, when the price has gone up k times. Also, say that the payoff at time t is the price has gone up k times is

$$X_t(k) = \{K - S\, u^k\, d^{t-k}\}_+. \tag{2.88}$$

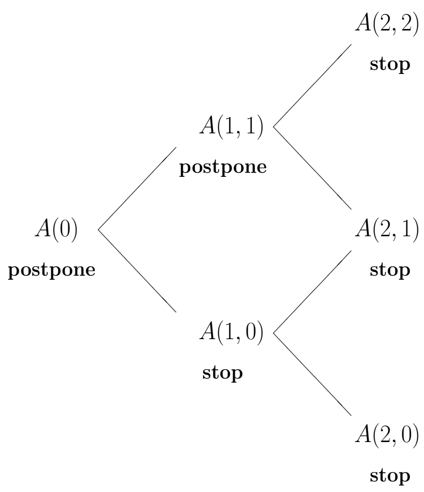

Figure 2.3: Binomial model for American price

We start with

$$A(T,k) = X_T(k), \qquad \forall\, 0 \le k \le T. \tag{2.89}$$

Now, for all $0 \le t \le T-1$, according to the formula (2.26)

$$A(t,k) = \max\{\, X_t(k),\ e^{-r}[q\, A(t+1,k+1) + (1-q)\, A(t+1,k)]\,\}, \tag{2.90}$$

for each $0 \le k \le t$. The value $A(0,0)$ (corresponding to $A(0)$ in (2.26)) is the fair price of the put.

Example 2.7. We give an explicit calculation. Set the initial price $S = 4$, the up factor $u = 2$, the down factor $d = 1/2$, the instantaneous interes rate $r = 1/4$, the strike price $K = 5$, and maturity $T = 2$. Then, we have the martingale measure and discount factor

$$q = \frac{e^{1/4} - 1/2}{2 - 1/2} \approx 0.52, \quad e^{-r} \approx 0.78$$

For ease of calculations, we suppose that $q = 1/2$ and $e^{-r} = 4/5$. The payoff process $\{X_t(k)\}$ in (2.88) is

$$\begin{aligned}
X_2(0) =& \{K - S\, d^2\}_+ = \{5-1\}_+ = 4 \\
X_2(1) =& \{K - S\, ud\}_+ = \{5-4\}_+ = 1 \\
X_2(2) =& \{K - S\, u^2\}_+ = \{5-16\}_+ = 0 \\
X_1(0) =& \{K - S\, d\}_+ = \{5-2\}_+ = 3 \\
X_1(1) =& \{K - S\, u\}_+ = \{5-8\}_+ = 0 \\
X_0(0) =& \{K - S\}_+ = \{5-4\}_+ = 1.
\end{aligned}$$

Now we figure out (2.90), starting at the terminal time $T = 2$ and then backwards. See Figure 2.3.

$$A(2,0) = X_2(0) = 4$$
$$A(2,1) = X_2(1) = 1$$
$$A(2,2) = X_2(2) = 0.$$

$$\begin{aligned}
A(1,0) &= \max\{X_1(0), e^{-r}[q\, A(2,1) + (1-q)\, A(2,0)]\} \\
&= \max\{3, \frac{4}{5}\left[\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 4\right]\} \\
&= \max\{3, 2\} = 3.
\end{aligned}$$

$$\begin{aligned}
A(1,1) &= \max\{X_1(1), e^{-r}[q\, A(2,2) + (1-q)\, A(2,1)]\} \\
&= \max\{0, \frac{4}{5}\left[\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1\right]\} \\
&= \max\{0, \frac{2}{5}\} = \frac{2}{5}.
\end{aligned}$$

$$\begin{aligned}
A(0,0) &= \max\{X_0(0), e^{-r}[q\, A(1,1) + (1-q)\, A(1,0)]\} \\
&= \max\{1, \frac{4}{5}\left[\frac{1}{2}\cdot \frac{2}{5} + \frac{1}{2}\cdot 3\right]\} \\
&= \max\{1, \frac{34}{25}\} = \frac{34}{25}.
\end{aligned}$$

Therefore, the fair price for this American put is $25/36 \approx 1.36$.

2.3 Continuous time

2.3.1 Arbitrage-free pricing

In analogy with Subsection 2.1.1, we develope the concepts of self-financing portfolio, arbitrage opportunity, and martingale measure, among others, and state the Fundamental Theorem of Asset Pricing in this context.

Previously, we worked with the binomial model (2.39) for the price process of a risky security evolving in discrete time. By means of the Central Limit Theorem, it can be proved that the binomial model converges in law to the Geometric Brownian Motion, whose differential form is

$$dS_t = S_t[\mu\, dt + \sigma\, dB_t].$$

We refer to Shreve [36, p. 91] or Kijima [19, Section 11.3] for a proof of this fact. This is the model for the price processes adopted by Black and Scholes in their work [3]. We will take this setting for pricing the European and American options in continuous time.

Consider a financial market with $n+1$ securities

$$\mathbf{S}_t = (\beta_t, S_t^1, \ldots, S_t^n), \qquad t \in [0, T], \tag{2.91}$$

defined on a probability space $(\Omega, \mathcal{F}, P)$, and let $\{\mathcal{F}_t\}_{0 \le t \le T}$ be the natural filtration. The price processes S_t^i of the risky securities are Itô diffusions satisfying

$$d\beta_t = r(t)\beta_t\, dt, \tag{2.92}$$

$$dS_t^i = S_t^i[\mu(t)\, dt + \sigma(t)\, dB_t], \quad i = 1, 2, \ldots, n, \tag{2.93}$$

where $\mu(t)$ and $\sigma(t)$ are continuous functions and B_t is a standard Brownian motion under P. Note from (2.92) that β_t is the deterministic function

$$\beta_t = \exp\{\int_0^t r(s)ds\}. \tag{2.94}$$

Portfolio processes

We assume that we can trade the securities continuously by using a portfolio process

$$\theta_t = (\theta_t^0, \theta_t^1, \ldots, \theta_t^n), \qquad t \in [0, T]. \tag{2.95}$$

Here, the portfolio $\{\theta_t\}_{0 \le t \le T}$ is adapted to $\{\mathcal{F}_t\}_{0 \le t \le T}$.

The portfolio value V_t is defined similarly to (2.5) by

$$V_t = \theta_t^0 \beta_t + \sum_{i=1}^n \theta_t^i\, S_t^i. \tag{2.96}$$

Since S_t^i and θ_t^i are Itô processes for each $i = 0, 1, \ldots, n$, from Itô's formula (see Friedman [10, pp. 80-83]) we have

$$dV_t = (\theta_t^0 d\beta_t + \beta_t d\theta_t^0 + d\theta_t^0 d\beta_t) + \sum_{i=1}^n \{\theta_t^i\, dS_t^i + S_t^i\, d\theta_t^i + d\theta_t^i\, dS_t^i\}. \tag{2.97}$$

If the portfolio θ_t does not depend on the security prices, then $d\theta_t^i\, dS_t^i = 0$. Moreover, if

$$\theta_t^0 \beta_t + \sum_{i=1}^n S_t^i\, d\theta_t^i = 0,$$

the changes in the portfolio value can only take place due to changes on the underlying security prices, not on the holding of the securities. In other words, there is no inflow or outflow of money. This is the intuitive feature of a self-financing portfolio.

Definition 2.8. A portfolio process $\{\theta_t\}_{0 \le t \le T}$ is *self-financing* if

$$dV_t = \theta_t^0 d\beta_t + \sum_{i=1}^n \theta_t^i\, dS_t^i. \tag{2.98}$$

The equation (2.98) becomes

$$V_t = V_0 + \int_0^t \theta_u^0 d\beta_t + \sum_{i=1}^{n} \int_0^t \theta_u^i \, dS_u^i. \tag{2.99}$$

The definition of arbitrage opportunity and contingent claim given in Definition 2.2 and 2.3 have the natural continuous-time analogue. Also, the Arbitrage-Free Pricing Theorem 2.1 is valid in this context.

The Fundamental Asset Pricing Theorem

Let X be a contingent claim and suppose that it is attainable by a replicating portfolio $\{\theta_t\}$. Then, from equation (2.99) we have

$$X = V_0 \int_0^T \theta_u^0 d\beta_t + \sum_{i=1}^{n} \int_0^T \theta_u^i \, dS_u^i, \tag{2.100}$$

which is the analogue of equation (2.13). In the abscense of arbitrage opportunities, Theorem 2.1 implies that the value V_0 is the fair price of the claim X. In the following we express V_0 as the discounted expected value of X under certain probability measure, using martingale methods.

From (2.96), it is easily seen that

$$d(V_t/\beta_t) = \sum_{i=1}^{n} \theta_t^i \, dD_t^i, \tag{2.101}$$

where

$$D_t^i = S_t^i/\beta_t, \qquad i = 1, \ldots, n. \tag{2.102}$$

Note that the sum starts with $i = 1$, since $dD_t^0 = 0$ for all t. Writing equation (2.101) in its integral form

$$\frac{V_t}{\beta_t} = V_0 + \sum_{i=1}^{n} \int_0^t \theta_i(u) \, dD_u^i, \tag{2.103}$$

for all $t \in [0, T]$. In particular, we have

$$\frac{V_T}{\beta_T} = \frac{V_t}{\beta_t} + \sum_{i=1}^{n} \int_t^T \theta_i(u) \, dD_u^i. \tag{2.104}$$

If we *assume* that there exists a probability measure Q, equivalent to the original probability P, so that the process $\{D_t^i\}$ is a martingale for each $i = 1, 2, \ldots, n$ under Q, then

$$E_Q \left(\int_t^T \theta_u^i \, dD_u^i \mid \mathcal{F}_t \right) = 0, \qquad \forall t \in [0, T]. \tag{2.105}$$

The measure Q is called a *martingale measure* . The Definition 2.5 of a martingale measure remains the same in the continuous-time setting.

Hence, by taking conditional expectation in (2.104) and setting $X = V_T$, we would get

$$V_t = \frac{\beta_t}{\beta_T} E_Q(X \mid \mathcal{F}_t). \tag{2.106}$$

In particular, for $t = 0$ we have $S_0^0 = 1$ and so

$$V_0 = \frac{E_Q(X)}{\beta_T}. \tag{2.107}$$

When the interest rate is constant (like in the Black-Scholes setting), say $r(t) = r$, then $S_0(t) = e^{rt}$ and (2.107) takes the form

$$V_0 = e^{-rT} E_Q(X),$$

which is the analogue of (2.18) in Section 2.1.1.

The Arbitrage-Free Pricing Theorem 2.1 states that if there are no arbitrage opportunities then the fair price of the contingent claim X is the initial value of a replicating portfolio, namely, V_0. But, how can we be sure that the market does not admit such arbitrage? The next result, due to the works of Harrison and Kreps [12] and Harrison and Pliska [13], answers this question.

Theorem 2.9. (First Fundamental Theorem of Asset Pricing)
Consider the market (2.91)-(2.93). If there exists a martingale measure (see Definition 2.5), then the market does not have arbitrage opportunities.

Proof. Let Q be a martingale measure for the market $\mathbf{S}_t$. We want to show that there is no arbitrage opportunities (recall Definition 2.2). To see this, let X be a contingent claim and V_t the portfolio value of a replicating portfolio θ_t, and suppose that $V_0 = 0$ and $P(V_T \geq 0) = 1$, where P is the original measure.

Given that the discounted process $D_t^i = S_t^i/\beta_t$ is a martingale for each $i = 1, 2, \ldots, n$ under Q, then the discounted portfolio value

$$\frac{V_t}{\beta_t} = \sum_{i=1}^{n} \theta_t^i D_t^i,$$

is also a martingale under Q. Thus, $E_Q(V_T/\beta_T) = V_0$, so that

$$E_Q(V_T/\beta_T) = 0. \tag{2.108}$$

Since $P \sim Q$ and $P(V_T \geq 0) = 1$, then $Q(V_T \geq 0) = 1$. This last fact together with (2.108), imply $Q(V_T > 0) = 0$. Again, by the equivalence of the measures we also have $P(V_T > 0) = 0$, concluding

$$E(V_T) = 0, \tag{2.109}$$

implying that θ_t does not allow an arbitrage opportunity. Therefore, there does not exist a replicating portfolio with $V_0 = 0$ and finishing with a positive probability of having made money. $\square$

Existence and uniqueness of the martingale measure

For our purposes and to simplify the exposition, we consider a market with a single risky security S_t.

Define $g(t,x) := x/\beta_t$, so that $D_t = g(t, S_t)$ is the discounted price process. Then $g_t = -r(t)\,g$, $g_x = \beta_t$, and $g_{xx} = 0$. By Itô's formula,

$$dD_t = [g_t + u\,g_x + \frac{v^2}{2}\,g_{xx}]dt + v\,g_x\,dB_t,$$

where $u(t,S_t) = \mu(t)S_t$ and $v(t,S_t) = \sigma(t)S_t$, we get

$$\begin{aligned} dD_t &= [-r(t)\,g + \mu(t)\frac{S_t}{\beta_t}]\,dt + \sigma(t)\frac{S_t}{\beta_t}\,dB_t \\ &= [-r(t)\,D_t + \mu(t)D_t]dt + \sigma(t)\,D_t\,dB_t. \end{aligned}$$

Thus, the dynamic of the discounted price process has the form

$$dD_t = D_t[\,\{\mu(t) - r(t)\}\,dt + \sigma(t)\,dB_t\,] \tag{2.110}$$

We want to find a measure that makes D_t a martingale. Proposition 1.5 ensures that a necessary and sufficient condition for this is

$$\mu(t) - r(t) = 0. \tag{2.111}$$

In other words, the discounted process D_t is a martingale if and only if the mean rate of return $\mu(t)$ of the stock is equal to the mean rate of return $r(t)$ of the risk-free security. This is why the martingale measure Q is also called *risk-neutral measure.*

If $\mu(t) = r(t)$ under the measure P, then we are done. In the case in which $\mu(t) \neq r(t)$ under P, we can make a change of variable as follows: rewrite (2.110) as

$$dD_t = \sigma(t)D_t d\left\{\frac{\mu(t) - r(t)}{\sigma(t)}t + B_t\right\} = \sigma(t)D_t d\tilde{B}_t. \tag{2.112}$$

The process

$$\tilde{B}_t = \tilde{\mu}(t)\,t + B_t, \tag{2.113}$$

is a Brownian motion with drift $\tilde{\mu}(t)$, where

$$\tilde{\mu}(t) = \frac{\mu(t) - r(t)}{\sigma(t)}. \tag{2.114}$$

The function $\tilde{\mu}$ is known as *the market price of risk* .

The process

$$M_t = \exp\left\{-\int_0^t \tilde{\mu}(s)\,dB_s - \frac{1}{2}\int_0^t \tilde{\mu}^2(s)\,ds\right\}, \tag{2.115}$$

is a martingale since the Novikov's condition (1.38) holds, that is, $\tilde{\mu}$ satisfies

$$E\left[\exp\{\frac{1}{2}\int_0^T \tilde{\mu}(s)ds\}\right] < \infty. \tag{2.116}$$

Girsanov's Theorem 1.10 implies that $\tilde{B} = \{\tilde{B}_t\}_{0\le t\le T}$ is a standard Brownian motion under the probability measure

$$Q(A) := E_P(I_A\, M_T), \qquad A \in \mathcal{F}_T. \tag{2.117}$$

Since D_t is a continuous process and it has the form

$$dD_t = \sigma(t) D_t\, d\tilde{B}_t, \tag{2.118}$$

then D_t is an Itô integral under Q, and therefore a martingale.

We have just proved the existence of the martingale measure.

Proposition 2.10. *Consider the market (2.91)-(2.93). Then, there exists a martingale measure, namely, the measure Q in (2.117).*

Remark 2.6. Recall that $dS_t = S_t[\mu(t)dt + \sigma(t)dB_t]$ under P. Using (2.113), we notice that the dynamic of the security price process, under Q, is

$$dS_t = S_t[r(t)\, dt + \sigma(t)\, d\tilde{B}_t]. \tag{2.119}$$

That is, the mean rate of return of the stock equals $r(t)$. Then, pricing with this measure is made in a risk-neutral world.

Concerning to the uniqueness of the martingale measure, we have the next result.

Theorem 2.11. (Second Fundamental Theorem of Asset Pricing)
Suppose that we can trade continuously in the time interval $[0, T]$. Also, suppose that the market has a martingale measure. Then, the market is complete if and only if the martingale measure is unique.

See Shreve [36, Theorem 5.4.9] for a sketch of proof and Dalang et al. [6] for a more general treatment.

It can be proved that the martingale measure Q (see (2.117)) in the market model (2.91)-(2.93) is unique. We refer to [2, Theorem 2.6.2 page 249] for a proof.

2.3.2 The Black-Scholes pricing formula for the European option

In the celebrated work of Black and Scholes [3], they considered a European call option with strike price K, constant interest rate $r > 0$, and maturity T, so that the payoff is given by

$$X = \{S_T - K\}_+,$$

and the market model (β_t, S_t) has the particular form

$$d\beta_t = r\,\beta_t\, dt, \tag{2.120}$$
$$dS_t = S_t[\mu\, dt + \sigma\, dB_t]. \tag{2.121}$$

where the constant parameters μ and σ are the drift and the volatility, respectively.

Using no-arbitrage arguments and solving a PDE, Black and Scholes arrived to the following important result.

Theorem 2.12. (The Black-Scholes pricing formula) *The price of the European call option at time t is*

$$C(t) = S_t\Phi(d_1(t)) - K\,e^{-r(T-t)}\Phi(d_2(t)), \tag{2.122}$$

where

$$d_1(t) = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\,\sqrt{T-t}}, \tag{2.123}$$

$$d_2(t) = \frac{\ln(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\,\sqrt{T-t}} \tag{2.124}$$

and $\Phi(x)$ is the Gaussian integral given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy.$$

Equation (2.122) is the so-called Black-Scholes formula .

Proof. Note that

$$d_2(t) = d_1(t) - \sigma\sqrt{T-t}. \tag{2.125}$$

According to the Arbitrage-Free Pricing Theorem, the fair price of the European call option is

$$C(t) = e^{-r(T-t)}E_Q(\{S_T - K\}_+ \mid \mathcal{F}_t), \tag{2.126}$$

for all $t \in [0, T]$, where the martingale measure Q is given by

$$Q(A) = E_P(I_A M_T), \qquad A \in \mathcal{F}_T, \tag{2.127}$$

with

$$M_T = \exp\left\{-\left(\frac{\mu - r}{\sigma}\right)B_T - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T\right\}. \tag{2.128}$$

Notice that the martingale measure Q exists because the Novikov's condition holds, that is, property (2.116) with $\tilde{\mu} = (\mu - r)/\sigma$ is satisfyied.

We shall verify in detail that (2.126) coincides with (2.122) for each t, using martingale methods instead of solving a PDE.

The property that defines Q is that $\tilde{B}_t = t(\mu - r)/\sigma + B_t$ is a Q-Brownian motion (see page 33). Then, we write S_T in terms of $\tilde{B}_t$ as follows:

$$\begin{aligned}
S_T &= S_0 \exp\{T(\mu - \sigma^2/2) + \sigma B_T\} \\
&= S_0 \exp\{T(\mu - \sigma^2/2) + \sigma(\tilde{B}_T + (r - \mu)T/\sigma)\} \\
&= S_0 \exp\{rT - \sigma^2 T/2 + \sigma\tilde{B}_T\} \\
&= S_t \exp\{r(T-t) - \sigma^2(T-t)/2 + \sigma(\tilde{B}_T - \tilde{B}_t)\}.
\end{aligned}$$

Thus, formula (2.126) becomes

$$C(t) = e^{-r(T-t)}E_Q[\,(S_t \exp\{r(T-t) + Y\} - K)_+ \mid \mathcal{F}_t], \tag{2.129}$$

where $Y := -\sigma^2(T-t)/2 + \sigma(\tilde{B}_T - \tilde{B}_t)$ is a normal variable with

$$\mu_Y := E_Q(Y) = -\sigma^2(T-t)/2, \tag{2.130}$$
$$\sigma_Y := Var_Q(Y) = \sigma^2(T-t), \tag{2.131}$$

and so its density is given by

$$d(x) = \frac{1}{\sqrt{2\pi\sigma_Y}} \exp\left\{-\frac{(x-\mu_Y)^2}{2\sigma_Y}\right\}. \tag{2.132}$$

Since $\tilde{B}_T - \tilde{B}_t$ is independent of $\mathcal{F}_t$ we have

$$C(t) = e^{-r(T-t)} E_Q(g(Y)), \tag{2.133}$$

where $g(y) := (S_t \exp\{r(T-t) + y\} - K)_+$ is non-zero when

$$y \geq \log(K/S_t) - r(T-t) := a(t). \tag{2.134}$$

Therefore, $E_Q(g(Y))$ can be written explicitly as

$$\begin{aligned} E_Q(g(Y)) &= \int_{-\infty}^{\infty} g(y) d(y) dy \\ &= \int_{a(t)}^{\infty} (S_t \exp\{r(T-t) + y\} - K) \exp\left\{-\frac{(y-\mu_Y)^2}{2\sigma_Y}\right\} \frac{dy}{\sqrt{2\pi\sigma_Y}}. \end{aligned}$$

From here, we rewrite V_t in (2.129) as the sum of two integrals:

$$V_t = S_t \int_{a(t)}^{\infty} \exp\left\{y - \frac{(y-\mu_Y)^2}{2\sigma_Y}\right\} \frac{dy}{\sqrt{2\pi\sigma_Y}} \tag{2.135}$$
$$- e^{-r(T-t)} K \int_{a(t)}^{\infty} \exp\left\{-\frac{(y-\mu_Y)^2}{2\sigma_Y}\right\} \frac{dy}{\sqrt{2\pi\sigma_Y}}. \tag{2.136}$$

We want to transform the expression above into the form (2.122) and we do this in two steps.

Step 1: Consider the integral in (2.136). First, from (2.130)-(2.131), note that

$$\frac{(y-\mu_Y)^2}{2\sigma_Y} = \frac{(y + \frac{\sigma^2}{2}(T-t))^2}{2\sigma^2(T-t)}.$$

Now, take the change of variable

$$z := \frac{y + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}, \tag{2.137}$$

so that

$$\frac{(y-\mu_Y)^2}{2\sigma_Y} \mapsto \frac{z^2}{2}, \quad \text{and} \quad dz = \frac{dy}{\sigma\sqrt{T-t}}. \tag{2.138}$$

Thus,

$$\int_{a(t)}^{\infty} \exp\left\{-\frac{(y-\mu_Y)^2}{2\sigma_Y}\right\} \frac{dy}{\sqrt{2\pi\sigma_Y}} = \int_{b(t)}^{\infty} \exp\left\{-\frac{z^2}{2}\right\} \frac{dz}{\sqrt{2\pi}}, \tag{2.139}$$

where

$$b(t) = \frac{a(t) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}. \tag{2.140}$$

The integral on the right-hand side of (2.139) is $1 - \Phi(b(t))$, which in turn equals to $\Phi(-b(t))$. Moreover,

$$\begin{aligned} -b(t) &= \frac{-a(t) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{-\log(K/S_t) + r(T-t) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\log(S_t/K) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\ &= d_2(t). \end{aligned} \tag{2.141}$$

Hence, the integral on (2.139) becomes $\Phi(d_2(t))$, as required.

Step 2: In order to transform the integral on (2.135) into $\Phi(d_1(t))$, we make two changes of variable. The first change of variable is the given in (2.137), so that we get

$$\begin{aligned} &\int_{a(t)}^{\infty} \exp\left\{ y - \frac{(y-\mu_Y)^2}{2\sigma_Y} \right\} \frac{dy}{\sqrt{2\pi\sigma_Y}} \\ &\quad = \int_{b(t)}^{\infty} \exp\left\{ \left(z\,\sigma\sqrt{T-t} - \frac{\sigma^2}{2}(T-t) \right) + \frac{z^2}{2} \right\} \frac{dz}{\sqrt{2\pi}}. \end{aligned} \tag{2.142}$$

Since

$$\begin{aligned} \left(z\,\sigma\sqrt{T-t} - \frac{\sigma^2}{2}(T-t) \right) + \frac{z^2}{2} &= \frac{1}{2}(2\,z\,\sigma\sqrt{T-t} - \sigma^2(T-t) + z^2) \\ &= \frac{1}{2}(z - \sigma\sqrt{T-t})^2, \end{aligned}$$

we further take the change of variable

$$w := z - \sigma\sqrt{T-t}, \tag{2.143}$$

so that

$$\int_{a(t)}^{\infty} \exp\left\{ y - \frac{(y-\mu_Y)^2}{2\sigma_Y} \right\} \frac{dy}{\sqrt{2\pi\sigma_Y}} = \int_{c(t)}^{\infty} \exp\left\{ \frac{w^2}{2} \right\} \frac{dw}{\sqrt{2\pi}}, \tag{2.144}$$

where, using (2.125) and (2.142), we get

$$\begin{aligned} c(t) &= b(t) - \sigma\sqrt{T-t} \\ &= -d_2(t) - \sigma\sqrt{T-t} \\ &= -d_1(t). \end{aligned} \tag{2.145}$$

Thus, the integral in (2.144) becomes $1 - \Phi(c(t))$ which in turn equals $\Phi(-c(t)) = \Phi(d_1(t))$, as required. □

Note 2.7. We obtained the Black-Scholes formula (2.122) using martingale methods. It can also be obtained as the weak limit of the binomial pricing formula (2.74) by means of the Central Limit Theorem. To see a proof of this we refer to Kijima [19, p. 181].

2.3.3 American option price as an optimal stopping problem

Consider the market model

$$d\beta_t = r\,\beta_t\,dt \tag{2.146}$$
$$dS_t = S_t[\mu\,dt + \sigma\,dB_t], \tag{2.147}$$

where $S = \{S_t\}_{0\le t\le T}$ is defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P)$. Let $X = \{X_t\}_{0\le t\le T}$ be the payoff of the American option and define

$$U_t := \text{ess sup}_{t\le\tau\le T} E_Q(e^{-r(\tau-t)}X_\tau \mid \mathcal{F}_t), \tag{2.148}$$

where Q is the martingale measure. See Appendix B for further treatment of the essential supremum.

The process $U = \{U_t\}_{0\le t\le T}$ is known as the *Snell's envelope* of the discounted payoff

$$Y_t := e^{-rt}X_t. \tag{2.149}$$

The Snell's envelope U can be replicated by a specific self-financing portfolio (see a proof in Myneni [26, Lemma 3.1] or Karatzas [17, Theorem 5.4]). Moreover, it is proved that if A_0 is the initial value of the American option, then

$$A_0 = U_0 \tag{2.150}$$

is necessary for no-arbitrage opportunities ([26, Theorem 3.1]). We state this important result.

Theorem 2.13. *The fair price of the American option with maturity T and strike price K is given by the optimal stopping problem*

$$A_0 = \sup_{0\le\tau\le T} E_Q(Y_\tau), \tag{2.151}$$

where the supremum is taken over all the stopping times from 0 to T. Moreover, the stopping time

$$\tau^* := \inf\{0 \le t \le T : U_t = Y_t\}, \tag{2.152}$$

is optimal in (2.151).

For a proof of the first part, see Myneni [26]. For a proof of the latter, see Peskir and Shiryaev [31, Theorem 2.2, (2.1.11)].

Remark 2.8. In some sense (see Appendix B)

$$U_t = \sup_{t\le\tau\le T} E_Q(e^{-r(\tau-t)}X_\tau \mid \mathcal{F}_t). \tag{2.153}$$

Note that $U_T = X_T$. From the discussion in Subsection 2.1.1 we must have $A_t = U_t$ for each $t \in [0, T]$.

Remark 2.9. If the American option is call-type, that is, the payoff X_t is of the form $X_t = \{S_t - K\}_+$, then the optimal stopping time to exercise the call is at maturity T. This can be proved as in the discrete-time case, see Subsection 2.1.2. Therefore, the fair price of the American call option

$$A_0 = \sup_{0 \le \tau \le T} E_Q[e^{-r\tau}\{S_t - K\}_+]$$

coincides with the Black-Scholes pricing formula (2.122).

We will give an approach to the value of the American put option in Chapter 4 by solving a free-boundary problem. Before that, we give the necessary background about the type of solution that we expect for optimal stopping problems in the next chapter.

For deeper insights into the results in this section, we refer to Karatzas [17].

Chapter 3

Optimal stopping problems

Suppose that $X = \{X_t\}_{t\geq 0}$ is a time-homogeneous strong Markov process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with values on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Consider the family of measures $\{P_x,\, x \in \mathbb{R}\}$ where P_x is the distribution law of the process X starting at $X_0 = x$, that is, $P_x(X_0 = x) = 1$. The transition probability is given by

$$P(X_t \in B \mid X_0 = x) = P_x(X_t \in B). \tag{3.1}$$

The family

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{P_x,\, x \in \mathbb{R}\}), \tag{3.2}$$

together with the process X form a Markov family (see Definition 1.5).

Without loss of generality, we take $\Omega = \mathbb{R}^{[0,\infty)}$, that is, as the class of all functions $\omega : [0,\infty) \to \mathbb{R}$ endowed with the σ-algebra $\mathcal{F}$ generated by the finite-dimensional cylinders

$$\{\omega \in \mathbb{R}^{[0,\infty)} : \omega(t_1) \in F_1, \dots, \omega(t_n) \in F_n; n \in \mathbb{N}\}, \qquad F_i \subset \mathbb{R},$$

Thus, the Markov process X is the projection process given by

$$X_t(\omega) = \omega(t)$$

for all $t \geq 0$ and $\omega \in \Omega$, and the shift operator θ_t is well-defined (see (1.9)).

In this chapter we solve the optimal stopping problem of the form

$$V(x) = \sup_{0\leq\tau\leq T} E_x G(X_\tau), \tag{3.3}$$

where $G : \mathbb{R} \to \mathbb{R}$ is a measurable function, the horizon T could be infinite, and the supremum is taken over all stopping times satisfying

$$P_x(\tau < \infty) = 1. \tag{3.4}$$

Solving the problem (3.3) means to find two things: an optimal stoping time τ^* and the value of V.

Note that, in order for equation (3.3) to make sense, it must to be assumed that G satisfies the condition

$$E_x\left(\sup_{0\leq t\leq T} |G(X_t)|\right) < \infty, \tag{3.5}$$

for all $x \in \mathbb{R}$, with $G(X_T) := 0$ if $T = \infty$.

We give sufficient conditions on the value and gain functions V and G, respectively, to show the existence of an optimal stopping time. As part of this scheme, we characterise V as the smallest superharmonic function dominating G. After that, we pass to the discounted problem with finite horizon

$$V(t,x) = \sup_{0 \le \tau \le T-t} E_{t,x} e^{-r\tau} G(t+\tau, X_{t+\tau}). \tag{3.6}$$

The existence of an optimal stopping time in (3.6) is a consequence of the solution in the former problem (3.3). Moreover, based on the results of the infinite horizon case, we find that V is the smallest r-superharmonic function dominanting G.

3.1 Assumptions and the problem

Along this chapter we will work under the following

Assumptions

(A1) The filtration $\{\mathcal{F}_t\}_{t \ge 0}$ satisfies the *usual conditions*: it is right-continuous, that is,

$$\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t, \tag{3.7}$$

and $\mathcal{F}_0$ contains all P-null sets from $\mathcal{F}$.

(A2) The sample paths of X are right-continuous, that is, if $X_0 = x$ then

$$t \downarrow s \quad \Rightarrow \quad X_t \to X_s, \qquad P_x - a.s. \tag{3.8}$$

(A3) The sample paths of X are left-continuous over stopping times, that is, if $X_0 = x$ then

$$\tau_n \uparrow \tau \quad \Rightarrow \quad X_{\tau_n} \to X_\tau, \qquad P_x - a.s. \tag{3.9}$$

The problem

Definition 3.1. Let $F : \mathbb{R} \to \mathbb{R}$ be a measurable function and $r > 0$. It is said that F is *r-superharmonic* if

$$E_x e^{-r\sigma} F(X_\sigma) \le F(x), \tag{3.10}$$

for all $x \in \mathbb{R}$ and all stopping times σ with $P_x(\sigma < \infty) = 1$. If (3.10) holds with $r = 0$, then we say that F is *superharmonic*.

Notice that if F is superharmonic, then $\{F(X_t)\}_{t \ge 0}$ is a supermartingale. Similarly, if F is r-superharmonic, then $\{e^{-rt} F(X_t)\}_{t \ge 0}$ is a supermartingale.

Recall that a function $F : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous (lsc) if for every real sequence $\{x_n\}$ with $\lim_{n \to \infty} x_n = x_0$ we have

$$\liminf_{n \to \infty} F(x_n) \ge F(x_0). \tag{3.11}$$

Similarly, F is said to be upper semicontinuous (usc) if for every real sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = x_0$ we have

$$\limsup_{n\to\infty} F(x_n) \le F(x_0). \tag{3.12}$$

Consider the problem (3.3) and assume that the gain function G is usc and the value function V is lsc. Introduce the *continuation set*

$$C := \{x \in \mathbb{R} : V(x) > G(x)\} \tag{3.13}$$

and the *stopping set*

$$D := \{x \in \mathbb{R} : V(x) = G(x)\}. \tag{3.14}$$

Also, define the first entry time τ_D of the process X into D by

$$\tau_D := \inf\{t \ge 0 : X_t \in D\}, \tag{3.15}$$

which is a stopping time in the light of Assumption (A1).

We will prove that

(i) V is the smallest superharmonic function dominating G.

(ii) The stopping time τ_D defined by (3.15) is optimal in (3.3) provided that $P_x(\tau_D < \infty) = 1$ for all $x \in \mathbb{R}$.

Then, solving the optimal stopping problem (3.3) is equivalent to the problem of finding the smallest superharmonic function $\hat{V}$ dominating G. In that case $\hat{V} = V$ and τ_D is optimal if $P_x(\tau_D < \infty) = 1$ for all $x \in \mathbb{R}$.

3.2 Superharmonic characterisation of the value function

The following proposition will be needed. See [31, p. 39] for a proof.

Proposition 3.1. *Let $F : \mathbb{R} \to \mathbb{R}$ be a superharmonic function. If F is lsc then $\{F(X_t)\}_{t\ge 0}$ is a right-continuous P_x-a.s. supermartingale for every $x \in \mathbb{R}$.*

Lemma 3.2. *Let $F : \mathbb{R} \to \mathbb{R}$ be an r-superharmonic with $r \ge 0$, and τ_D the first entry time to a measurable set D. Assume that $P_x(\tau_D < \infty) = 1$ for all $x \in \mathbb{R}$. Then the mapping*

$$x \mapsto E_x(e^{-r\tau_D} F(X_{\tau_D})) \tag{3.16}$$

is also an r-superharmonic function.

Proof. The strong Markov property of the process $X = \{X_t\}_{t\ge 0}$ and properties of the shift operator imply

$$\begin{aligned} E_x(e^{-r\sigma}\, E_{X_\sigma} e^{-r\tau_D} F(X_{\tau_D})) &= E_x[e^{-r\sigma}\, E_x(e^{-r\tau_D\circ\theta_\sigma} F(X_{\tau_D}\circ\theta_\sigma) \mid \mathcal{F}_\sigma)\,] \\ &= E_x[E_x(e^{-r\sigma}\, e^{-r\tau_D\circ\theta_\sigma} F(X_{\tau_D\circ\theta_\sigma+\sigma}) \mid \mathcal{F}_\sigma)\,] \\ &= E_x e^{-r(\tau_D\circ\theta_\sigma+\sigma)} F(X_{\tau_D\circ\theta_\sigma+\sigma}). \end{aligned} \tag{3.17}$$

Note that

$$\tau_D \circ \theta_\sigma + \sigma = \inf\{t \geq \sigma : X_t \in D\} \geq \tau_D.$$

Since $e^{-rt}F(X_t)$ is a supermartingale, we conclude that

$$E_x e^{-r(\tau_D \circ \theta_\sigma + \sigma)} F(X_{\tau_D \circ \theta_\sigma + \sigma}) \leq E_x e^{-r\tau_D} F(X_{\tau_D}), \tag{3.18}$$

as required. □

Lemma 3.3. *For each $x \in \mathbb{R}$ and each stopping time σ, the family*

$$\mathcal{S} = \{E_x(G(X_\tau \circ \theta_\sigma) \mid \mathcal{F}_\sigma) : \tau \textit{ is a stopping time}\} \tag{3.19}$$

is upwards directed (see Definition A.1 in the Appendix).

Proof. Fix $x \in \mathbb{R}$ and a stopping time σ. We have to show that given stopping times τ_1, τ_2, there exists a stopping time τ such that

$$E_x(G(X_\tau \circ \theta_\sigma) \mid \mathcal{F}_\sigma) \geq E_x(G(X_{\tau_1} \circ \theta_\sigma) \mid \mathcal{F}_\sigma) \vee E_x(G(X_{\tau_2} \circ \theta_\sigma) \mid \mathcal{F}_\sigma). \tag{3.20}$$

Recall that $X_\tau \circ \theta_\sigma = X_{\tau \circ \theta_\sigma + \sigma}$. Define $\beta_i := \tau_i \circ \theta_\sigma + \sigma$ for $i = 1, 2$, and consider the event

$$B = [\, E_x(G(X_{\beta_1}) \mid \mathcal{F}_\sigma) \geq E_x(G(X_{\beta_2}) \mid \mathcal{F}_\sigma)\,], \tag{3.21}$$

Note that $B \in \mathcal{F}_\sigma$ since the expectations conditioned to $\mathcal{F}_\sigma$ are $\mathcal{F}_\sigma$-measurable. Define

$$\beta := \beta_1 I_B + \beta_2 I_{B^c}, \tag{3.22}$$

so that β is a stopping time. To see this, note that $\beta \leq t$ implies $\sigma \leq t$, so that we have $[\beta \leq t] = [\beta \leq t] \cap [\sigma \leq t]$ and then for each $t \geq 0$ we have that

$$[\beta \leq t] = \{\, [\beta_1 \leq t] \cap B \cap [\sigma \leq t]\,\} \bigcup \{\, [\beta_2 \leq t] \cap B^c \cap [\sigma \leq t]\,\} \in \mathcal{F}_t. \tag{3.23}$$

Moreover, β is of the form $\beta = \tau \circ \theta_\sigma + \sigma$, for some stopping time τ. Indeed,

$$\begin{aligned}\beta &= (\tau_1 \circ \theta_\sigma + \sigma) I_B + (\tau_2 \circ \theta_\sigma + \sigma) I_{B^c} \\ &= [\,(\tau_1 \circ \theta_\sigma) I_B + (\tau_2 \circ \theta_\sigma) I_{B^c}\,] + \sigma \\ &= [\tau_1 I_{\theta_\sigma(B)} + \tau_2 I_{\theta_\sigma(B^c)}] \circ \theta_\sigma + \sigma.\end{aligned} \tag{3.24}$$

It only remains to prove that $\tau := \tau_1 I_{\theta_\sigma(B)} + \tau_2 I_{\theta_\sigma(B^c)}$ is a stopping time. Notice that

$$\theta_\sigma(B) = [\, E_{X_0} G(X_{\tau_1}) \geq E_{X_0} G(X_{\tau_2})\,] \in \mathcal{F}_0, \tag{3.25}$$

where the latter can be seen by writting $B = [E_{X_\sigma} G(X_{\tau_1}) \geq E_{X_\sigma} G(X_{\tau_2})]$ by the strong Markov property of X. Thus,

$$[\tau \leq t] = \{\, [\tau_1 \leq t] \cap [\theta_{\sigma(B)}]\,\} \bigcup \{\, [\tau_2 \leq t] \cap [\theta_{\sigma(B^c)}]\,\} \in \mathcal{F}_t, \tag{3.26}$$

for all $t \geq 0$, implying that τ is a stopping time.

Therefore,

$$\begin{aligned} E_x(G(X_\beta) \mid \mathcal{F}_\sigma) &= E_x(G(X_{\beta_1})I_B \mid \mathcal{F}_\sigma) + E_x(G(X_{\beta_2})I_{B^c} \mid \mathcal{F}_\sigma) \\ &= E_x(G(X_{\beta_1}) \mid \mathcal{F}_\sigma)I_B + E_x(G(X_{\beta_2}) \mid \mathcal{F}_\sigma)I_{B^c} \\ &= E_x(G(X_{\beta_1}) \mid \mathcal{F}_\sigma) \vee E_x(G(X_{\beta_2}) \mid \mathcal{F}_\sigma), \end{aligned} \tag{3.27}$$

as we require. We used the fact that $G(X_\beta) = G(X_{\beta_1})I_B + G(X_{\beta_2})I_{B^c}$.

□

Theorem 3.4. *Consider the optimal stopping problem with infinite horizon $T = \infty$, that is,*

$$V(x) = \sup_{\tau \geq 0} E_x G(X_\tau), \tag{3.28}$$

Assume that V is lsc, and G is usc and bounded. Then V is the smallest superharmonic function dominating G, and τ_D in (3.15) is optimal in (3.28) provided that $P_x(\tau_D < \infty) = 1$ for all $x \in \mathbb{R}$.

Proof. To see that V dominates G, it is enough to take $\tau \equiv 0$ in (3.28). We split up the rest of the proof into two parts. First, we prove that V is superharmonic, that is, we verify that

$$E_x V(X_\sigma) \leq V(x), \tag{3.29}$$

for all stopping times σ, and all $x \in \mathbb{R}$ and that V is the minimal in the class of superharmonic functions dominating G. Second, we show that τ_D is optimal.

First part. Since V is lsc, it can be written as the supremum of a sequence of continuous functions, implying that V is a measurable function. Then the composition $V(X_\sigma)$ is also measurable.

Fix $x \in \mathbb{R}$ and assume that $X_0 = x$. For each stopping time σ we have

$$V(X_\sigma) = \sup_{\tau \geq 0} E_{X_\sigma} G(X_\tau). \tag{3.30}$$

By the strong Markov property of X,

$$E_{X_\sigma} G(X_\tau) = E_x(G(X_\tau \circ \theta_\sigma) \mid \mathcal{F}_\sigma). \tag{3.31}$$

The supremum of the right hand side in (3.31) may be not well defined. However, by Proposition A.1, we can write

$$V(X_\sigma) = \text{ess sup}_{\tau \geq 0} E_x(G(X_\tau \circ \theta_\sigma) \mid \mathcal{F}_\sigma). \tag{3.32}$$

By Lemma 3.3, the family

$$\mathcal{S} = \{E_x(G(X_\tau \circ \theta_\sigma) \mid \mathcal{F}_\sigma) : \tau \text{ is a stopping time}\} \tag{3.33}$$

is upwards directed, then by Proposition A.2, there exists a sequence of stopping times $\{\tau_n : n \geq 1\}$ such that

$$V(X_\sigma) = \lim_{n \to \infty} E_x(G(X_{\tau_n} \circ \theta_\sigma) \mid \mathcal{F}_\sigma), \tag{3.34}$$

where $\{E_x(G(X_{\tau_n} \circ \theta_\sigma) \mid \mathcal{F}_\sigma) : n \geq 1\}$ is increasing P_x-a.s. Finally, by the Monotone Convergence Theorem and the definition of V we have

$$E_x V(X_\sigma) = \lim_{n\to\infty} E_x G(X_{\tau_n \circ \theta_\sigma + \sigma}) \leq V(x), \tag{3.35}$$

proving that V is superharmonic. We used that $X_{\tau_n} \circ \theta_\sigma = X_{\tau_n \circ \theta_\sigma + \sigma}$.

If $\tilde{V}$ is another superharmonic function dominating G, we have

$$E_x G(X_\tau) \leq E_x \tilde{V}(X_\tau) \leq \tilde{V}(x), \tag{3.36}$$

and taking supremum over all stopping times τ we get $V(x) \leq \tilde{V}(x)$, proving the minimality property of V.

Second part. Fix $x \in \mathbb{R}$ and suppose that $P_x(\tau_D < \infty) = 1$. We want to prove that τ_D is optimal in (3.28), that is,

$$V(x) = E_x G(X_{\tau_D}), \tag{3.37}$$

where τ_D is the first entry time to the stopping set $D = \{x \in \mathbb{R} : V(x) = G(x)\}$. By definition of V, it is clear that $V(x) \geq E_x G(X_{\tau_D})$. In what follows we prove the reverse inequality.

Fix $\epsilon > 0$ and define the sets

$$C_\epsilon := \{x \in \mathbb{R} : V(x) > G(x) + \epsilon\}, \tag{3.38}$$
$$D_\epsilon := \{x \in \mathbb{R} : V(x) \leq G(x) + \epsilon\}. \tag{3.39}$$

Note that $C_\epsilon \uparrow C$ and $D_\epsilon \downarrow D$, as $\epsilon \downarrow 0$. Also, define the stopping times

$$\tau_{D_\epsilon} := \inf\{t \geq 0 : X_t \in D_\epsilon\}. \tag{3.40}$$

We will show two facts in order to prove $V(x) \leq E_x G(X_{\tau_D})$. First, we show that

$$V(x) = E_x V(X_{\tau_{D_\epsilon}}), \qquad \forall\, x \in \mathbb{R}. \tag{3.41}$$

Second, we will check that

$$\tau_{D_\epsilon} \uparrow \tau_D \quad \text{as} \quad \epsilon \downarrow 0. \tag{3.42}$$

Thus, if (3.41) and (3.42) hold, we will have

$$V(x) \leq E_x G(X_{\tau_{D_\epsilon}}) + \epsilon \tag{3.43}$$
$$\leq \limsup_{\epsilon\downarrow 0} E_x G(X_{\tau_{D_\epsilon}}) \tag{3.44}$$
$$\leq E_x \limsup_{\epsilon\downarrow 0} G(X_{\tau_{D_\epsilon}}) \tag{3.45}$$
$$\leq E_x G\left(\limsup_{\epsilon\downarrow 0} X_{\tau_{D_\epsilon}}\right) \tag{3.46}$$
$$= E_x G(X_{\tau_D}), \tag{3.47}$$

as we require to prove. Equation (3.43) holds by definition of D_ϵ; in (3.45) we used Fatou's Lemma (recall that G is bounded); in (3.46) we used that G is usc; and finally, (3.47) follows from the facts that $\tau_{D_\epsilon} \uparrow \tau_D$ as $\epsilon \downarrow 0$ and that the process X is left-continuous over stopping times (see condition (d) in Section 3.1) were necessary.

Now, to prove (3.41) set

$$c := \sup_{x\in\mathbb{R}}\{G(x) - E_x V(X_{\tau_{D_\epsilon}})\} < \infty. \tag{3.48}$$

We can see that c is finite since $V(X_{\tau_{D_\epsilon}}) \leq G(X_{\tau_{D_\epsilon}}) + \epsilon$ and G is bounded. Then

$$G(x) \leq E_x V(X_{\tau_{D_\epsilon}}) + c, \qquad \forall\, x \in \mathbb{R}. \tag{3.49}$$

Given that $D_\epsilon \supset D$, then $\tau_{D_\epsilon} \leq \tau_D$ and so $P_x(\tau_{D_\epsilon} < \infty) = 1$. From here, Lemma 3.2 implies that $x \mapsto E_x V(X_{\tau_{D_\epsilon}})$ is superharmonic, so also is the mapping $x \mapsto E_x V(X_{\tau_{D_\epsilon}}) + c$. Since V is the smallest superharmonic function dominating G, we must have

$$V(x) \leq E_x V(X_{\tau_{D_\epsilon}}) + c, \qquad \forall\, x \in \mathbb{R}. \tag{3.50}$$

By definition of supremum, given $\delta > 0$ with $\delta \leq \epsilon$, we can choose $y \in \mathbb{R}$ such that

$$G(y) - E_y V(X_{\tau_{D_\epsilon}}) \geq c - \delta. \tag{3.51}$$

Thus, from (3.50) and (3.51),

$$V(y) \leq G(y) + \delta \leq G(y) + \epsilon, \tag{3.52}$$

which implies that $y \in D_\epsilon$, and then starting the process at y, it enters to D_ϵ immediately, that is $\tau_{D_\epsilon} = 0$. Therefore $E_y V(X_{\tau_{D_\epsilon}}) = E_y V(X_0) = V(y)$, so that (3.51) becomes

$$0 \geq G(y) - V(y) \geq c - \delta. \tag{3.53}$$

Given that δ is arbitrarily small, we conclude that $c \leq 0$. Hence, from (3.49),

$$G(x) \leq E_x V(X_{\tau_{D_\epsilon}}). \tag{3.54}$$

Since $E_x V(X_{\tau_{D_\epsilon}})$ is superharmonic and V is the smallest one which dominates G, we can see that

$$V(x) \leq E_x V(X_{\tau_{D_\epsilon}}), \tag{3.55}$$

The inverse inequality in (3.55) follows by the superharmonic definition of V.

Now, let's proceed to prove (3.42). To do this, first note that over the set D_ϵ, we have

$$V(X_{\tau_{D_\epsilon}}) \leq G(X_{\tau_{D_\epsilon}}) + \epsilon. \tag{3.56}$$

Since $D_\epsilon \downarrow D$, then there exists a stopping time $\tau_0 \leq \tau_D$ such that $\tau_{D_\epsilon} \uparrow \tau_0$. From the left-continuity of X over stopping times and letting $\epsilon \downarrow 0$, we have

$$V(X_{\tau_{D_\epsilon}}) \to V(X_{\tau_0}), \quad G(X_{\tau_{D_\epsilon}}) \to G(X_{\tau_0}) \qquad P_x - \text{a.s.} \tag{3.57}$$

for all $x \in \mathbb{R}$. Since V is lsc, G is usc, and (3.56) holds, it follows that

$$V(X_{\tau_0}) \leq \liminf_{\epsilon\downarrow 0} V(X_{\tau_{D_\epsilon}}) \leq \limsup_{\epsilon\downarrow 0} G(X_{\tau_{D_\epsilon}}) \leq G(X_{\tau_0}), \tag{3.58}$$

which implies $V(X_{\tau_0}) = G(X_{\tau_0})$. Thus, by the definition of D, the process X is in D at the time τ_0. This shows that $\tau_D \leq \tau_0$ and therefore $\tau_D = \tau_0$.

□

Remark 3.1. If $P_x(\tau_D < \infty) < 1$ for some $x \in \mathbb{R}$, then there is no optimal stopping time in (3.28). To see this, suppose that there is an optimal stopping time, say τ^*. Then $V(X_{\tau^*}) = G(X_{\tau^*})$ P_x-a.s., otherwise $P_x(V(X_{\tau^*}) > G(X_{\tau^*})) < 1$ implies $V(x) \geq E_x V(X_{\tau^*}) > E_x G(X_{\tau^*})$, but in such case τ^* is not optimal. Since τ_D is the first entry time to the set D, from the identity $V(X_{\tau^*}) = G(X_{\tau^*})$ we must have

$$\tau_D \leq \tau^*, \quad P_x - \text{a.s.}$$

The latter implies $P_x(\tau^* < \infty) < 1$ for some $x \in \mathbb{R}$, which contradicts (3.4).

3.3 The discounted case with finite horizon

Passing to the discounted case

We are interested in solving the optimal stopping problem

$$V(x) = \sup_{\tau \geq 0} E_x(e^{-r\tau} G(X_\tau)). \tag{3.59}$$

Based on the results obtained in the last section, we will show that V is the smallest *r-superharmonic* function which dominates G, and the stopping time

$$\tau_D := \inf\{t \geq 0 : X_t \in D\}, \tag{3.60}$$

with $D = \{x \in \mathbb{R} : V(x) = G(x)\}$ is optimal in (3.59).

We state the analogue results of the last section for the discounted case and indicate only the less natural changes that we should make in their proofs.

The following lemma is the analogue of Lemma 3.3.

Lemma 3.5. *For each $x \in \mathbb{R}$ and stopping time σ, the family*

$$\mathcal{S} = \{E_x(e^{-r\tau\circ\theta_\sigma} G(X_\tau \circ \theta_\sigma) \mid \mathcal{F}_\sigma) : \tau \textit{ is a stopping time}\} \tag{3.61}$$

is upwards directed (see Definition A.1).

Proof. Fix $x \in \mathbb{R}$ and a stopping time σ. We have to show that given stopping times τ_1, τ_2, there exists a stopping time τ such that

$$\begin{aligned} &E_x(e^{-r(\tau\circ\theta_\sigma)} G(X_\tau \circ \theta_\sigma) \mid \mathcal{F}_\sigma) \\ &\quad \geq E_x(e^{-r(\tau_1\circ\theta_\sigma)} G(X_{\tau_1} \circ \theta_\sigma) \mid \mathcal{F}_\sigma) \vee E_x(e^{-r(\tau_2\circ\theta_\sigma)} G(X_{\tau_2} \circ \theta_\sigma) \mid \mathcal{F}_\sigma). \end{aligned} \tag{3.62}$$

Recall that $X_\tau \circ \theta_\sigma = X_{\tau\circ\theta_\sigma+\sigma}$. Instead of (3.21), consider the event

$$B = [\, e^{-r(\tau_1\circ\theta_\sigma)} E_x(G(X_{\beta_1}) \mid \mathcal{F}_\sigma) \geq E_x(e^{-r(\tau_2\circ\theta_\sigma)} G(X_{\beta_2}) \mid \mathcal{F}_\sigma)\,], \tag{3.63}$$

where $\beta_i = \tau_i \circ \theta_\sigma + \sigma$ for $i = 1, 2$. Next, it can be shown that $\beta := \beta_1 I_B + \beta_2 I_{B^c}$ is a stopping time and that it takes the particular form

$$\beta = [\tau_1 I_{\theta_\sigma(B)} + \tau_2 I_{\theta_\sigma(B^c)}] \circ \theta_\sigma + \sigma, \tag{3.64}$$

where $\theta_\sigma(B) = [E_{X_0} e^{-r\tau_1} G(X_{\tau_1}) \geq E_{X_0} e^{-r\tau_2} G(X_{\tau_2})] \in \mathcal{F}_0$ and

$$\tau := [\tau_1 I_{\theta_\sigma(B)} + \tau_2 I_{\theta_\sigma(B^c)}] \tag{3.65}$$

is a stopping time. The arguments in this part are similar to the ones used from (3.22) to (3.26).

Since

$$\begin{aligned} G(X_\tau \circ \theta_\sigma) &= G(X_{\tau\circ\theta_\sigma+\sigma}) = G(X_\beta) \\ &= G(X_{\beta_1} I_B + X_{\beta_2} I_{B^c}) \\ &= G(X_{\beta_1}) I_B + G(X_{\beta_2}) I_{B^c}, \end{aligned} \tag{3.66}$$

then multiplying by $e^{-r(\tau\circ\theta_\sigma)}$, we can verify that

$$\begin{aligned} &e^{-r(\tau\circ\theta_\sigma)} G(X_\tau \circ \theta_\sigma) \\ &\qquad = e^{-r(\tau_1\circ\theta_\sigma)} G(X_{\beta_1}) I_B + e^{-r(\tau_2\circ\theta_\sigma)} G(X_{\beta_2}) I_{B^c} \end{aligned} \tag{3.67}$$

Therefore, using that $B \in \mathcal{F}_\sigma$,

$$\begin{aligned} &E_x[e^{-r(\tau\circ\theta_\sigma)} G(X_\tau \circ \theta_\sigma) \mid \mathcal{F}_\sigma] \\ &\quad = E_x[e^{-r(\tau_1\circ\theta_\sigma)} G(X_{\beta_1}) \mid \mathcal{F}_\sigma] I_B + E_x[e^{-r(\tau_2\circ\theta_\sigma)} G(X_{\beta_2}) \mid \mathcal{F}_\sigma] I_{B^c} \\ &\quad = E_x(e^{-r(\tau_1\circ\theta_\sigma)} G(X_{\tau_1} \circ \theta_\sigma) \mid \mathcal{F}_\sigma) \vee E_x(e^{-r(\tau_2\circ\theta_\sigma)} G(X_{\tau_2} \circ \theta_\sigma) \mid \mathcal{F}_\sigma). \end{aligned}$$

□

The following result was established by Dynkin [7].

Theorem 3.6. *Consider the optimal stopping problem with infinite horizon $T = \infty$, that is,*

$$V(x) = \sup_{\tau \geq 0} E_x e^{-r\tau} G(X_\tau) \tag{3.68}$$

where $r > 0$. Assume that V is lsc, and G is usc and bounded. Then V is the smallest r-superharmonic function dominating G, and τ_D in (3.60) is optimal in (3.68) provided that $P_x(\tau_D < \infty) = 1$ for all $x \in \mathbb{R}$.

Proof. First part. We shall prove that V is r-superharmonic, that is, we verify that

$$E_x e^{-r\sigma} V(X_\sigma) \leq V(x), \tag{3.69}$$

for all stopping times σ, and all $x \in \mathbb{R}$.

In equations (3.30)-(3.34) we make the natural changes, like using Lemma 3.5 instead of Lemma 3.3 in equation (3.33). From there, we can write

$$V(X_\sigma) = \lim_{n\to\infty} E_x(e^{-r\tau_n\circ\theta_\sigma} G(X_{\tau_n} \circ \theta_\sigma) \mid \mathcal{F}_\sigma), \tag{3.70}$$

where $\{E_x(e^{-r\tau_n\circ\theta_\sigma} G(X_{\tau_n} \circ \theta_\sigma) \mid \mathcal{F}_\sigma) : n \geq 1\}$ is increasing P_x-a.s. Thus, by the MCT and the notation $X_{\tau_n} \circ \theta_\sigma = X_{\tau_n\circ\theta_\sigma+\sigma}$,

$$\begin{aligned} E_x e^{-r\sigma} V(X_\sigma) &= \lim_{n\to\infty} E_x e^{-r\sigma} E_x(e^{-r\tau_n\circ\theta_\sigma} G(X_{\tau_n\circ\theta_\sigma+\sigma}) \mid \mathcal{F}_\sigma) &(3.71)\\ &= \lim_{n\to\infty} E_x e^{-r(\tau_n\circ\theta_\sigma+\sigma)} G(X_{\tau_n\circ\theta_\sigma+\sigma}) \leq V(x). &(3.72) \end{aligned}$$

Then, V is r-superharmonic. If $\tilde{V}$ is another r-superharmonic function dominating G, then we have

$$E_x e^{-r\tau} G(X_\tau) \le E_x e^{-r\tau} \tilde{V}(X_\tau) \le \tilde{V}(x), \tag{3.73}$$

and taking supremum over all stopping times τ we get $V(x) \le \tilde{V}(x)$.

Second part. Now, we want to prove that

$$V(x) = E_x(e^{-r\tau_D} G(X_{\tau_D})). \tag{3.74}$$

The definition of the sets C_ϵ and D_ϵ, and the stopping time τ_{D_ϵ} are (3.38), (3.39), and (3.42) respectively.

It is clear that $V(x) \ge E_x(e^{-r\tau_D} G(X_{\tau_D}))$ by definition of V. It remains to show the reverse inequality.

First, the following can be verified:

$$V(x) = E_x(e^{-r\tau_{D_\epsilon}} V(X_{\tau_{D_\epsilon}})), \qquad \forall\, x \in \mathbb{R}. \tag{3.75}$$

To show (3.75), we set

$$c = \sup_{x \in \mathbb{R}} \{G(x) - E_x(e^{-r\tau_{D_\epsilon}} V(X_{\tau_{D_\epsilon}}))\}, \tag{3.76}$$

so that the rest of the arguments are verifyied according to the corresponding part from (3.48) to (3.55) in Theorem 3.4 with the natural changes. Note that here, we use Lemma 3.2 in place of Lemma 3.2.

Similarly, the fact that

$$\tau_{D_\epsilon} \uparrow \tau_D \quad \text{as} \quad \epsilon \downarrow 0, \tag{3.77}$$

is proved in exactly the same way as in equations (3.56)-(3.58).

Therefore, from (3.75) and (3.77) and the definition of τ_{D_ϵ}, we obtain

$$\begin{aligned}
V(x) &= E_x(e^{-r\tau_{D_\epsilon}} V(X_{\tau_{D_\epsilon}})) && (3.78)\\
&\le E_x(e^{-r\tau_{D_\epsilon}} G(X_{\tau_{D_\epsilon}})) + \epsilon\, E_x e^{-r\tau_{D_\epsilon}} && (3.79)\\
&\le \limsup_{\epsilon \downarrow 0} E_x(e^{-r\tau_{D_\epsilon}} G(X_{\tau_{D_\epsilon}})) + \limsup_{\epsilon \downarrow 0} \epsilon\, E_x e^{-r\tau_{D_\epsilon}} && (3.80)\\
&\le E_x \limsup_{\epsilon \downarrow 0} e^{-r\tau_{D_\epsilon}} G(X_{\tau_{D_\epsilon}}) && (3.81)\\
&= E_x(e^{-r\tau_D} G(X_{\tau_D})), && (3.82)
\end{aligned}$$

as required. In (3.81) we used Fatou's lemma (G is bounded) and $P_x(\tau_{D_\epsilon} < \infty) = 1$. □

The discounted with finite horizon case

Suppose the horizon is finite, that is, $T < \infty$. Replace the Markov process X_t by the Markov process (t, X_t) with state space $[0,T] \times \mathbb{R}$. Thus, the problem (3.59) becomes

$$V(t,x) = \sup_{0 \le \tau \le T-t} E_{t,x} e^{-r\tau} G(t+\tau, X_{t+\tau}), \tag{3.83}$$

where (t, x) is the initial state, that is, $X_t = x$ under $P_{t,x}$. The stopping time defined in (3.15) becomes

$$\tau_D := \inf\{0 \leq s \leq T - t : (t + s, X_{t+s}) \in D\}, \tag{3.84}$$

where D is given by

$$D = \{(t, x) \in [0, T] \times \mathbb{R} : V(t, x) = G(t, x)\}. \tag{3.85}$$

Set $X_t = x$ and define the process $Y = \{Y_u\}_{u \geq 0}$ as

$$Y_u := (t + u, X_{t+u}), \qquad t \leq u \leq T. \tag{3.86}$$

Then Y is a strong Markov process with $Y_0 = (t, x) = y$. Thus, the problem (3.83) reads

$$V(y) = \sup_{0 \leq \tau \leq T-t} E_y(e^{-r\tau} G(Y_\tau)). \tag{3.87}$$

Hence, the following theorem can be proved in exactly the same way as Theorem 3.6 with a few changes. For example, instead of taking supremum over all stopping times $\tau \geq 0$, we take supremum over all stopping times $0 \leq \tau \leq T - t$, where t is fixed.

Theorem 3.7. *Consider the optimal stopping problem (3.83). Assume that V is lsc, and G is usc and bounded. Then V is the smallest r-superharmonic function dominating G and τ_D given by (3.84) is optimal in (3.83).*

A particular case

Jaillet, Lamberton, and Lapeyre [16] studied the optimal stopping problem (3.83) assuming that G is a time-independent function, and X an Itô difussion solving

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t, \tag{3.88}$$

where B_t is a standard Brownian motion with $B_0 = 0$. Note that X is a time-homogeneous strong Markov process, so that we can apply the results in the previous section.

The following property of the value function $V(t, x)$ (see [16, Proposition 2.2] for a proof) will be used in the next chapter.

Proposition 3.8. (Continuity of V) *Assume that $\mu(x)$ and $\sigma(x)$ are bounded C^1 functions from $\mathbb{R}$ into $\mathbb{R}$ with bounded derivatives. Also, assume that G is a continuous function and that $|G(x)| \leq Me^{M|x|}$ for some $M > 0$. Then, the value function $V(t, x)$ given by*

$$V(t, x) = \sup_{0 \leq \tau \leq T-t} E_x e^{-r\tau} G(X_{t+\tau}), \tag{3.89}$$

is continuous in $[0, T] \times (0, \infty)$.

Chapter 4

American put option pricing

We know, from Theorem 2.13 and Remark 2.8, that the price of the American put option with strike price K and finite horizon T, is given by the probabilistic expression

$$A(t) = \sup_{t \le \tau \le T} E[e^{-r(\tau - t)}\{K - S_\tau\}_+ \mid \mathcal{F}_t], \tag{4.1}$$

where the stock price process $S = \{S_t\}_{t \ge 0}$ obeys the SDE

$$dS_t = rS_t\, dt + \sigma S_t\, dB_t, \tag{4.2}$$

under the martingale measure (see Remark 2.6), and the constant parameters $r > 0$ and $\sigma > 0$ are the interest rate in the market and the volatility of the stock, respectively.

It can be seen ([27, p. 64]) that the unique solution to (4.2) is the *geometric Brownian motion*

$$S_t = S_0 \exp\{(r - \frac{1}{2}\sigma^2)t + \sigma B_t\}, \qquad \forall\, t \ge 0. \tag{4.3}$$

Since $S = \{S_t : t \ge 0\}$ is an Itô diffusion, Proposition 1.7 implies that S is a strong Markov process. Also, note that S is time-homogeneous and its infinitesimal generator (see Proposition 1.8) is

$$\mathbb{L}_S = r\, x\, \frac{\partial}{\partial x} + \frac{\sigma^2}{2} x^2\, \frac{\partial^2}{\partial^2 x}. \tag{4.4}$$

Suppose that $S_t = x$ and consider the optimal stopping problem

$$V(t,x) = \sup_{0 \le \tau \le T-t} E_{t,x} e^{-r\tau}\{K - S_{t+\tau}\}_+, \tag{4.5}$$

where $P_{t,x}$ is the law of S starting at $S_t = x$. Then, the above problem and the one in (4.1) are related as follows:

$$A(t) = V(t, S_t), \quad \forall\, t \in [0,T]. \tag{4.6}$$

Set $G(x) = (K - x)_+$ which satisfies the condition $|G(x)| \le Me^{M|x|}$ for some $M > 0$, and then Proposition 3.8 implies that the value function $V(t,x)$ is continuous on $[0,T] \times (0,\infty)$. Thus, the hypothesis of Theorem 3.7 are covered, so we conclude that the stopping time

$$\tau_D = \inf\{0 \le u \le T - t : (t+u, S_{t+u}) \in D\}, \tag{4.7}$$

is optimal in the problem (4.5), where D is the stopping set

$$D = \{(t, x) \in [0, T] \times (0, \infty) : V(t, x) = G(x)\}, \tag{4.8}$$

and its complement, called the continuation set, takes the form

$$C = \{(t, x) \in [0, T) \times (0, \infty) : V(t, x) > G(x)\}. \tag{4.9}$$

The boundary ∂D separating C and D is the *optimal stopping boundary.* Then, if we *know* this boundary, not only the pricing problem is solved (our primary problem), but also we are able to give the optimal stopping strategy to exercise the option.

So far, the optimal stopping boundary is given in an implicit way in terms of the unknown value function V. Thus, our aim in this Chapter is to describe (and maybe compute) this boundary analytically to get more information about it.

In Section 4.1 we prove further properties of the value function V and the boundary. After that, we transfer the optimal stopping problem (4.5) to an analytical one involving a non-linear system of partial differential equations with a boundary condition which is initially unknown. Such a system corresponds to a *free-boundary problem* and it is obtained by means of the Markovian structure of the price process.

In Section 4.2 we derive an integral equation of V known as the *exercise premium representation* of V, as a simple (but powerful) consequence of the free-boundary formulation.

In Section 4.3 we prove that the optimal stopping boundary is the unique solution, in the class of continuous increasing functions satisfying certain conditions, to an integral equation derived from the early exercise premium representation.

4.1 Free-boundary problem formulation

Seeking a solution to a free-boundary problem means to solve a (system of) partial differential equation in a domain whose boundary is a priori unknown (see Proposition 4.4).

The relashionship between optimal stopping and free-boundary problems comes from the stochastic representation of solutions to certain partial differential equations. The first work applying such a relashionship to the pricing of the put option is due McKean [22].

Most of the results exposed in this section are taken from the book of Peskir and Shiryayev [31] and the article of Jacka [15].

Sometimes will be convenient to adopt the notation $S_{t+u}(t, x)$ to represent S_{t+u} with the process starting at $S_t = x$. In this case,

$$S_{t+u} = x \exp\{(r - \frac{1}{2}\sigma^2)u + \sigma B_u\}, \tag{4.10}$$

where $B = \{B_u\}_{u \geq 0}$ is a Brownian motion started at zero under $P_{t,x}$.

Then, the problem (4.5) can be expressed as

$$V(t, x) = \sup_{0 \leq \tau \leq T-t} E\, e^{-r\tau}\{K - S_{t+\tau}(t, x)\}_+. \tag{4.11}$$

Strictly speaking, the expectation above is the conditional expectation under the martingale measure with respect to S_t.

Proposition 4.1. (i) *The function $V(t,x) > 0$ for all $0 \le t < T$ and $x > 0$.*

(ii) *The mapping $x \mapsto V(t,x)$ is convex and decreasing for each t.*

(iii) *The mapping $t \mapsto V(t,x)$ is decreasing for each x.*

Proof. (i) Fix $0 \le t < T$. For all $x < K$, $V(t,x) \ge G(x) = (K-x)_+ > 0$. Then, suppose that $x \ge K$. Define

$$\sigma := \inf\{u \ge 0 : S_{t+u} \le K/2\} \wedge (T-t). \tag{4.12}$$

Then, we have that

$$\begin{aligned} V(t,x) &\ge E_{t,x} e^{-r\sigma} G(S_{t+\sigma}) \\ &\ge E_{t,x} e^{-r\sigma} \{\{K - S_T\}_+ I_{[\sigma = T-t]} + K/2 I_{[\sigma < T-t]}\} \\ &\ge K/2 E_{t,x} e^{-r\sigma} I_{[\sigma < T-t]}. \end{aligned}$$

Since

$$[\sigma < T-t] = [S_{t+u} \le K/2, \text{ for any } u < T-t]$$

and $S_{t+u} \sim$ Log-N$(\,(r-\sigma^2/2)u, \sigma^2 u)$ under $P_{t,x}$, then $P_{t,x}[S_{t+u} \le K/2] > 0$ for all $u < T-t$ which implies $P_{t,x}[\sigma < T-t] > 0$ and therefore

$$E_{t,x} e^{-r\sigma} I_{[\sigma < T-t]} > 0,$$

as required.

(ii) The decreasing property follows from the expression in (4.10), while the convexity of V follows direclty from the convexity of $G(x) = (K-x)_+$, as we will see. Take $\lambda \in (0,1)$, and $x, y \in (0,\infty)$, then

$$\begin{aligned} &\lambda E_{t,x} e^{-r\tau} \{K - S_{t+\tau}\}_+ + (1-\lambda) E_{t,y} e^{-r\tau} \{K - S_{t+\tau}\}_+ \\ &\qquad = E\, e^{-r\tau} \{\lambda \{K - S_{t+\tau}(t,x)\}_+ + (1-\lambda)\{K - S_{t+\tau}(t,y)\}_+\} \\ &\qquad \ge E\, e^{-r\tau} \{\, K - S_{t+\tau}(t, \lambda x + (1-\lambda) y)\,\}_+ \\ &\qquad = E_{t,\lambda x + (1-\lambda) y}\, e^{-r\tau} \{K - S_{t+\tau}\}_+ \end{aligned}$$

Thus,

$$\begin{aligned} &\lambda V(t,x) + (1-\lambda) V(t,y) \\ &\qquad \ge \sup_{0 \le \tau \le T-t} \{\lambda E_{t,x} e^{-r\tau} \{K - S_{t+\tau}\}_+ + (1-\lambda) E_{t,y} e^{-r\tau} \{K - S_{t+\tau}\}_+\} \\ &\qquad \ge \sup_{0 \le \tau \le T-t} E_{t,\lambda x + (1-\lambda) y}\, e^{-r\tau} \{K - S_{t+\tau}\}_+ \\ &\qquad = V(t, \lambda x + (1-\lambda) y). \end{aligned}$$

(iii) We have to show that if $s \le t$ then $V(t,x) \le V(s,x)$. This can be seen by noting that if τ is a stopping time taken into account in the problem $V(t,x)$, so is in the problem $V(s,x)$. In other words, if $\tau \le T-t$ then $\tau \le T-s$. $\square$

Remark 4.1. It sounds reasonable that there is a non-zero cost for entering to an American put contract giving you the right to make a profit, so Proposition 4.1 (i) confirms our intuition it. Also, note that at time T

$$V(T,x) = G(x) = \{K - x\}_+, \tag{4.13}$$

which is zero for all $x \geq K$. Intuitively if we are the holder of a put contract, we would chose to exercise the same day if $S_T = x$ is such that $x < K$, otherwise we do not exercise the option.

Our aim is to determine an analytical expression for the stopping rule τ_D. This goal takes us to asking about the form of the boundary of the stopping set D, which in turn coincides with the boundary of C. In what follows we shall show that there exists a function $t \mapsto b(t)$ which describes the boundary separating C and D. This function possesses nice properties and will help us to understand the form of the solution to the optimal stopping problem.

Proposition 4.2. (i) *There exists a function* $b : [0,T] \to \mathbb{R}_+$, *such that*

$$C = \{(t,x) \in [0,T) \times (0,\infty) : x > b(t)\}. \tag{4.14}$$

(ii) *The mapping* $t \mapsto b(t)$ *is increasing and bounded above by* K, *that is,* $0 < b(t) \leq K$ *for all* $0 \leq t \leq T$ *with* $b(T) = K$.

Proof. (i) Fix $0 \leq t < T$. Consider the t-section of C,

$$C_t := \{x \in (0,\infty) : (t,x) \in C\}. \tag{4.15}$$

Note that C_t is bounded below by 0. Thus, we take $b(t)$ to be the lowest bound of C_t. Next, to see that C takes the form in (4.14), it is enough to prove that if $x \in C_t$, then $y \in C_t$ for any $y > x$.

Let τ_D be the optimal stopping time for the problem $V(t,x)$. Then,

$$\begin{aligned}
V(t,y) - V(t,x) &= V(t,y) - E[e^{-r\tau_D}\{K - S_{t+\tau_D}(t,x)\}_+] \\
&\geq E[e^{-r\tau_D}\{K - S_{t+\tau_D}(t,y)\}_+ - \{K - S_{t+\tau_D}(t,x)\}_+] \\
&\geq E[e^{-r\tau_D}(S_{t+\tau_D}(t,x) - S_{t+\tau_D}(t,y))] \\
&\qquad + E[e^{-r\tau_D}\{K - S_{t+\tau_D}(t,y)\}_- - \{K - S_{t+\tau_D}(t,x)\}_-],
\end{aligned}$$

since $\{a-b\}_+ + \{a-b\}_- = a - b$. Given that $S_{t+\tau_D}(t,y) \geq S_{t+\tau_D}(t,x)$ because of the path form in (4.10), the second expectation on the right-hand side above is nonnegative, so that we obtain

$$\begin{aligned}
V(t,y) - V(t,x) &\geq E[e^{-r\tau_D}(S_{t+\tau_D}(t,x) - S_{t+\tau_D}(t,y))] \\
&= E[e^{-r\tau_D}(x-y)\exp\{(r - \frac{1}{2}\sigma^2)\,\tau_D + \sigma B_{\tau_D}\}] \\
&= (x-y)E[\exp\{\sigma B_{\tau_D} - \frac{1}{2}\sigma^2\,\tau_D\}] \\
&= (x-y).
\end{aligned}$$

The latter equality holds because $\exp\{\sigma B_t - \frac{1}{2}\sigma^2 t\}$ is a martingale starting at one. Since $(t,x) \in C$, we see that $V(t,x) > (K-x)_+$ and thus

$$\begin{aligned} V(t,y) &\geq (x-y) + V(t,x) \\ &> (x-y) + \{K-x\}_+ \\ &\geq (x-y) + (K-x) \\ &= K - y. \end{aligned} \tag{4.16}$$

Finally, by Proposition 4.1 (i) one see that $V(t,y) > 0$. Thus, $V(t,y) > \{K-y\}_+$ and so $(t,y) \in C$, implying $y \in C_t$.

Note that necessarily we have the strict inequality $x > b(t)$ for all $(t,x) \in C$. This is because if $x = b(t)$ happens, it would imply that C is closed, which is not.

(ii) Fix $s,t \geq 0$ such that $t+s < T$ and let $\epsilon > 0$. Then $(t+s, b(t+s)+\epsilon) \in C$, so that

$$V(t+s, b(t+s)+\epsilon) > G(b(t+s)+\epsilon).$$

By Proposition 4.1 (iii), $V(t,\cdot) \geq V(t+s,\cdot)$ and thus

$$V(t, b(t+s)+\epsilon) > G(b(t+s)+\epsilon),$$

which implies that $(t, b(t+s)+\epsilon) \in C$ as well. Hence, one obtain

$$b(t+s)+\epsilon > b(t),$$

and taking $\epsilon \downarrow 0$ we conclude that b is increasing.

To see that b is bounded above by K, recall that the pairs $(t, b(t))$, with $0 \leq t < T$, belong to the stopping set D. That is,

$$V(t, b(t)) = G(b(t)).$$

Since $V(t,b(t)) > 0$, we must have $G(b(t)) > 0$. But, the only points $x \in (0,\infty)$ satisfying $G(x) > 0$ belong to the interval $(0,K)$. Moreover, $b(t) > 0$ for all t since D contains all the points (t,x) with $0 < x \leq b(t)$. Therefore, we conclude that $0 < b(t) < K$ for all $0 \leq t < T$.

All the pairs (T,x) with $x \in (0,\infty)$ belong to the stopping set D. As we explained in Remark 4.1, we will make a profit at T (starting/ending the contract at the same time T) only if $x < K$, so that we may guess that $b(T) = K$. This will follow once we show later that b is continuous so that in the meantime we take $b(T) := b(T-)$. □

The last proposition implies that the stopping set takes the form

$$D = \{(t,x) \in [0,T] \times (0,\infty) : x \leq b(t)\} \cup \{(T,x) : x > b(T)\}, \tag{4.17}$$

and that the optimal stopping time in (4.7), which we denote by τ_b from now on, is the first time that the process S falls below the boundary $t \mapsto b(t)$, that is, the stopping time

$$\tau_b = \inf\{0 \leq u \leq T-t : S_{t+u} \leq b(t+u)\} \tag{4.18}$$

is optimal in (4.5). For this reason the boundary $t \mapsto b(t)$ is called the *optimal stopping boundary.* .

For ease of notation, define the set

$$\mathcal{S} := \{(t,x) \in [0,T] \times (0,\infty) : x < b(t)\}, \tag{4.19}$$

so that $D = \bar{\mathcal{S}} \cup \{(T,x) : x > b(T)\}$.

In what follows we will prove that the *smooth fit condition* holds, which claims that $V(t,x)$ is continuously differentiable at the boundary $b(t)$. In the next section, we will see that holding this condition is crucial to derive an important representation of the value function.

Proposition 4.3. (Smooth fit condition) *The function $V(t,x)$ satisfies the smooth fit condition, that is,*

$$V_x(t,x+) = V_x(t,x-), \qquad \textit{for } x = b(t),\ 0 < t < T.$$

In particular,

$$V_x(t,x) = G'(x) = -1, \qquad \textit{for } x = b(t). \tag{4.20}$$

Proof. Fix $(t,x) \in (0,T) \times (0,\infty)$ such that $x = b(t)$. Since $b(t) < K$, we have $V(t,y) = K - y$ for all $0 \le y \le b(t)$, so that $V_x(t,y) = -1$. Thus,

$$V_x(t,x-) = -1. \tag{4.21}$$

Now, let $\epsilon > 0$ be small enough so that $x + \epsilon < K$. Given that $G(x+\epsilon) - G(x) = \epsilon$, then we get

$$\frac{V(t,x+\epsilon) - V(t,x)}{\epsilon} \ge -1$$

since V dominates G. By taking $\epsilon \downarrow 0$, the limit on the left-hand side exists because $V(t,x)$ is convex, so that we obtain

$$V_x(t,x+) \ge -1. \tag{4.22}$$

Now, we want to verify the reverse inequality $V_x(t,x+) \le -1$. Let $\epsilon > 0$ be such that $x + \epsilon < K$ and let τ_ϵ be the optimal stopping time for the problem $V(t,x+\epsilon)$. Then, recalling (4.10), we have

$$\begin{aligned}
&V(t,x+\epsilon) - V(t,x) \\
&\le E[\,e^{-r\tau_\epsilon}\{K - S_{t+\tau_\epsilon}(t,x+\epsilon)\}_+] - E[\,e^{-r\tau_\epsilon}\{K - S_{t+\tau_\epsilon}(t,x)\}_+] \\
&\le E[\,e^{-r\tau_\epsilon}(\,\{K - S_{t+\tau_\epsilon}(t,x+\epsilon)\}_+ - \{K - S_{t+\tau_\epsilon}(t,x)\}_+) I_{[S_{t+\tau_\epsilon}(t,x+\epsilon)<K]}] \\
&= E[\,e^{-r\tau_\epsilon}(S_{t+\epsilon}(t,x) - S_{t+\epsilon}(t,x+\epsilon)) I_{[S_{t+\tau_\epsilon}(t,x+\epsilon)<K]}] \\
&= -\epsilon E[\,e^{-r\tau_\epsilon}\exp\{\sigma B_{\tau_\epsilon} + (r - \frac{1}{2}\sigma^2)\tau_\epsilon\} I_{[S_{t+\tau_\epsilon}(t,x+\epsilon)<K]}] \\
&\le -\epsilon E[\exp\{\sigma B_{\tau_\epsilon} - \frac{1}{2}\sigma^2\tau_\epsilon\}] \\
&= -\epsilon,
\end{aligned}$$

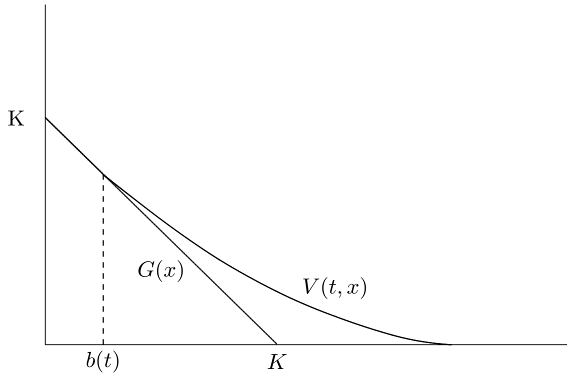

Figure 4.1: Smooth-fit

since $\exp\{\sigma B_t - \frac{1}{2}\sigma^2 t\}$ is a martingale starting at one. Thus, after division by $-\epsilon$ and taking $\epsilon \downarrow 0$, we obtain

$$V_x(t, x+) \leq -1. \tag{4.23}$$

□

Remark 4.2. The smooth fit condition does not hold at $t = T$, since $b(T) = K$ and $V(T, x) = \{K - x\}_+$, so that $V_x(T, x-) = -1$ while $V_x(T, x+) = 0$.

It is important to note that the smooth-fit condition, together with the decreasing and convex properties of the function $x \mapsto V(t, x)$, imply

$$V_x(t, x) \in [-1, 0]. \tag{4.24}$$

See Figure 4.1.

Proposition 4.4. *The value function V is the unique solution $\phi \in C^{1,2}$ in C to the free-boundary problem*

$$\mathcal{L}\phi(t, x) = 0, \qquad \textit{in } C \tag{4.25}$$

$$\phi(t, x) = K - x, \quad x = b(t), \tag{4.26}$$

where $\mathcal{L}$ is the operator

$$\mathcal{L} = rx\,\frac{\partial}{\partial x} + \frac{\sigma^2 x^2}{2}\,\frac{\partial^2}{\partial^2 x} + \frac{\partial}{\partial t} - r. \tag{4.27}$$

In particular, V_x, V_{xx}, V_t exist and are continuous in C.

Proof. The proof relies on Theorem 3.6 of Friedman [1] [10, p. 138], see also Karatzas and Shreve [18, Theorem 7.7].

Take $(t, x) \in C$. Since C is open, we can choose an open rectangle $R = (t_1, t_2) \times (x_1, x_2)$ such that $(t, x) \in R \subset C$. Denote by ∂R_0 the set $\{t = t_2\} \times (x_1, x_2)$. Thus,

[1] Set $f(x, t) = 0$ and $g(x, t) = K - x$ in (3.5) and (3.7) of [10].

by the result in Friedman we obtain that there exists a unique solution $\phi \in \mathcal{C}^{1,2}$ to the free-boundary problem

$$\mathcal{L}\,\phi(t,x) = 0 \qquad \text{in } R, \tag{4.28}$$

$$\phi(t,x) = V(t,x) \quad \text{on } \partial R_0 \tag{4.29}$$

Use the notation $S_{t+u} = S_{t+u}(t,x)$. Consider the stopping time

$$\tau_R := \inf\{0 \le u \le T-t : S_{t+u} \notin R\}, \tag{4.30}$$

representing the first exit time from R. Now, we can apply Dynkin's formula to the process $e^{-r\,u}\phi(t+u, S_{t+u})$, to obtain

$$\begin{aligned} &E[e^{-r\tau_R}\phi(t+\tau_R, S_{t+\tau_R})] - \phi(t,x) \\ &\quad = \int_t^{t+\tau_R} \left(\mathbb{L}_S + \frac{\partial}{\partial t}\right)\{e^{-r\,u}\phi(t+u, S_{t+u})\}du, \end{aligned} \tag{4.31}$$

Given that $(t+u, S_{t+u})$ is in R for all $u \le \tau_R$, then equation (4.28) tells us that $\mathcal{L}\,\phi(t+u, S_{t+u}) = 0$, which in turn implies that the integrand in the right-hand side of (4.31) vanishes as well (after some calculations). Thus, using the boundary condition (4.29), we have that

$$\phi(t,x) = E[e^{-r\tau_R}\phi(t+\tau_R, S_{t+\tau_R})] = E[e^{-r\tau_R}V(t+\tau_R, S_{t+\tau_R})]. \tag{4.32}$$

Since the process $M_u := e^{-r(u\wedge\tau_D)}V(t+u\wedge\tau_D, S_{t+u\wedge\tau_D})$ is a martingale[2], and $\tau_R \wedge \tau_D = \tau_R$, then

$$E[e^{-r\tau_R}V(t+\tau_R, S_{t+\tau_R})] = E[e^{-r\tau_D}V(t+\tau_D, S_{t+\tau_D})] = V(t,x). \tag{4.33}$$

Therefore, we conclude that

$$\phi(t,x) = V(t,x), \tag{4.34}$$

and $\mathcal{L}\,V(t,x) = 0$ hold for arbitrary $(t,x) \in C$. That V satisfies equation (4.26) is direct.

□

Remark 4.3. As a consequence of Proposition 4.4, we have that the value and boundary functions, V and b respectively, satisfy

$$\mathbb{L}_S V + V_t = r\,V \qquad \text{in } C, \tag{4.35}$$

$$V(t,x) = K - x \quad \text{for } x = b(t), \tag{4.36}$$

where $\mathbb{L}_S$ is given by

$$\mathbb{L}_S = r\,x\,\frac{\partial}{\partial x} + \frac{\sigma^2}{2}x^2\,\frac{\partial^2}{\partial^2 x}. \tag{4.37}$$

We also see that $V \in C^{1,2}$ in $\mathcal{S}$, since $G(x) = K - x$ for $x < b(t)$ so that $G \in C^2$ and $V = G$ on this region. Thus $V \in C^{1,2}$ in $C \cup \mathcal{S}$. Moreover, using the fact that $V(t,x) = G(x)$ for $x < b(t)$, we see that

$$\mathcal{L}\,V(t,x) = -r\,K, \qquad \text{for all } x < b(t). \tag{4.38}$$

[2]See Theorem 5.3 page 56 and equations (7.7)-(7.8) of [18]

Note that (4.38) simplifies to

$$\mathbb{L}_S V + V_t - rV = -r\,K \qquad \text{for all } x < b(t). \tag{4.39}$$

We are now able to prove a further atribute of the boundary function.

Proposition 4.5. *The mapping $t \mapsto b(t)$ is continuous on $[0,T]$.*

Proof. (a) The boundary b is left-continuous:

Fix $t \in (0,T)$. Since $V_t \leq 0$ and $V_x \leq 0$ by the monotone properties of V (see Proposition 4.1), and

$$\mathcal{L}\,V = rx\,V_x + \frac{\sigma^2 x^2}{2}\,V_{xx} + V_t - rV = 0 \qquad \text{in } C \tag{4.40}$$

by Proposition 4.4, we must have that

$$\frac{\sigma^2 x^2}{2}\,V_{xx} - r\,V \geq 0. \tag{4.41}$$

Define $C_n := C \cap \{[1/n, n] \times [0,K]\}$. Since $V > 0$ and continuous in C_n,

$$\inf_{(t,x)\in C_n} \frac{\sigma^2 x^2}{2} V_{xx}(t,x) \geq \inf_{(t,x)\in C_n} r\,V(t,x) \geq \epsilon_n > 0. \tag{4.42}$$

for $\epsilon_n > 0$ small enough.

Now, let N be small enough such that $t \in (1/N, N]$. Then, for any s with $1/N \leq s < t$ and $x \in [b(t-) + \eta, K]$ with $0 < \eta < K - b(t-)$, we have that $(s,x) \in C_N$: it is clear that $(s,x) \in [1/N, N] \times [0,K]$; to see that $(s,x) \in C$, note that $x > b(t-) \geq b(s)$, since b is increasing.

Given that $V(t,\cdot), G(\cdot) \in C^2$ in C and using the Fundamental Theorem of Calculus (FTC), we have that

$$V(s,x) - V(s,b(s)) = \int_{b(s)}^{x} V_x(s,u)du, \tag{4.43}$$

and

$$G(x) - G(b(s)) = \int_{b(s)}^{x} G'(u)du. \tag{4.44}$$

Thus, using that $V(s,b(s)) = G(b(s))$ and summing the last two equalities we get

$$V(s,x) - G(x) = \int_{b(s)}^{x} \{V_x(s,u) - G'(u)\}du. \tag{4.45}$$

Applying the FTC to the derivatives V_x and G' and summing up the resulting equations, we obtain

$$V(s,x) - G(x) = \int_{b(s)}^{x} \int_{b(s)}^{u} \{V_{xx}(s,v) - G''(v)\}dv\,du. \tag{4.46}$$

Now, note that $G''(x) = 0$ in C_N (since $x \in [0, K]$), thus from equation (4.42) we see that

$$\begin{aligned} V(s,x) - G(x) &= \int_{b(s)}^{x} \int_{b(s)}^{u} V_{xx}(s,v)\, dv\, du \\ &\geq \int_{b(s)}^{x} \int_{b(s)}^{u} \frac{2}{\sigma^2 v^2} \epsilon_N \, dv\, du \\ &= \frac{2}{\sigma^2} \epsilon_N \int_{b(s)}^{x} \left\{ \frac{1}{b(s)} - \frac{1}{u} \right\} du \\ &> \frac{a}{\sigma^2} \epsilon_N . \end{aligned} \tag{4.47}$$

where $a := \int_{b(s)}^{x} \left\{ \frac{1}{b(s)} - \frac{1}{u} \right\} du > 0$. Hence, take a sequence $\{s_n\}$ with $s_n \uparrow t$. It follows from the continuity of $V - G$ that

$$\begin{aligned} V(t, b(t-) + \eta) - &G(b(t-) + \eta) \\ &= \lim_{n\to\infty} \{V(s_n, b(t-) + \eta) - G(b(t-) + \eta)\} \\ &\geq \frac{a}{\sigma^2} \epsilon_N > 0, \end{aligned} \tag{4.48}$$

implying that $(t, b(t-) + \eta) \in C$ so that

$$b(t-) + \eta \geq b(t), \tag{4.49}$$

for $\eta > 0$ arbitrarily small. Hence, we conclude that $b(t-) \geq b(t)$ and therefore $b(t-) = b(t)$.

Now, we will set $t = T$ and suppose that $b(T) < K$. We can proceed with the arguments as before, until we arrive to (4.48) concluding

$$V(T, b(T-) + \eta) - G(b(T-) + \eta) > 0$$

so that $(T, b(T-) + \eta) \in C$, which is a contradiction since $(T, x) \in D$ for all x. Thus, given that b is bounded above by K we see that $b(T) = K$.

(b) The boundary b is right-continuous:

Fix $t \in [0, T)$ and consider a sequence $\{s_n\}$ with $s_n \downarrow t$. Since $(s_n, b(s_n)) \in D$ and D is closed, we see that $(t, b(t+)) \in D$ and thus (see equation (4.17))

$$b(t+) \leq b(t).$$

Since b is increasing, we obtain the equality $b(t+) = b(t)$.

The result holds from the conclusions in (a) and (b). □

Until recent years, numerical simulations suggested that the boundary $t \mapsto b(t)$ is convex. In 2004, Ekström [8] and Chen et al. [4] independently showed that this is certainly true.

Finally, we arrive to the following facts. If $S_t = x$, then the price of the American put option is given by

$$V(t, x) = E_{t,x}\, e^{-r\tau_b} \{K - S_{t+\tau_b}\}_+, \tag{4.50}$$

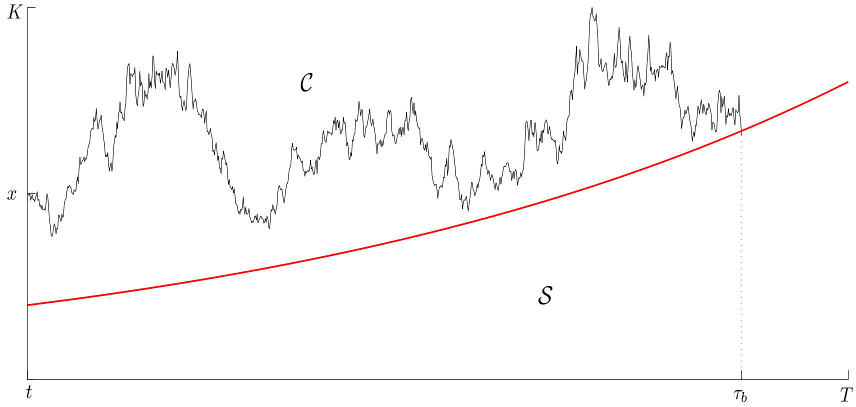

Figure 4.2: Geometric Brownian motion S_{t+u} hitting the optimal stopping boundary $t \mapsto b(t)$ at τ_b.

where the optimal strategy is

$$\tau_b = \inf\{0 \leq u \leq T - t : S_{t+u} \leq b(t+u)\}, \tag{4.51}$$

and the optimal stopping boundary $t \mapsto b(t)$ has the following properties: it is increasing, continuous, convex, and $0 < b(t) \leq K$ for all $0 < t \leq T$ with $b(T) = K$.

Thus, the optimal strategy to stop the option is to look at the time (in mean) at which the geometric Brownian motion $S = \{S_{t+u}(t,x)\}_{u \geq 0}$ hits the optimal stopping boundary $t \mapsto b(t)$. See Figure 4.2.

Then, the problem of pricing the American put option reduces to find the optimal stopping boundary. In the following sections we present an integral equation involving the boundary $b(t)$.

The free-boundary problem

We are lead to the free-boundary problem for the unkown value function V and the unknown boundary $b : [0,T] \to \mathbb{R}_+$:

$$\mathbb{L}_S V + V_t = r\,V \qquad \text{in } C, \tag{4.52}$$

$$V(t,x) = K - x \qquad \text{for } x = b(t), \tag{4.53}$$

$$V(t,x) > G(x) \qquad \text{in } C, \tag{4.54}$$

$$V(t,x) = G(x) \qquad \text{in } \mathcal{S}, \tag{4.55}$$

$$V_x(t,x) = -1 \qquad \text{for } x = b(t), \tag{4.56}$$

where the continuation set C and the set $D = \bar{\mathcal{S}}$ are given by

$$C = \{(t,x) \in [0,T) \times (0,\infty) : x > b(t)\}, \tag{4.57}$$

$$\mathcal{S} = \{(t,x) \in [0,T] \times (0,\infty) : x < b(t)\}. \tag{4.58}$$

The differential operator $\mathbb{L}_S$ corresponds to the infinitesimal generator of S given by

$$\mathbb{L}_S = r\,x\,\frac{\partial}{\partial x} + \frac{\sigma^2}{2}x^2\,\frac{\partial^2}{\partial^2 x}. \tag{4.59}$$

Also, recall for further reference that

$$\mathbb{L}_S\,V + V_t - r\,V = -r\,K \quad \text{in } \mathcal{S}. \tag{4.60}$$

We will use (4.60) to derive the so called *early exercise premium representation* of the value function in the next section.

The smooth-fit condition (4.56) was first established by Samuelson [34] and then proved by McKean [22], who solved a similar free-boundary problem for the American call option.

4.2 The early exercise premium representation

In this section we characterise the American put price as a nonlinear integral equation decomposed as the sum of the European put price and the *early exercise premium*, the latter involving the boundary $b(t)$ implicitly. This decomposition, called the *early exercise premium representation of* V, is widely known, probably because it has a clear economic meaning. It was derived in Kim [20, Equation (12)], Jacka [15, Equation (2.8)], and others.

The first appearance of the value function as an integral equation with the stopping boundary $b(t)$ implicit, is due to McKean [22], who also used the derivative b'. The problem with the McKean's equation is that it is hardly treatable in the context of numerical valuation because the derivative b' tends to infinity at maturity T.

The following important result derived by Peskir in [28, Theorem 3.1 and Remark 3.2] [3] will be crucial in the rest of the Section. It serves as a generalized Itô's formula when the differentiabily of the process presents difficulties on certain boundary, which is our case. To be specific, recall that $V \in C^{1,2}$ on $C \cup \mathcal{S}$. From here, we cannot conclude directly that $V \in C^{1,2}$ in the entire domain (needed to use Itô's formula in its standar form), because in principle, the partial derivatives V_t and V_{xx} may diverge when we approach to the boundary $b(t)$ from C.

Proposition 4.6. (Change-of-variable formula) *Consider the Itô diffusion* $X = (X_t)_{t\geq 0}$ *solving*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t.$$

Let $c : \mathbb{R}_+ \to \mathbb{R}$ *be a continuous function of bounded variation and define the sets* H_1 *and* H_2 *as follows:*

$$H_1 :=\{(t, x) \in \mathbb{R}_+ \times \mathbb{R} : x > c(t)\}, \tag{4.61}$$

$$H_2 :=\{(t, x) \in \mathbb{R}_+ \times \mathbb{R} : x < c(t)\}. \tag{4.62}$$

[3] Conditions (3.27) and (3.28) in [28] are given, which we replace by the sufficient conditions (3.35) and (3.36). In this thesis, (3.35) and (3.36) correspond to (4.65) and (4.66) respectively

Suppose that $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *is a continuous function satisfying the following conditions:*

$$
\begin{aligned}
&F \in C^{1,2} \quad \text{on } H_1 \cup H_2, && (4.63)\\
&\mathbb{L}_X F + F_t \quad \text{is locally bounded on } H_1 \cup H_2, && (4.64)\\
&t \mapsto F_x(t, c(t)\pm) \quad \text{is continuous}, && (4.65)\\
&x \mapsto F(t,x) \quad \text{is convex or concave}, && (4.66)
\end{aligned}
$$

where $\mathbb{L}_X$ *is the infinitesimal generator of* X*, and the signs* $\pm$ *are simultaneously equal to either* $+$ *or* $-$*. Condition (4.66) can be relaxed with the condition*

$$F_{xx} = F_1 + F_2 \quad \text{on } H_1 \cup H_2 \tag{4.67}$$

where F_1 *is nonnegative and* F_2 *is continuous on* $\bar{H}_1$ *and* $\bar{H}_2$*. Then, the following change-of-variable formula holds*

$$
\begin{aligned}
F(t, X_t) = F(0, X_0) &+ \int_0^t (\mathbb{L}_X F + F_t)(u, X_u)\, I_{[X_u \neq c(u)]}\, du \\
&+ \int_0^t (\sigma F_x)(u, X_u)\, I_{[X_u \neq c(u)]}\, dB_u \\
&+ \frac{1}{2} \int_0^t (F_x(u, X_u+) - F_x(u, X_u-))\, I_{[X_u = c(u)]} d\ell_u^c(X),
\end{aligned} \tag{4.68}
$$

where $\ell_u^c(X)$ *is the local time of* X *at the curve* c *given by*

$$\ell_u^c(X) = P - \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^u I_{[c(s)-\epsilon < X_s < c(s)+\epsilon]} \sigma^2\, X_s^2 ds. \tag{4.69}$$

The proof of Proposition 4.6 is somewhat long and technical, so we refer to [28] for details and generalizations.

Remark 4.4. The measure $\ell_u^c(X)$ takes into account the possible jumps of $F_x(t, c(t))$ at the boundary $c(t)$. The last integral in (4.68) is a Lebesgue-Stieltjes integral with respect to the continuous increasing function $u \mapsto \ell_u^c(X)$.

Now, consider the Itô diffusion $S = (S_t)_{t \geq 0}$ corresponding to the stock price process satisfying

$$dS_t = r\, S_t dt + \sigma\, S_t dB_t, \tag{4.70}$$

so that

$$\mu(t, S_t) = r\, S_t, \qquad \text{and} \qquad \sigma(t, S_t) = \sigma\, S_t. \tag{4.71}$$

Also, consider the continuous function $b : [0,T] \to \mathbb{R}$ which is bounded monotone (implying bounded variation), and the continuous function V (see Proposition 3.8). Set $X = S$; $c = b$; and for t, x fixed with $0 \leq t < T$ and $x > 0$, suppose that $S_t = x$, and set

$$F(t+u, S_{t+u}) = e^{-r\, u} V(t+u, S_{t+u}). \tag{4.72}$$

In this case, $H_1 \equiv C$ and $H_2 \equiv \mathcal{S}$, where C and $\mathcal{S}$ are given in (4.57)-(4.58).

Conditions (4.63), (4.65), and (4.66) hold by Proposition 4.4, Proposition 4.1 (ii), and Proposition 4.3, respectively. To verify (4.64), we need to check that $\mathbb{L}_S F + F_t$ is bounded on $K \cap (C \cup \mathcal{S})$ for each compact set $K \subset [0,T] \times (0,\infty)$. Note that

$$\mathbb{L}_S F + F_t = e^{-r\,u}(\mathbb{L}_S V + V_t - rV) = \begin{cases} 0 & \text{on } C \\ -rK\,e^{-r\,u} & \text{on } \mathcal{S}. \end{cases} \tag{4.73}$$

Now is clear that $\mathbb{L}_S F + F_t$ is locally bounded. Hence, we can apply the change-of-variable formula to F in (4.72) and together with (4.73), obtain

$$\begin{aligned} e^{-r\,u}V(t+u, S_{t+u}) &= V(t,x) - rK \int_0^u e^{-r\,v} I_{[S_{t+v} \le b(t+v)]} dv \\ &+ \int_0^u \sigma S_{t+v} e^{-r\,v} V_x(t+v, S_{t+v}) I_{[S_{t+v} \neq b(t+v)]} dB_v \\ &+ \frac{1}{2} \int_0^u e^{-r\,v} [V_x(t+v, S_{t+v}+) - V_x(t+v, S_{t+v}-)] I_{[S_{t+v} = b(t+v)]} d\ell_v^b(S). \end{aligned} \tag{4.74}$$

The last integral above is zero because of the smooth fit condition, that is, the continuity of F_x on the boundary (see Proposition 4.3). Then, equation (4.74) becomes

$$\begin{aligned} V(t,x) = e^{-r\,u}V(t+u, S_{t+u}) &+ rK \int_0^u e^{-r\,v} I_{[S_{t+v} \le b(t+v)]} dv \\ &- \int_0^u \sigma S_{t+v} e^{-r\,v} V_x(t+v, S_{t+v}) I_{[S_{t+v} \neq b(t+v)]} dB_v \end{aligned} \tag{4.75}$$

Take expectation with respect to $P_{t,x}$. Then, the last integral in (4.75) vanishes since $V_x \in [-1, 0]$. Thus, after taking $u = T - t$ and recalling that $V(T,x) = G(x)$, we obtain the following important result.

Theorem 4.7. (Early exercise premium representation) *The value function V of the American put option with strike price K, interest rate r, and horizon T, has the representation*

$$\begin{aligned} V(t,x) = e^{-r(T-t)} & E_{t,x}\{K - S_T\}_+ \\ &+ r\,K E_{t,x}\left[\int_0^{T-t} e^{-r\,v}\, I_{[S_{t+v} \le b(t+v)]} dv\right], \end{aligned} \tag{4.76}$$

for all $0 \le t < T$, where $b(t)$ is the optimal stopping boundary.

Note that the first term in (4.76) corresponds to the price of the European put option, according to the Arbitrage-free pricing theory developed in Section 2.3. The second term, which is non-negative, is known as the *early exercise premium*, and it represents the cost for the advantage over the European option to stop before maduration T.

4.3 The free-boundary equation

Since $V(t,x) = K - x$ for all $x = b(t)$, from the early exercise premium representation of V, the following integral equation for $b(t)$ holds:

$$\begin{aligned} K - b(t) = & e^{-r(T-t)} E_{t,b(t)}\{K - S_T\}_+ \\ & + r\, K E_{t,b(t)} \left[\int_0^{T-t} e^{-r\,v}\, I_{[S_{t+v} \le b(t+u)]} dv \right]. \end{aligned} \tag{4.77}$$

This is called the *free boundary equation*. It is a nonlinear problem and there is no a closed form solution.

We are prompted to ask about the uniqueness of the solution of the free boundary equation, not only for the natural theoretical scheme, but also because this is important when approaching the solution with numerical methods. Thanks to the recent work of Peskir in [29], it turns out that the optimal stopping boundary $b(t)$ is the unique solution of (4.77). We shall state and prove this result in the rest of the Section.

It is remarkable that a necessary hypothesis to obtain the representation in (4.76), and then (4.77), is that the smooth fit condition holds for V. Now, we ask about the inverted reasoning, that is, if a function $\check{V}$ is defined as V in (4.76), with $b(t)$ replaced by $c(t)$ and $c(t)$ holding the same properties that $b(t)$ (we formalise this idea later), can we conclude that $\check{V}$ satisfies the smooth fit condition at $c(t)$? The following lemma answers this question.

Lemma 4.8. *Let $c : [0,T] \to \mathbb{R}$ be a continuous increasing function satisfying $0 < c(t) < K$ for all $0 \le t < T$ and suppose that c solves the free boundary equation (4.77) with $c(t)$ instead of $b(t)$. Let us define the function $\check{V}$ as*

$$\begin{aligned} \check{V}(t,x) := & e^{-r(T-t)} E_{t,x}\{K - S_T\}_+ \\ & + r\, K E_{t,x} \left[\int_0^{T-t} e^{-r\,v}\, I_{[S_{t+v} \le c(t+v)]} dv \right], \end{aligned} \tag{4.78}$$

for all $(t,x) \in [0,T) \times (0,\infty)$. Then $\check{V}_x$ is continuous on $[0,T) \times (0,\infty)$.

Proof. Define the functions

$$V^1(t,x) : = E_{t,x}\{K - S_T\}_+, \tag{4.79}$$

$$V^2(t,x) : = \int_0^{T-t} e^{-rv} P_{t,x}(S_{t+v} \le c(t+v)) dv, \tag{4.80}$$

so that

$$\check{V}(t,x) = e^{-r(T-t)} V^1(t,x) + rKV^2(t,x). \tag{4.81}$$

We give an analytical expression for V_1 and V_2. First, recall that

$$S_{t+u} = x \exp\{(r - \frac{\sigma^2}{2})u + \sigma B_u\} \sim \text{Log-N}((r - \sigma^2/2)u, \sigma^2 u), \tag{4.82}$$

and thus

$$\log \frac{S_{t+u}}{x} \sim N((r - \sigma^2/2)u, \sigma^2 u). \tag{4.83}$$

From here we arrive to

$$Z := \frac{1}{\sigma\sqrt{u}}\left\{\log\frac{S_{t+u}}{x} - (r - \frac{\sigma^2}{2})u\right\} \sim N(0,1) \tag{4.84}$$

Thus

$$\begin{aligned} P_{t,x}(S_{t+u} \leq c(t+u)) &= P_{t,x}\left(\log\frac{S_{t+u}}{x} \leq \log\frac{c(t+u)}{x}\right) \\ &= P_{t,x}\left(Z \leq \frac{1}{\sigma\sqrt{u}}\left\{\log\frac{c(t+u)}{x} - (r-\frac{\sigma^2}{2})u\right\}\right) \\ &= \Phi\left(\frac{1}{\sigma\sqrt{u}}\left\{\log\frac{c(t+u)}{x} - (r-\frac{\sigma^2}{2})u\right\}\right), \end{aligned} \tag{4.85}$$

From (4.80) and (4.85) obtain

$$V^2(t,x) = \int_0^{T-t} e^{-rv}\Phi\left(\frac{1}{\sigma\sqrt{v}}\left\{\log\frac{c(t+v)}{x} - (r-\frac{\sigma^2}{2})v\right\}\right)dv. \tag{4.86}$$

With similar arguments it can be seen that

$$V^1(t,x) = \int_0^K \Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left\{\log\frac{K-z}{x} - (r-\frac{\sigma^2}{2})(T-t)\right\}\right)dz. \tag{4.87}$$

Now, after some calculations, the partial derivatives of V_1 and V_2 are

$$V_x^1(t,x) = -\frac{1}{\sigma x\sqrt{T-t}}\int_0^K \phi\left(\frac{1}{\sigma\sqrt{T-t}}\left\{\log\frac{K-z}{x} - (r-\frac{\sigma^2}{2})(T-t)\right\}\right) \tag{4.88}$$

$$V_x^2(t,x) = -\frac{1}{\sigma x}\int_0^{T-t}\frac{e^{-rv}}{\sqrt{v}}\phi\left(\frac{1}{\sigma\sqrt{v}}\left\{\log\frac{c(t+v)}{x} - (r-\frac{\sigma^2}{2})v\right\}\right)dv, \tag{4.89}$$

where $\phi(x) = \Phi'(x)$. Since $\phi(x)$ is continuous for all $0 \leq t < T$ and $x > 0$, it follows that V_x^1 and V_x^2 are continuous. Hence,

$$\check{V}_x(t,x) = e^{-r(T-t)}V_x^1(t,x) + rKV_x^2(t,x), \tag{4.90}$$

is continuous on $[0,T) \times (0,\infty)$ as well. □

As a consequence of the continuity of the functions in (4.88) and (4.89) in the proof of the last lemma, the function $\check{V}(t,x)$ satisfies the smooth fit condition at $c(t)$, that is, $x \mapsto \check{V}(t,x)$ is C^1 at $c(t)$.

Lemma 4.9. *Consider the hypothesis of Lemma 4.8 and also take $\check{V}(T,x) = G(x)$ for all $x > 0$. Then $\check{V}(t,x) = G(x)$ for all $x \leq c(t)$.*

Proof. Consider the sets H_1, H_2 defined by

$$H_1 := \{(t,x) \in [0,T) \times (0,\infty) : x > c(t)\}, \tag{4.91}$$

$$H_2 := \{(t,x) \in [0,T) \times (0,\infty) : x < c(t)\}. \tag{4.92}$$

The following facts are verified in the Appendix C:

$$\check{V} \text{ is } C^{1,2} \text{ on } H_1 \text{ and } \mathbb{L}_S\check{V} + \check{V}_t - r\check{V} = 0 \text{ on } H_1, \tag{4.93}$$

$$\check{V} \text{ is } C^{1,2} \text{ on } H_2 \text{ and } \mathbb{L}_S\check{V} + \check{V}_t - r\check{V} = -rK \text{ on } H_2. \tag{4.94}$$

Also, it can be seen (by direct calculation) that

$$\check{V}_{xx} = W_1 + W_2, \tag{4.95}$$

where W_1 is nonnegative and W_2 is continuous on $\bar{H}_1$ and $\bar{H}_2$.

Set

$$F(t+u, S_{t+u}) = e^{-ru}\check{V}(t+u, S_{t+u}). \tag{4.96}$$

Then

$$\mathbb{L}_S F + F_t = e^{-ru}(\mathbb{L}_S\check{V} + \check{V}_t - r\check{V}) = \begin{cases} 0 & \text{on } H_1, \\ -rKe^{-ru} & \text{on } H_2. \end{cases} \tag{4.97}$$

The conditions (4.63)-(4.65) are verified for F using (4.91)-(4.92) and Lemma 4.8. Also, condition (4.66) holds for F using (4.95). Thus, we apply the change-of-variable formula (4.68) to F to obtain

$$\begin{aligned} e^{-r\,u}\check{V}(t+u, S_{t+u}) &= \check{V}(t,x) - rK\int_0^u e^{-r\,v} I_{[S_{t+v}\le c(t+v)]}dv \\ &+ \int_0^u (\sigma S_{t+v})e^{-r\,v}\check{V}_x(t+v, S_{t+v}) I_{[S_{t+v}\neq c(t+v)]}dB_v \\ &+ \frac{1}{2}\int_0^u e^{-r\,v}[\check{V}_x(t+v, S_{t+v}+) - \check{V}_x(t+v, S_{t+v}-)] I_{[S_{t+v}=c(t+v)]}d\ell_v^c(S). \end{aligned} \tag{4.98}$$

Now, suppose that $0 < x \le c(t)$ and define the stopping time

$$\sigma_c := \inf\{0 \le u \le T-t : S_{t+u} \ge c(t+u)\}. \tag{4.99}$$

Take $u = \sigma_c$ and expectation $E_{t,x}$ in (4.98). Since $\check{V}_x$ is continuous in all the domain and $\check{V}_x \in [-1, 0]$ (this can be verified as (4.23), the last two integrals vanish yielding

$$\check{V}(t,x) = E_{t,x}e^{-r\,\sigma_c}\check{V}(t+\sigma_c, S_{t+\sigma_c}) + rKE_{t,x}\left[\int_0^{\sigma_c} e^{-r\,v} I_{[S_{t+v}\le c(t+v)]}dv\right]. \tag{4.100}$$

Also apply the change-of-variable formula to G:

$$\begin{aligned} e^{-r\,u}G(S_{t+u}) &= G(x) - rK\int_0^u e^{-r\,v} I_{[S_{t+v}<K]}dv \\ &+ \int_0^u (\sigma S_{t+v})e^{-r\,v}G'(S_{t+v}) I_{[S_{t+v}\neq c(t+v)]}dB_v \\ &+ \frac{1}{2}\int_0^u e^{-r\,v}[G'(S_{t+v}+) - G'(S_{t+v}-)] I_{[S_{t+v}=c(t+v)]}d\ell_v^c(S). \end{aligned} \tag{4.101}$$

Since $0 < c(t) < K$ for all $0 \le t < T$, we have that $G'(c(t)) = -1$. Then $\int_0^u \sigma S_{t+v}e^{-r\,v}G'(S_{t+v})I_{[S_{t+v}\neq c(t+v)]}dB_v = -\int_0^u \sigma S_{t+v}e^{-r\,v}I_{[S_{t+v}<K]}dB_v$ is a martingale under $P_{t,x}$, and the last integral vanishes. Thus, by making $u = \sigma_c$ and taking expectation $E_{t,x}$ we find

$$G(x) = E_{t,x}e^{-r\,\sigma_c}G(S_{t+\sigma_c}) + rKE_{t,x}\int_0^{\sigma_c} e^{-r\,v} I_{[S_{t+v}<K]}dv. \tag{4.102}$$

On the other hand, c solves the free boundary equation (4.77) and so $\check{V}(t, c(t)) = G(c(t))$. This implies that

$$\check{V}(t+\sigma_c, S_{t+\sigma_c}) = G(S_{t+\sigma_c}).$$

Thus, upon substituying (4.102) into (4.100), we have

$$\begin{aligned}\check{V}(t,x) &= E_{t,x}e^{-r\,\sigma_c}G(S_{t+\sigma_c}) + rKE_{t,x}\left[\int_0^{\sigma_c} e^{-r\,v}I_{[S_{t+v}\le c(t+v)]}dv\right],\\ &= G(x) - rKE_{t,x}\int_0^{\sigma_c} e^{-r\,v}I_{[S_{t+v}<K]}dv + rKE_{t,x}\left[\int_0^{\sigma_c} e^{-r\,v}I_{[S_{t+v}\le c(t+v)]}dv\right]\\ &= G(x),\end{aligned} \tag{4.103}$$

for all $0 < x \le c(t)$. □

Theorem 4.10. *Consider the American put option price*

$$V(t,x) = e^{-r(T-t)}E_{t,x}(K-S_T)_+ + r\,KE_{t,x}\left[\int_0^{T-t} e^{-r\,v}\,I_{[S_{t+v}\le b(t+v)]}dv\right].$$

The stopping boundary function $t \mapsto b(t)$ *is the unique continuous increasing solution* $c : [0,T] \to \mathbb{R}$ *of the free boundary equation (4.77) satisfying* $0 < c(t) < K$ *for all* $0 < t < T$.

Proof. Let $c : [0,T] \to \mathbb{R}$ be a continuous increasing function satisfying $0 < c(t) < K$ for all $0 < t < T$ and suppose that c solves the free boundary equation (4.77). Consider the function $\check{V}$ defined in (4.78) and define the stopping time

$$\tau_c := \inf\{0 \le u \le T-t : S_{t+u} \le c(t+u)\}. \tag{4.104}$$

The proof is organized in three steps. First, we show that $\check{V}(t,x) \le V(t,x)$ for all $(t,x) \in [0,T) \times (0,\infty)$; second, it is proved that $c(t) \ge b(t)$; and third, we conclude that $c(t) = b(t)$.

First step: consider the formula (4.98) for $\check{V}$. After taking $u = \tau_c$ and expectation $E_{t,x}$, the last two integrals vanish obtaining

$$\check{V}(t,x) = E_{t,x}e^{-r\,\tau_c}\check{V}(t+\tau_c, S_{t+\tau_c}) + rKE_{t,x}\left[\int_0^{\tau_c} e^{-r\,v}I_{[S_{t+v}\le c(t+v)]}dv\right]. \tag{4.105}$$

On one hand, Lemma 4.9 implies that $\check{V}(t,x) = G(x)$ for all $x \le c(t)$ and then $\check{V}(t+\tau_c, S_{t+\tau_c}) = G(S_{t+\tau_c})$. On the other hand, by the definition of τ_c, the integral in brackets is zero. Thus,

$$\check{V}(t,x) = E_{t,x}e^{-r\,\tau_c}G(S_{t+\tau_c}). \tag{4.106}$$

Now, upon recalling that $\tau_b = \inf\{0 \le u \le T-t : S_{t+u} \le b(t+u)\}$ is the optimal stopping time for $V(t,x)$,

$$V(t,x) = E_{t,x}e^{-r\,\tau_b}G(S_{t+\tau_b}). \tag{4.107}$$

Hence,

$$\check{V}(t,x) \leq V(t,x), \quad \forall\, (t,x) \in [0,T) \times (0,\infty). \tag{4.108}$$

Second step: for $0 < x \leq b(t) \wedge c(t)$ consider the stopping time

$$\sigma_b := \inf\{0 \leq u \leq T-t : S_{t+u} \geq b(t+u)\}. \tag{4.109}$$

Recall equation (4.75) for V. Taking $u = \sigma_b$ and expectation $E_{t,x}$, leads to

$$E_{t,x} e^{-r\,\sigma_b} V(t+\sigma_b, S_{t+\sigma_b}) = V(t,x) - rKE_{t,x} \int_0^{\sigma_b} e^{-r\,v} dv. \tag{4.110}$$

Note that $I_{[S_{t+v} \leq b(t+v)]} = 1$ in $[0, \sigma_b]$.

Now, consider again equation (4.98) for $\check{V}$. After taking $u = \sigma_b$ and expectation $E_{t,x}$ we obtain

$$E_{t,x} e^{-r\,\sigma_b} \check{V}(t+\sigma_b, S_{t+\sigma_b}) = \check{V}(t,x) - rKE_{t,x} \left[\int_0^{\sigma_b} e^{-r\,v} I_{[S_{t+v} \leq c(t+v)]} dv \right]. \tag{4.111}$$

Given that $x \leq b(t) \wedge c(t)$, then $V(t,x) = \check{V}(t,x) = G(x)$. From (4.108) and comparing (4.110) with (4.111) one has

$$E_{t,x} \int_0^{\sigma_b} e^{-r\,v} dv \leq E_{t,x} \left[\int_0^{\sigma_b} e^{-r\,v} I_{[S_{t+v} \leq c(t+v)]} dv \right]. \tag{4.112}$$

The left-hand side is positive because $\sigma_b > 0$. On the other hand, $S_{t+u} \leq b(t+u)$ from 0 to σ_b, so in order for equation (4.112) to hold, it must happen that $b(t) \leq c(t)$ by the continuity of the functions b and c.

Third step: finally we prove that $c(t) = b(t)$. Assume that there is $t \in (0,T)$ such that $c(t) > b(t)$ and take $x \in (b(t), c(t))$. Replace τ_b in (4.105) in lieu of τ_c to obtain

$$\check{V}(t,x) = E_{t,x} e^{-r\,\tau_b} \check{V}(t+\tau_b, S_{t+\tau_b}) + rKE_{t,x} \left[\int_0^{\tau_b} e^{-r\,v} I_{[S_{t+v} \leq c(t+v)]} dv \right]. \tag{4.113}$$

Since $\check{V}(t,x) = G(x)$ for all $x \leq c(t)$ and $b(t) < c(t)$, we have that $\check{V}(t+\tau_b, S_{t+\tau_b}) = G(S_{t+\tau_b})$. Hence,

$$\check{V}(t,x) = E_{t,x} e^{-r\,\tau_b} G(S_{t+\tau_b}) + rKE_{t,x} \left[\int_0^{\tau_b} e^{-r\,v} I_{[S_{t+v} \leq c(t+v)]} dv \right]. \tag{4.114}$$

From here and using (4.106) and (4.107),

$$E_{t,x} \left[\int_0^{\tau_b} e^{-r\,v} I_{[S_{t+v} \leq c(t+v)]} dv \right] \leq 0. \tag{4.115}$$

But $\sigma_b > 0$, so it follows by the continuity of the functions b and c that the expression above is impossible, and then such a point x does not exist implying that $c(t) = b(t)$. $\square$

Chapter 5

Conclusion

The main concern of our work is about pricing the American put and finding an optimal stopping rule for the holder. Let us give a summary of the results.

After applying the martingale measure one study the problem using the following model for the stock prices S_t,

$$dS_t = rS_t\,dt + \sigma S_t\,dB_t, \tag{5.1}$$

where r is the instantaneous interest rate, σ is the volatility of the stock, and B_t is a standard Brownian motion under the martingale measure (see Chapter 2).

Suppose that the stock price starts at time t with the value $S_t = x$. In Chapter 4 we found that the fair price of the American put option is given by

$$V(t,x) = E_{t,x}\, e^{-r\tau_b}\{K - S_{t+\tau_b}\}_+, \tag{5.2}$$

where

$$\tau_b = \inf\{0 \le u \le T-t : S_{t+u} \le b(t+u)\} \tag{5.3}$$

is the first passage time of the geometric Brownian motion S_t to the stopping region $\mathcal{S}$. That is, τ_b is the first time that the price process falls below the the optimal stopping boundary $t \mapsto b(t)$. See Figure 5.1.

Note that the American put price and the optimal strategy to exercise the option are determined by the optimal stopping boundary. As a consequence, it is very important to obtain an accurate curve describing $t \mapsto b(t)$. From Chapter 4, it turns out that such function $b(t)$ is the unique solution of the integral equation

$$\begin{aligned} K - b(t) &= e^{-r(T-t)} E_{t,b(t)}\{K - S_T\}_+ \\ &\quad + r\,K E_{t,b(t)}\left[\int_0^{T-t} e^{-r\,v}\, I_{[S_{t+v}\le b(t+u)]} dv\right] \\ &= e^{-r(T-t)} \int_0^K \Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left\{\log\frac{K-z}{b(t)} - (r-\frac{\sigma^2}{2})(T-t)\right\}\right) dz \\ &\quad + r\,K \int_0^{T-t} e^{-rv}\Phi\left(\frac{1}{\sigma\,\sqrt{v}}\left\{\log\frac{b(t+v)}{b(t)} - (r-\frac{\sigma^2}{2})v\right\}\right) dv, \end{aligned} \tag{5.4}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2} dz. \tag{5.5}$$

To obtain the last equality in (5.4) recall equations (4.86)-(4.87) and replace x by $b(t)$, and $c(t+v)$ by $b(t+v)$.

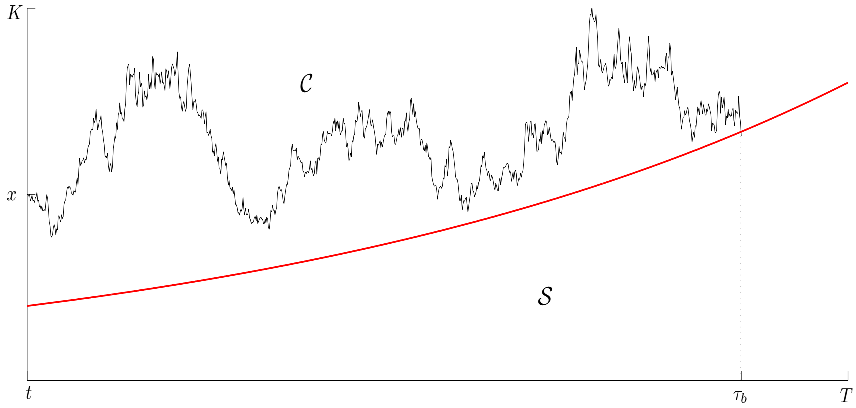

Figure 5.1: Geometric Brownian motion S_{t+u} hitting the optimal stopping boundary $t \mapsto b(t)$ at τ_b. Note that $\mathcal{C}$ and $\mathcal{S}$ stand for continuation and stopping region, respectively.

Although the exact analytical expresion for $b(t)$ seems difficult to obtain, one can approximate such barrier with numerical methods. This is feasible because $b(t)$ uniquely solves this integral equation, as proved in Theorem 4.10.

Thus, it is relevant to develope efficient numerical methods to approximate the solution of (5.4). We refer to Goodman and Ostrov [11], Huang et al [14], and the references therein for a review of different numerical methods.

Appendix A

The essential supremum

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{S} = \{f_\alpha : \alpha \in I\}$ be a non-empty family of random variables $f_\alpha : \Omega \to \bar{\mathbb{R}}$, where I is an arbitrary index set.

If I is countable, then the supremum

$$f = \sup_{\alpha \in I} f_\alpha \tag{A.1}$$

is measurable.

If I is uncountable, then f is not necessarily a measurable function. Yet, it can be shown the existence of a measurable function $f^* : \Omega \to \bar{\mathbb{R}}$ such that

(i) $f^* \geq f_\alpha$ P-a.s. for each $\alpha \in I$.

(ii) If $g : \Omega \to \bar{\mathbb{R}}$ is measurable and $g \geq f_\alpha$ P-a.s. for each $\alpha \in \mathcal{I}$, then $g \geq f^*$ P-a.s.

The function f^* is called the *essential supremum* and is denoted by

$$f^* = \text{ess sup}\, \mathcal{S}. \tag{A.2}$$

Proposition A.1. Existence of the essential supremum
Let $\mathcal{S} = \{f_\alpha : \alpha \in I\}$ be a family of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, where I is an arbitrary index set. Then there exists a countable set $J \subset I$ such that the measurable function

$$f^* = \sup_{\alpha \in J} f_\alpha \tag{A.3}$$

is the essential supremum of $\mathcal{S}$, i.e., f^* satisfies the conditions (i)-(ii) above.

Proof. Consider the family

$$\bar{\mathcal{S}} := \{g : g = \max_{\alpha \in J} f_\alpha,\ J \subset I \,\text{finite}\}. \tag{A.4}$$

Let $h : \bar{\mathbb{R}} \to \mathbb{R}$ be any continuous, strictly increasing function. Then $h \circ g$ is measurable and bounded for each $g \in \bar{\mathcal{S}}$ and

$$\beta := \sup_{\bar{\mathcal{S}}} E(h \circ g) \tag{A.5}$$

exists and is finite. We can choose an increasing sequence $\{g_n\}$ in $\bar{\mathcal{S}}$ such that

$$\beta = \sup_n E(h \circ g_n). \tag{A.6}$$

Define

$$f^* := \sup_n g_n = \lim_{n\to\infty} g_n. \tag{A.7}$$

Since g_n is the maximum of a finite sequence of functions, say $g_n = \max_{\alpha\in J_n} f_\alpha$ with $J_n \subset I$ finite. Then $J = \bigcup_n J_n$ is countable, say

$$J = \{\alpha_n : n \geq 1\}, \tag{A.8}$$

so that

$$f^* = \sup_n f_{\alpha_n}. \tag{A.9}$$

In the following we will show that f^* is the essential supremum.

To prove (i), note that the Monotone Convergence Theorem and equation (A.7) imply

$$E[h \circ (f^* \vee f)] = \lim_{n\to\infty} E[h \circ (g_n \vee f)] \leq \beta, \tag{A.10}$$

for each $f \in \mathcal{S}$. Thus the non-negative function

$$h \circ (f^* \vee f) - h \circ f^* \tag{A.11}$$

has non-positive expectation, implying that $f^* \vee f = f^*$ P-a.s. Therefore $f^* \geq f$ P-a.s. for each $f \in \mathcal{S}$.

Regarding condition (ii), suppose that $g : \Omega \to \bar{\mathbb{R}}$ satisfies (i) as well. Then $g \geq f_{\alpha_n}$ for each n since $f_{\alpha_n} \in \mathcal{S}$. This implies

$$g \geq \sup_n f_{\alpha_n} = f^*, \quad P - \text{a.s.} \tag{A.12}$$

□

Sometimes is helpful to express the essential supremum as a limit of functions. To do this, we need to assume a condition over the family $\mathcal{S}$.

Definition A.1. The family $\{f_\alpha : \alpha \in I\}$ is *upwards directed* if for any $\alpha, \beta \in I$ there exists $\gamma \in I$ such that

$$f_\gamma \geq f_\alpha \vee f_\beta, \qquad P - \text{a.s.} \tag{A.13}$$

We will use the following proposition when proving important results of optimal stopping problems in Chapter 3.

Proposition A.2. *If the family $\mathcal{S} = \{f_\alpha : \alpha \in I\}$ is upwards directed, then the countable set $J = \{\gamma_n : n \geq 1\}$ can be chosen so that*

$$f^* = \lim_{n\to\infty} f_{\gamma_n}, \quad P - a.s. \tag{A.14}$$

with $f_{\gamma_1} \leq f_{\gamma_2} \leq \cdots$ P_x-a.s.

Proof. Let J_0 be the initial countable set in (A.8), that is,

$$J_0 = \{\alpha_n : n \geq 1\},$$

and $f^* = \sup_n f_{\alpha_n}$. We construct the set J as follows. Set $\gamma_1 = \alpha_1$. Since $\mathcal{S}$ is upwards directed, there exists $\gamma_2 \in I$ such that

$$f_{\gamma_2} \geq f_{\gamma_1} \vee f_{\alpha_1}.$$

Then, inductively choose $\gamma_n \in I$ such that $f_{\gamma_n} \geq f_{\gamma_{n-1}} \vee f_{\alpha_{n-1}}$. Thus,

$$f_{\gamma_1} \leq f_{\gamma_2} \leq \cdots, \qquad P - \text{a.s.}$$

and given that $f_{\gamma_n} \geq f_{\alpha_n}$ we have

$$\lim_{n\to\infty} f_{\gamma_n} = \sup_n f_{\gamma_n} \geq f^*.$$

Therefore $\square$

Appendix B

Optimal stopping problems in discrete time

Set $\mathcal{T} = \{0, 1, \ldots, T\}$ and let $Y = \{Y_t\}_{t\in\mathcal{T}}$ be an adapted, integrable process. Consider the *optimal stopping problem*

$$V = \sup_{\tau \leq T} E(Y_\tau). \tag{B.1}$$

A solution to (B.1) consists in two things: finding the value of V, and a stopping time τ^* which is *optimal* in the sense that the supremum is attained. The process $U = \{U_t\}_{t\in\mathcal{T}}$, defined below, plays a key role in the solution of the optimal stopping problem (B.1). Define

$$U_T = Y_T \tag{B.2}$$
$$U_t = \max\{Y_t, \quad E(U_{t+1} \mid \mathcal{F}_t)\}, \quad t = 0, 1, \ldots, T-1. \tag{B.3}$$

The process U is adapted since $E(U_{t+1} \mid \mathcal{F}_t)$ is $\mathcal{F}_t$-measurable.

The next result will be used in the following.

Proposition B.1. (Optional Stopping) *Let τ be a stopping time taking values on $\{0, 1, \ldots\}$ and let $X = \{X_n\}_{n\geq 0}$ be a martingale (submartingale, supermartingale). Then the stopped process $\{X_{n\wedge\tau}\}_{n\geq 0}$ is also a martingale (submartingale, supermartingale).*

See Klebaner [21, p. 85] for a proof.

Theorem B.2. *The process U defined by (B.2)-(B.3) is the smallest supermartingale which dominates Y, that is, $U_t \geq Y_t$ for all $t \in \mathcal{T}$, a.s.*

Proof. The supermartingale property follows directly from the definition. Suppose that $W = \{W_t\}_{t\in\mathcal{T}}$ is another supermartingale that dominates Y. Since $U_T = Y_T$, is clear that $W_T \geq U_T$. This, combined with the fact that W is a supermartingale, yields

$$W_{T-1} \geq E(W_T \mid \mathcal{F}_{T-1}) \geq E(U_T \mid \mathcal{F}_{T-1}), \quad \text{a.s.}$$

Moreover, since $W_{T-1} \geq Y_{T-1}$, it results that

$$W_{T-1} \geq \max\{Y_{T-1}, \quad E(U_T \mid \mathcal{F}_{T-1})\} = U_{T-1}, \quad \text{a.s.}$$

Thus, the proof is completed by backwards induction. $\square$

Define

$$\tau^* := \inf\{0 \leq k \leq T : U_k = Y_k\}. \tag{B.4}$$

Then τ^* is a stopping time. To see this, we have to show that $[\tau^* = t]$ is $\mathcal{F}_t$-measurable for all $t \in \mathcal{T}$. Indeed,

$$[\tau^* = 0] = [U_0 = Y_0] \in \mathcal{F}_0,$$

since U_0 and Y_0 are $\mathcal{F}_0$-measurable. In general, for $t = 1, 2, \ldots, T$ we have

$$[\tau^* = t] = \left[\bigcap_{k=0}^{t-1} [U_k > Y_k]\right] \bigcap [U_t = Y_t].$$

Each of the events in the intersection is $\mathcal{F}_t$-measurable, so does the event $[\tau^* = t]$.

Proposition B.1 ensures $\{U_{t\wedge\tau}\}_{t\in\mathcal{T}}$ is supermartingale for each stopping time $\tau \leq T$. In particular, the stopped process $\{U_{t\wedge\tau^*}\}_{t\in\mathcal{T}}$ is a martingale.

Lemma B.3. *Let τ be a stopping time taking values on $\mathcal{T}$, and $X = \{X_t\}_{t\in\mathcal{T}}$ an arbitrary process. For each $t = 1, 2, \ldots, T$,*

$$X_{t\wedge\tau} = X_0 + \sum_{k=1}^{t} I_{[\tau\geq k]}\{X_k - X_{k-1}\}. \tag{B.5}$$

Proof. We observe the sum on the right-hand side under the following situations.

If $\tau \geq t$, then

$$\begin{aligned}
\sum_{k=1}^{t} I_{[\tau\geq k]}\{X_k - X_{k-1}\} &= \sum_{k=1}^{t}\{X_k - X_{k-1}\} \\
&= X_t - X_0 \\
&= X_{t\wedge\tau} - X_0.
\end{aligned}$$

If $\tau = m < t$, then

$$\begin{aligned}
\sum_{k=1}^{t} I_{[\tau\geq k]}\{X_k - X_{k-1}\} &= \sum_{k=1}^{m}\{X_k - X_{k-1}\} \\
&= X_m - X_0 = X_\tau - X_0 \\
&= X_{t\wedge\tau} - X_0.
\end{aligned}$$

□

Theorem B.4. *Let $\{U_t\}_{t\in\mathcal{T}}$ be the process defined by (B.2)-(B.3) and consider τ^* given by (B.4). Then, the stopped process*

$$\{U_{t\wedge\tau^*}\}_{t\in\mathcal{T}} \tag{B.6}$$

is a martingale.

Proof. We want to verify

$$E(\,U_{(t+1)\wedge\tau^*} - U_{t\wedge\tau^*} \mid \mathcal{F}_t) = 0, \tag{B.7}$$

for each $t = 0, 1, \ldots, T-1$. According to equation (B.5), we have

$$U_{(t+1)\wedge\tau^*} - U_{t\wedge\tau^*} = I_{[\tau^*\geq t+1]}\{U_{t+1} - U_t\}.$$

Since $[\tau^* \geq t+1] \in \mathcal{F}_t$, by taking conditional expectation we obtain

$$E(\,U_{(t+1)\wedge\tau^*} - U_{t\wedge\tau^*} \mid \mathcal{F}_t) = I_{[\tau^*\geq t+1]}E(U_{t+1} - U_t \mid \mathcal{F}_t). \tag{B.8}$$

If $\tau^* < t+1$, the right-hand side becomes zero and we are done. If $\tau^* \geq t+1$, then by definition of τ^* we must have $U_t > Y_t$ and so

$$U_t = E(U_{t+1} \mid \mathcal{F}_t).$$

This implies that the right-hand side in (B.8) becomes zero, as required. $\square$

Theorem B.5. *Let $U = \{U_t\}_{t\in\mathcal{T}}$ be the process defined by (B.2)-(B.3) and consider τ^* given by (B.4). Then, τ^* is optimal for the problem (B.1). Moreover, we conclude that*

$$V = E(Y_{\tau^*}) = U_0. \tag{B.9}$$

Proof. Let $0 \leq \tau \leq T$ be an arbitrary stopping time. Since U dominates Y by Theorem B.2, and U is a supermartingale, then

$$E(Y_\tau) \leq E(U_\tau) \leq E(U_{t\wedge\tau}).$$

Theorem B.1 implies that $\{U_{t\wedge\tau}\}$ is a supermartingale. Thus,

$$E(U_{t\wedge\tau}) \leq E(U_{0\wedge\tau}) = U_0,$$

since $\mathcal{F}_0$ is the trivial σ-algebra and U_0 is $\mathcal{F}_0$-measurable. Hence, we get

$$V = \sup_{\tau\leq T} E(Y_\tau) \leq U_0. \tag{B.10}$$

Now, by definition, $U_{\tau^*} = Y_{\tau^*}$. Thus

$$E(Y_{\tau^*}) = E(U_{\tau^*}) = E(U_{T\wedge\tau^*}) = U_{0\wedge\tau^*} = U_0.$$

where we used that the stopped process $\{U_{t\wedge\tau^*}\}$ is a martingale. Therefore, τ^* is optimal and

$$V = E(Y_{\tau^*}) = U_0.$$

$\square$

Appendix C

The Cauchy problem

We want to verify the statements

$$\check{V} \text{ is } C^{1,2} \text{ on } H_1 \text{ and } \mathbb{L}_S\check{V} + \check{V}_t - r\check{V} = 0 \text{ on } H_1, \tag{C.1}$$
$$\check{V} \text{ is } C^{1,2} \text{ on } H_2 \text{ and } \mathbb{L}_S\check{V} + \check{V}_t - r\check{V} = -rK \text{ on } H_2. \tag{C.2}$$

stated in the proof of Lemma 4.9.

Consider the *Cauchy problem*

$$\mathcal{L}u + \frac{\partial u}{\partial t} = f(t,x) \quad \text{in } [0,T) \times \mathbb{R}, \tag{C.3}$$
$$u(T,x) = \phi(x) \qquad \text{in } \mathbb{R}, \tag{C.4}$$

where $\mathcal{L}$ is given by

$$\mathcal{L}u(t,x) := rx\,\frac{\partial u(t,x)}{\partial x} + \frac{\sigma^2 x^2}{2}\,\frac{\partial^2 u(t,x)}{\partial^2 x} + \frac{\partial u(t,x)}{\partial t} - r\,u(t,x), \tag{C.5}$$

and the Itô diffusion $dS_t = \mu S_t\,dt + \sigma S_t\,dB_t$. Then (C.5) takes the form

$$\mathcal{L}u(t,x) = \mathbb{L}_S u + \frac{\partial u}{\partial t} - ru. \tag{C.6}$$

Under certain conditions (see [10, Theorem 5.3 page 148]), the unique solution $u \in C^{1,2}$ of the Cauchy problem (C.3)-(C.4) is given by

$$u(t,x) = E_{t,x}e^{-r(T-t)}\phi(S_T) - E_{t,x}\int_t^T e^{-r(v-t)}f(v,S_v)dv. \tag{C.7}$$

Now, we verify (C.1)-(C.2). To do this, set

$$f(t,x) = -rKI_{[x\le c(t)]}, \tag{C.8}$$
$$\phi(x) = G(x). \tag{C.9}$$

Then, the unique solution to

$$\mathbb{L}_S u + \frac{\partial u}{\partial t} - ru = -rKI_{[x\le c(t)]} \quad \text{in } [0,T) \times \mathbb{R}, \tag{C.10}$$
$$u(T,x) = G(x) \qquad \text{in } \mathbb{R}, \tag{C.11}$$

is given by

$$u(t,x) = E_{t,x} e^{-r(T-t)} G(S_T) + rKE_{t,x} \int_0^{T-t} e^{-rv} I_{[S_{t+v} \leq c(t+v)]} dv, \tag{C.12}$$

which coincides with $\check{V}(t,x)$ (see (4.78)). Therefore $u \equiv \check{V}$ and from (C.10) one concludes that: on the set $H_1 = \{(t,x) \in [0,T) \times (0,\infty) : x > c(t)\}$,

$$\mathbb{L}_S \check{V} + \check{V}_t - r\check{V} = 0, \tag{C.13}$$

and on the set $H_2 = \{(t,x) \in [0,T) \times (0,\infty) : x < c(t)\}$,

$$\mathbb{L}_S \check{V} + \check{V}_t - r\check{V} = -rK. \tag{C.14}$$

Index

Bibliography

[1] Bensoussan, A. On the theory of option pricing. *Acta Applicandae Mathematicae 2* (1984), 139–158.

[2] Bingham, N. H., and Kiesel, R. *Risk-Neutral Valuation, Pricing and Hedging of Financial Derivatives*, second ed. Springer, 2004.

[3] Black, F., and Scholes, M. The pricing of options and corporate liabilities. *The Journal of Political Economy 81* (1973), 637–654.

[4] Chen, X., Chadam, J., Jiang, L., and Zheng, W. Convexity of the exercise boundary of the american put option on a zero dividend asset. *Mathematical Finace 18*, 1 (2008), 185–197.

[5] Cox, J., Ross, S., and Rubinstein, M. Option pricing: a simplified approach. *Journal of Financial Economics 7* (1979), 229–261.

[6] Dalang, R., Morton, A., and Willinger, W. Equivalent martingale measures and no-arbitrage in stochastic securities market models. *Stochastics and Stochastics Reports 29* (1990), 185–201.

[7] Dynkin, E. Optimal choice of the stopping moment of a markov process. *Doklady Akademii Nauk SSSR 150* (1963), 238–240. In Russian.

[8] Ekstrm, E. Convexity of the optimal stopping boundary for the american put option. *Journal of Mathematical Analysis and Applications 299* (2004), 147–156.

[9] Elliott, R. J., and Kopp, E. *Mathematics of Financial Markets*, second ed. Springer, 2005.

[10] Friedman, A. *Stochastic Differential Equations and Applications*, vol. 1. Academic Press, 1975.

[11] Goodman, J., and Ostrov, D. On the early exercise boundary of the american put option. *SIAM Journal on Applied Mathematics 62* (2002), 1823–1835.

[12] Harrison, J., and Kreps, D. Martingales and arbitrage in multiperiod security markests. *Journal of Economic Theory 20*, 3 (June 1979), 381–408.

[13] Harrison, M., and Pliska, S. Martingales and stochastic integrals in the theory of continuous trading. *Stochastics Processes and their Applications 11* (1981), 215–260.

[14] Huang, J., Subrahmanyam, M. G., and Yu, G. Pricing and hedging american options: a recursive integration method. *Review of Financial Studies 9* (1996), 277–330.

[15] Jacka, S. D. Optimal stopping and the american put. *Mathematical Finace 1* (1991), 1–14.

[16] Jaillet, P., Lamberton, D., and Lapeyre, B. Variational inequalities and the pricing of american options. *Acta Applicandae Mathematicae 21* (1990), 263–289.

[17] Karatzas, I. On the pricing of american options. *Applied Mathematics and Optimization 17* (1988), 37–60.

[18] Karatzas, I., and Shreve, S. *Methods of Mathematical Finance*. Springer-Verlag New York, Inc., 1998.

[19] Kijima, M. *Stochastic Processes with Applications to Finance*. Chapman & Hall/CRC, 2003.

[20] Kim. The analytic valuation of american options. *The Review of Financial Studies 3* (1990), 547–572.

[21] Klebaner, C. F. *Introduction to Stochastic Calculus with Applications*. Imperial College Press, 2005.

[22] McKean. Appendix: A free boundary problem for the heat equation arising from a problem in mathematical economics. *Industrial Management Review 6* (1965), 32–39.

[23] Merton, R. Theory of rational option pricing. *The Bell Journal of Economics and Management Science 4* (1973), 141–183.

[24] Meyer, M. *Continuous Stochastic Calculus with Applications to Finance*. No. 17 in Applied Mathematics. Chapman & Hall/CRC, Michael Meyer.

[25] Musiela, M., and Rutkowski, M. *Martingale Methods in Financial Modelling*, second ed. Stochastic Modelling and Applied Probability. Springer, 2005.

[26] Myneni, R. The pricing of the american option. *The Annals of Applied Probability 2* (1992), 1–23.

[27] Ø ksendal, B. *Stochastic Differential Equations: An Introduction with Applications*. Springer, 2002.

[28] Peskir, G. A change-of-variable formula with local time on curves. *Journal of Theoretical Probability 18*, 3 (2005), 499–535.

[29] Peskir, G. On the american option problem. *Mathematical Finace 15* (2005), 169–181.

[30] Peskir, G., and Shiryaev, A. A note on the call-put parity and a call-put duality. *Theory of Probability and its Applications 46* (2002), 167–170.

[31] Peskir, G., and Shiryaev, A. *Optimal Stopping and Free-Boundary Problems*. Lectures in Mathematics. ETH Zrich. Birkhuser Basel, 2006.

[32] Pliska, S. *Introduction to Mathematical Finance: Discrete Time Models*. Wiley, 1997.

[33] Rogers, L. Equivalent martingale measures and no-arbitrage. *Stochastics 51* (1994), 41–49.

[34] Samuelson, P. A. Rational theory of warrant pricing. *Industrial Management Review 6* (1965), 13–31.

[35] Schachermayer, W. A hilbert space proof of the fundamental theorem of asset pricing in finite discrete time. *Insurance: Mathematics and Economics 11* (1992), 1–9.

[36] Shreve, S. *Stochastic Calculus for Finance*, vol. 2. Springer, 2004.

[37] Steele, J. M. *Stochastic Calculus and Financial Applications*. Springer, 2001.

[38] van Moerbeke. An optimal stopping problem with linear reward. *Acta Mathematica 132* (1974), 111–154.