Extendability of Eigenvalues for Continuous Families of Self-Adjoint Operators

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Abstract

We consider a continuous family $\{A_s\}_{s\in S}$ of self-adjoint operators in a separable Hilbert space parametrized by a complete metric space S. It is well known that simple isolated eigenvalues behave "well" under small changes of the parameter – they change continuously and do not disappear. On the other hand, eigenvalues embedded in the essential spectrum can display very "bad" behavior. Nevertheless, it turns out that for a Baire typical s all eigenvalues of A_s (if any) are *extendable*, that is, each of them belongs to a continuous branch of the (multivalued) eigenvalue function.

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1 Introduction and the main result

Suppose $\{A_s\}_{s\in S}$ is a family of self-adjoint operators in a Hilbert space $\mathcal{H}$ parametrized by a metric space S and Q is a subset of $\mathbb{R}$. If $E^* \in Q$ is an eigenvalue of A_{s^*}, we say E^* is *Q-extendable* provided there is an open neighborhood U of s^* and a continuous mapping $s \mapsto E(s)\colon U \to Q$ with the property that for each $s \in U$, $E(s)$ is an eigenvalue of A_s and $E(s^*) = E^*$.

The main result of the present work is the following statement.

Theorem 1 *Let S be a complete metric space, $\mathcal{H}$ a separable Hilbert space, Q an F_σ set in $\mathbb{R}$, and $\{A_s\}_{s\in S}$ a family of self-adjoint operators that is continuous in the strong resolvent sense (i.e., for each $z_0 \in \mathbb{C}\setminus\mathbb{R}$ and $x \in \mathcal{H}$, the mapping $s \mapsto (A_s - z_0 I)^{-1}x\colon S \to \mathcal{H}$ is continuous). Then there is a dense G_δ set $S_0 \subset S$ such that for all $s^* \in S_0$ any eigenvalue E^* of A_{s^*} that belongs to Q is Q-extendable.*

Remark. Suppose the operators A_s are bounded, have only simple isolated eigenvalues, and the mapping $s \mapsto A_s$ is continuous in the norm sense. Then the conclusion of Theorem 1 is obvious. It becomes nontrivial if there may be eigenvalues embedded in the essential spectrum (a common phenomenon in the theory of random and almost-periodic operators [1], [7]).

A property of a point $s \in S$ is said to be *Baire typical* (or *generic*) if there is a dense G_δ subset of S whose points all have that property.

Theorem 1 strengthens the main result of [4] where the following alternative was established: in the setting of Theorem 1, *either* (i) *for a Baire typical s the operator A_s has no eigenvalues, or* (ii) *there is an open set $U \subset S$ and an eigenvalue $E(s)$ of A_s ($s \in U$) which is coninuous in s.*

Applications of this result include the generic absence of eigenvalues for certain ergodic families of one-dimensional Schrödinger operators (see [4]). Another application of this statement and of a closely related statement from

[4] involving eigenvectors (see Corollary 2 below) is a unified explanation of such seemingly unrelated results as that of [2], [3], on the one hand, and that of [5], on the other (see [4]).

The rest of the paper is organized as follows. In Section 2 we prove Theorem 1. In Section 3, we consider the case where the operators A_s can have only simple eigenvalues. Here, we will formulate and prove an extension of Theorem 1 which includes eigenvectors.

2 The proof

1. The F_σ set Q can be represented as the union of an increasing sequence of compact sets. Therefore, it suffices to prove Theorem 1 under the assumption that Q *is compact.* From now on we assume that this is the case.

Consider the topological product

$$\mathcal{L} := S \times Q \times B,$$

where $B = \{y \in \mathcal{H} \mid \ \|y\| \leq 1\}$ is the unit ball in the Hilbert space $\mathcal{H}$ endowed with its weak topology. Note that $\mathcal{L}$ is a product of two completely metrizable spaces S and $Q \times B$, the latter being compact.

Consider a set

$$\mathcal{R} := \{\eta = \langle s, E, y\rangle \in \mathcal{L} \mid A_s y = Ey,\ y \neq 0\}.$$

Let $\{g_k\}_{k=1}^{\infty}$ be a dense sequence in the unit sphere of $\mathcal{H}$, and consider the following subset of $\mathcal{R}$:

$$\mathcal{R}_k := \left\{\eta = \langle s, E, y\rangle \in \mathcal{L} \;\middle|\; A_s y = Ey,\ \ (y, g_k) = \frac{3}{4}\right\}. \tag{1}$$

2. *The set* $\mathcal{R}_k$ *is closed in* $\mathcal{L}$.

To show this, we should verify that if

$$\mathcal{R}_k \ni \eta_j = \langle s_j, E_j, y_j \rangle \to \eta = \langle s, E, y \rangle \in \mathcal{L} \quad \text{as} \quad j \to \infty,$$

then $\eta \in \mathcal{R}_k$. That $(y, g_k) = \frac{3}{4}$ is obvious, since $y_j \stackrel{w}{\to} y$ (by $\stackrel{w}{\to}$ we denote the weak convergence in $\mathcal{H}$). To check that $A_s y = Ey$, or equivalently, $(A_s - z_0 I)^{-1} y = (E - z_0)^{-1} y$, where $z_0 \in \mathbb{C} \setminus \mathbb{R}$, we pass to the limit in the equation $(A_{s_j} - z_0 I)^{-1} y_j = (E_j - z_0)^{-1} y_j$, using the strong convergence of the operator $(A_{s_j} - z_0 I)^{-1}$ to $(A_s - z_0 I)^{-1}$ and relations $y_j \stackrel{w}{\to} y$ and $E_j \to E$. ■

3. Let

$$Z_k := \mathrm{pr}_S \left(\mathcal{R}_k\right) \tag{2}$$

(by pr_S, $\mathrm{pr}_{S \times Q}$, etc. we denote the corresponding projections of the product $\mathcal{L} = S \times Q \times \mathcal{H}$ onto S, $S \times Q$, etc.).

The set Z_k is closed in S.

This is a special case of the following statement.

Lemma 1 *If X and Y are topological spaces, Y is compact, and C is a closed subset of $X \times Y$, then $\mathrm{pr}_X(C)$ is closed.* ■

Denote by F_k the boundary of the set Z_k. Since F_k is closed and nowhere dense, *the set*

$$S_0 := S \setminus \bigcup_{k=1}^{\infty} F_k \tag{3}$$

is a dense G_δ set.

4. *If $\eta = \langle s, E, y \rangle \in \mathcal{R}_k$ and $\tilde{\eta} = \langle s, \tilde{E}, \tilde{y} \rangle \in \mathcal{R}_k$, then $E = \tilde{E}$.*

Proof. We have $A_s y = Ey$ and $A_s \tilde{y} = \tilde{E} \tilde{y}$, so that if $E \neq \tilde{E}$, then y and $\tilde{y}$ are orthogonal. Putting $z = y/\|y\|$ and $\tilde{z} = \tilde{y}/\|\tilde{y}\|$, we have $\|z\| = \|\tilde{z}\| = 1$, $(z, \tilde{z}) = 0$ and $|(z, g_k)| \geq \frac{3}{4}$, $|(\tilde{z}, g_k)| \geq \frac{3}{4}$. By Bessel's inequality $\|g_k\|^2 \geq \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^2 > 1$, which is impossible since $\|g_k\| = 1$. ■

The statement just proved implies that in the set

$$\mathcal{G}_k := \mathrm{pr}_{S\times Q}(\mathcal{R}_k)$$

the second component of each pair $\langle s, E\rangle$ is uniquely determined by its first component; hence $\mathcal{G}_k$ is the graph of a function

$$s \mapsto E_k(s)\colon Z_k \to Q \tag{4}$$

(obviously, $\mathrm{pr}_S(\mathcal{G}_k) = \mathrm{pr}_S(\mathcal{R}_k) = Z_k$).

Again by Lemma 1, the set $\mathcal{G}_k$ is closed. Therefore, in view of the following lemma, the function $E_k(\cdot)$, having a closed graph, is continuous.

Lemma 2 *If X and Y are topological spaces, Y is compact, and a mapping $f\colon X \to Y$ has a closed graph, then f is continuous.* ■

5. Suppose s^* belongs to the set S_0 defined by (3) and $E^* \in Q$ is an eigenvalue of A_{s^*}. Then there exists a vector $y \in \mathcal{H}$ with $\|y\| = 1$ such that $A_{s^*}y = E^*y$. Pick an element g_k of the sequence $\{g_l\}_{l=1}^{\infty}$ so close to y that $|a| > \frac{3}{4}$ where $a := (y, g_k)$. Let $y^* := \left(\frac{3}{4a}\right) y$. Then $(y^*, g_k) = \frac{3}{4}$ and $\|y^*\| < 1$ which implies that $\langle s^*, E^*, y^*\rangle \in \mathcal{R}_k$.

It follows that $s^* \in Z_k$ and $\langle s^*, E^*\rangle \in \mathcal{G}_k$. The latter inclusion means that $E^* = E_k(s^*)$, where the function $E_k\colon Z_k \to Q$ is continuous. The inclusions $s^* \in Z_k$ and $s^* \in S_0$ imply that s^* belongs to the set $U_k = Z_k \setminus F_k$ – the interior of Z_k. Putting $U := U_k$ and defining the function $E(\cdot)$ as the restriction of $E_k(\cdot)$ to U completes the proof of Theorem 1. ■

Corollary 1 [4] *Let $\{A_s\}_{s\in S}$ be a family of operators which satisfies the conditions of Theorem 1, and Q an F_σ subset of $\mathbb{R}$. Let Z be the set of all $s \in S$ for which the operator A_s has at least one eigenvalue in Q.*

An alternative takes place: either

(i) *Z is a meager set, or*

(ii) *there exist a nonempty open set $U \subset Z$ and a continuous function*

$$s \mapsto E(s)\colon\ U \to Q$$

such that for each $s \in U$, $E(s)$ is an eigenvalue of A_s.

Proof. Suppose the set Z is not meager. Then its intersection with the dense G_δ set S_0 of Theorem 1 is nonempty. Let $s^* \in Z \cap S_0$. Since $s^* \in Z$, the operator A_{s^*} has at least one eigenvalue E^* in the set Q; and since $s^* \in S_0$, that eigenvalue is Q-extendable, which gives (ii). ■

3 The case of simple eigenvalues

In the case where all eigenvalues of the operators A_s are simple, the statement of Theorem 1 can be strengthened by including eigenvectors.

Theorem 2 *Let $\{A_s\}_{s\in S}$ be a family of operators that satisfies the conditions of Theorem 1, and Q an F_σ subset of $\mathbb{R}$. Assume, in addition, that all eigenvalues of the operators A_s are simple. Then there is a dense G_δ set $S_1 \subset S$ such that for any triple $\langle s^*, E^*, y^*\rangle \in S_1 \times Q \times (\mathcal{H} \setminus \{0\})$ with $A_{s^*}y^* = E^*y^*$ there exist an open neighborhood U of s^* and a mapping $s \mapsto \langle E(s), y(s)\rangle\colon U \to Q \times (\mathcal{H} \setminus \{0\})$ with the following properties:*

(a) $A_s y(s) = E(s)y(s)$ for all $s \in U$;

(b) *the mapping $s \mapsto E(s)$ is continuous;*

(c) *the mapping $s \mapsto y(s)$ is weakly continuous;*

(d) $E(s^*) = E^*$;

(e) $y(s^*) = y^*$;

(f) $\|y(s) - y(s^*)\| \to 0$ *as* $U \ni s \to s^*$.

Proof. In what follows, we use the objects and notation introduced in the proof of Theorem 1. We can assume again that the set Q is compact.

1. Suppose s^* belongs to the set S_0 defined by (3) and $A_{s^*}y^* = E^*y^*$, where $E^* \in Q$ and $y^* \neq 0$. We can select k and scale y^* (like we did before) so that $\langle s^*, E^*, y^*\rangle \in \mathcal{R}_k$.

For each point s of $Z_k = \mathrm{pr}_S(\mathcal{R}_k)$, there is only one triple $\langle s, E, y\rangle \in \mathcal{R}_k$ with the given s. (The uniqueness of E was established in the proof of Theorem 1; the uniqueness of y follows from the simplicity of the eigenvalue E of A_s and the equality $(y, g_k) = \frac{3}{4}$ (see (1)).)

Consequently, the set $\mathcal{R}_k$ is the graph of a mapping $s \mapsto \langle E_k(s), y_k(s)\rangle :$ $Z_k \to Q \times B$, which is continuous by Lemmas 1 and 2. This means that the functions $E_k(\cdot)$ and $y_k(\cdot)$ on Z_k are continuous and weakly continuous, respectively. Therefore, the restriction of the mapping $Z_k \ni s \mapsto \langle E_k(s), y_k(s)\rangle \in Q \times B$ to the open set $U_k = \mathrm{int} Z_k$ has properties (a) – (e).

2. Fix an orthonormal basis $e_1, e_2, \dots$ of the Hilbert space $\mathcal{H}$. The function $q_k(s) := \|y_k(s)\|^2$ $(s \in Z_k)$ is the pointwise limit of a sequence of continuous functions: $q_k(s) = \lim_{n\to\infty} q_k^{(n)}(s)$, where

$$q_k^{(n)}(s) = \sum_{j=1}^{n} |(y_k(s), e_j)|^2.$$

Consequently, there is a dense G_δ set $X_k \subset Z_k$ such that the function $q_k(\cdot)$ is continuous at all points of X_k (see [6, Theorem 7.3]).

Let $Y_k := Z_k \setminus X_k$. This set is meager. Therefore, the set $S_0 \setminus \cup_{k=1}^{\infty} Y_k$ contains a dense G_δ set, S_1.

If $s^* \in Z_k$ belongs to S_1, then $s^* \in X_k \cap U_k$. Therefore, as $U_k \ni s \to s^*$, $y_k(s) \stackrel{w}{\to} y_k(s^*)$ and $\|y_k(s)\| \to \|y_k(s^*)\|$ hence $\|y_k(s) - y_k(s^*)\| \to 0$. This completes the proof of Theorem 2. ■

Corollary 2 *Let $\{A_s\}_{s\in S}$ be a family of operators that satisfies the conditions of Theorem 1, and Q an F_σ subset of $\mathbb{R}$. Also assume that all eigenvalues of the operators A_s are simple. Then an alternative takes place: either*

(i) *for all Baire typical* s*, the operator* A_s *has no eigenvalues in* Q*, or*

(ii) *there is a non-empty open set* $U \subset Z$ *and a mapping* $U \ni s \mapsto \langle E(s), y(s)\rangle \in Q \times (\mathcal{H} \setminus \{0\})$ *such that*

(a) $A_s y(s) = E(s)y(s)$ for all $s \in U$*;*

(b) *the function* $E(\cdot)$ *is continuous;*

(c) *the function* $y(s)$ *is weakly continuous;*

(d) *for all points* s^* *in a dense* G_δ *subset of* U *we have* $\|y(s) - y(s^*)\| \to 0$ *as* $U \ni s \to s^*$*.* ■

This result was used in [4] to derive (in a very different way than their original proof) the following theorem of Jitomirskaya and Simon [5]. *Let* $v_0(\cdot)$ *be a real-valued almost periodic function on* $\mathbb{R}$*, and* $H(v_0)$ *its hull (i.e., the set of uniform limits of its shifts endowed with the topology of uniform convergence). If the function* $v_0(\cdot)$ *is even, then for a Baire typical element* v *of* $H(v_0)$ *the Schrödinger operator* $A_v = -d^2/dx^2 + v(x)$ *acting in* $L^2(\mathbb{R})$ *has no eigenvalues.* The $l^2(\mathbb{Z})$ version of this result was also proved in [5].

The question whether the same is true for *any* real-valued almost periodic function $v_0(\cdot)$ on $\mathbb{R}$ (or $\mathbb{Z}$) remains open.

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