A Spectral Alternative for Continuous Families of Self-Adjoint Operators

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Abstract

We consider a continuous family of self-adjoint operators A_s in a separable Hilbert space, the parameter s being a point of a complete metric space S. It is well known that isolated simple eigenvalues (assuming that the operators are bounded and the mapping $s \mapsto A_s$ is continuous in the norm sense) behave "well": under small changes of the parameter they do not disappear and change continuously. Unlike this, the eigenvalues embedded in the essential spectrum can display a "very bad" behavior. It turns out, nevertheless, that if the set of eigenvalues is non-empty for a topologically rich (e.g., open) set of values of the parameter, then the (multi-valued) eigenvalue function has continuous branches.

One application is as follows. Suppose a one-dimensional quasi-periodic Schrödinger operator has Cantor spectrum; then a Baire generic operator in its hull does not have eigenvalues.

1 Introduction

Let $\{A_s\}_{s\in S}$ be a family of self-adjoint operators in a separable Hilbert space $\mathcal{H}$ parametrized by a point s of a complete metric space S. We need to assume some kind of continuity of the mapping $s \mapsto A_s$. Let first the operators A_s be of the form

$$A_s = A + B_s, \quad s \in S, \tag{1}$$

where A is a self-adjoint operator (bounded or not) and B_s is a bounded self-adjoint operator that depends on s strongly continuously. Here are two examples:

(a) *Rank one perturbations:*

$$A_s = A + sP, \quad s \in \mathbb{R}, \tag{2}$$

where A is a cyclic self-adjoint operator and $P = (\cdot, \varphi)\,\varphi$ is the orthogonal projection onto a one-dimensional cyclic subspace $\mathbb{C}\varphi$ $(\|\varphi\| = 1)$ of A;

(b) *Almost periodic Schrödinger operators in $L^2(\mathbb{R})$ or $l^2(\mathbb{Z})$:*

$$A_s = -\Delta + v_s, \quad s \in \Omega, \tag{3}$$

where Δ is the one-dimensional Laplacian or lattice Laplacian, Ω is the Bohr compact of an almost periodic real-valued function $v(\cdot)$, and $v_s(\cdot)$ is the element of its hull corresponding to $s \in \Omega$; the second summand in (3) is the operator of multiplication by the function $v_s(\cdot)$.

In many cases the parameter space S is also endowed with a natural Borel measure, μ. In the example (a), it is the Lebesgue measure on $\mathbb{R}$; in (b), it is the Haar measure on Ω. In the theory of random and almost periodic operators (see [5], [19]) it often turns out that for μ-almost every $s \in S$ the spectrum of the operator (1) is pure point (i.e., there exists an orthonormal basis consisting of eigenvectors). In the case of rank one perturbations (2), a necessary and sufficient condition for the spectrum to be pure point for almost

all $s \in \mathbb{R}$ is given in [22]. Applications of that result to random operators include the one-dimensional Anderson model (see [22]), where, under mild conditions, the spectrum turns out to be pure point for a.e. $s \in \mathbb{R}$. The almost periodic operator (3) in $l^2(\mathbb{Z})$, in the case of the celebrated almost Mathieu potential (see [16])

$$v_s(n) = \lambda \cos(2\pi\alpha n + s), \quad s \in \mathbb{R}/2\pi\mathbb{Z}, \tag{4}$$

has pure point spectrum for almost all s if the irrational α is Diophantine and $|\lambda| > 2$ [14]. In both these examples the eigenvalues embedded in the essential spectrum (which has the cardinality of the continuum) are dense there. The same holds in a number of other cases, see [3] and references therein.

Therefore, if by "small" sets we understand those subsets of S whose μ-measure is zero, and remove a suitable one, then for the remaining "measure typical" s in these examples the operator A_s has a pure point spectrum and the eigenvalues embedded in the essential spectrum are dense in it.

On the other hand, since S is a complete metric space, there is another notion of "smallness" for its subsets: we can consider as small the first category sets (also called *meager* sets), i.e., countable unions of nowhere dense sets. What structure of the spectrum (if any) is *Baire generic*, i.e., holds for all s except those in some meager set (or equivalently, for all s in a dense G_δ set)?

In the cases, where the answer is known, it is quite different from the above one. In particular, it turns out in the examples (2) and (3), (4) that for a Baire generic s there are *no* eigenvalues embedded in the essential spectrum. (For the operator family (2), this is proved in [12] and independently in [7]; for the family (3), (4), this follows from the result of [15].)

Note that these results on absence of eigenvalues in the essential spectrum for rank one perturbations (2) and for almost periodic operators (3), (4) are of the same kind. Nevertheless, the proofs in these two cases are completely

different and do not reveal any common mechanism behind these phenomena.

In the present work we find an obstruction that in many cases prevents the existence of eigenvalues embedded in the essential spectrum for *all* values of the parameter (even if they are dense there for *almost all* values of the parameter with respect to a suitable measure). Now the results of [12] and [7], on the one hand, and of [15], on the other, get a common explanation.

The families of self-adjoint operators that we consider in this work do not necessarily have the form (1). The only requirement is that A_s depend on s continuously in the strong resolvent sense. In the rest of the paper, the term "Baire generic" is abbreviated to "generic". A property of an element of a complete metric space S is, therefore, *generic* if there exists a dense G_δ subset of S all of whose elements have that property.

The remaining part of the paper is organized as follows. In Section 2 we state the main result – Theorem 1. It is proved in Section 3, which also contains some generalizations of the main result. Section 4 provides additional information (Theorem 2) about eigenvectors in the case where the eigenvalues are simple. Section 5 contains a sufficient condition for genericity of purely continuous spectrum (Theorem 3). In Section 6 some applications are considered. First we prove the genericity of absence of eigenvalues for certain ergodic families of one-dimensional Schrödinger operators with Cantor spectrum. Then we show how results obtained in [12], [7] and in [15] can be derived from the results of this work.

2   The main result

Theorem 1 *Let S be a complete metric space and $\{A_s\}_{s \in S}$ be a family of self-adjoint operators in a separable Hilbert space $\mathcal{H}$ such that the mapping $s \mapsto A_s$ is continuous in the strong resolvent sense (that is, the mapping $S \ni s \mapsto (A_s - z_0 I)^{-1} x \in \mathcal{H}$ is continuous for any $z_0 \in \mathbb{C} \setminus \mathbb{R}$ and any $x \in \mathcal{H}$). Let F be a closed subset of the real line $\mathbb{R}$. Denote by Z the set of*

all $s \in S$ such that the operator A_s has at least one eigenvalue in F.

An alternative takes place: either

(i) Z is a meager set, or

(ii) Z has a non-empty interior; in this case there exist a non-empty open set $U \subset Z$ and a continuous function

$$s \mapsto \mathcal{E}_s : \ U \to F$$

such that for all $s \in U$, $\mathcal{E}_s$ is an eigenvalue of A_s.

Remarks. 1. The first part of this theorem (stating that either Z is meager, or it contains an open set) can also be derived from [21, Theorem 1.1]. What we emphasize is the existence, in the latter case, of an eigenvalue that depends continuously on the parameter (on some smaller open set).

2. Let the operator A_s be bounded for all s and depend on s continuously in the norm sense. Assume that, for some $s = s_0$, A_s has a simple isolated eigenvalue E_0. Then, as is well known, for all s close enough to s_0, A_s also has a simple isolated eigenvalue $E(s)$ close to E_0; it is continuous in s and satisfies $E(s_0) = E_0$. (The corresponding eigenspace may be determined as the range of a suitable Riesz projection.) So in this case our statement is obvious. But if the eigenvalues are not isolated – e.g., if they are dense in some interval, – we do not have such a natural way to identify individual eigenvalues for different values of s, and it is not clear whether it is possible to pick a point $E(s)$ in this dense set (depending on s) that will be continuous in s on some open subset of S. Of course, such a selection is impossible if for all s in some dense subset of S the operator A_s has no eigenvalues at all. The theorem implies that this is the *only* case where continuous selection of an eigenvalue is impossible.

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3 The proof

1. Consider the topological product

$$\mathcal{T} := S \times F \times \mathcal{H},$$

where the Hilbert space $\mathcal{H}$ is endowed with the weak topology. Furthermore, for any $m \in \mathbb{N}$ define a topological subspace of $\mathcal{T}$:

$$\mathcal{T}_m := S \times F_m \times B_1,$$

where $F_m = F \cap [-m, m]$ and $B_1 = \{y \in \mathcal{H} \mid \|y\| \leq 1\}$. Note that $\mathcal{T}_m$ endowed with the induced topology is a product of two completely metrizable spaces S and $F_m \times B_1$, the latter being compact.

Also consider a subset of $\mathcal{T}$

$$\mathrm{Sol} := \{\eta = (s, E, y) \in \mathcal{T} \mid A_s y = Ey, \ y \neq 0\}.$$

Clearly,

$$\mathrm{pr}_S(\mathrm{Sol}) = Z$$

(by pr_S, $\mathrm{pr}_{S \times F}$, etc. we denote the corresponding projections of the product $\mathcal{T} = S \times F \times \mathcal{H}$ onto S, $S \times F$, etc.). Furthermore, fix a sequence $\{g_k\}_{k=1}^{\infty}$ that is dense in the unit sphere of $\mathcal{H}$; let

$$\mathrm{Sol}_m^k := \left\{ \eta = (s, E, y) \in \mathcal{T} \mid A_s y = Ey, \ (y, g_k) = \frac{3}{4}, \ y \in B_1, \ E \in F_m \right\}, \tag{5}$$

or equivalently,

$$\mathrm{Sol}_m^k := \mathrm{Sol} \cap \left\{ \eta = (s, E, y) \in \mathcal{T}_m \mid (y, g_k) = \frac{3}{4} \right\}. \tag{6}$$

2. We will need the following simple fact.

Lemma 1 Sol_m^k *is closed in* $\mathcal{T}_m$.

Proof. Suppose

$$\text{Sol}_m^k \ni \eta_j = (s_j, E_j, y_j) \to \eta = (s, E, y) \in \mathcal{T}_m \quad \text{as} \quad j \to \infty.$$

We should verify that $\eta \in \text{Sol}_m^k$. That $(y, g_k) = \frac{3}{4}$ is obvious, since $y_j \xrightarrow{w} y$ as $j \to \infty$ (by $\xrightarrow{w}$ we denote the weak convergence in $\mathcal{H}$).

According to (6), it remains to check that $\eta \in \text{Sol}$, i.e., $A_s y = Ey$, or equivalently, $(A_s - z_0 I)^{-1} y = (E - z_0)^{-1} y$, where $z_0 \in \mathbb{C} \setminus \mathbb{R}$ is fixed. We have $A_{s_j} y_j = E_j y_j$, hence

$$(A_{s_j} - z_0 I)^{-1} y_j = (E_j - z_0)^{-1} y_j. \tag{7}$$

The left-hand side converges weakly to $(A_s - z_0 I)^{-1} y$. (Indeed, if φ is an arbitrary vector in $\mathcal{H}$, then $((A_{s_j} - z_0 I)^{-1} y_j, \varphi) = (y_j, (A_{s_j} - \overline{z_0} I)^{-1} \varphi)$ converges to $(y, (A_s - \overline{z_0} I)^{-1} \varphi)$, in view of the strong continuity of the resolvent of A_s in s.) At the same time, the right-hand side of (7) converges weakly to $(E - z_0)^{-1} y$. Thus $(A_s - z_0 I)^{-1} y = (E - z_0)^{-1} y$. ∎

3. Let

$$Z_m^k := \text{pr}_S \left(\text{Sol}_m^k \right). \tag{8}$$

The set Z_m^k is closed in S.

Proof. This is a special case of the following general statement.

Lemma 2 *If M and N are topological spaces, N is compact and B is a closed subset of $M \times N$, then the projection of B to M is closed.* ∎

Assume that the case (i) of the theorem does not take place, i.e., the set Z is not a countable union of nowhere dense sets. Since

$$Z = \bigcup_{m,\,k=1}^{\infty} Z_m^k,$$

there is a pair $(k, m) \in \mathbb{N}^2$ such that Z_m^k is not nowhere dense in S. Being closed, Z_m^k contains some open set U; this implies the first part of (ii).

4. Now we are going to derive the existence of an eigenvalue $\mathcal{E}_s \in F$ of A_s depending continuously on $s \in Z_m^k$.

Lemma 3 If $\eta = (s, E, y) \in \mathrm{Sol}_m^k$ and $\tilde{\eta} = (s, \tilde{E}, \tilde{y}) \in \mathrm{Sol}_m^k$, then $E = \tilde{E}$.

Proof. We have $A_s y = E y$ and $A_s \tilde{y} = \tilde{E} \tilde{y}$, so that if $E \neq \tilde{E}$, then y and $\tilde{y}$ are orthogonal. It follows then from the Bessel inequality and relations $(y, g_k) = (\tilde{y}, g_k) = \frac{3}{4}$, $\|y\| \leq 1$ and $\|\tilde{y}\| \leq 1$ that

$$\|g_k\|^2 \geq \left| \left(g_k, \frac{y}{\|y\|} \right) \right|^2 + \left| \left(g_k, \frac{\tilde{y}}{\|\tilde{y}\|} \right) \right|^2 \geq \left(\frac{3}{4} \right)^2 + \left(\frac{3}{4} \right)^2 > 1,$$

which is impossible because $\|g_k\| = 1$. ∎

Lemma 3 implies that on the set

$$G_m^k := \mathrm{pr}_{S \times F_m}(\mathrm{Sol}_m^k)$$

the second component of a pair (s, E) is uniquely determined by its first component. In other words, G_m^k is the graph of a function

$$s \mapsto \mathcal{E}_s \colon Z_m^k \to F_m \tag{9}$$

(we use the fact that $\mathrm{pr}_S(G_m^k) = \mathrm{pr}_S(\mathrm{Sol}_m^k) = Z_m^k$).

The set G_m^k is closed, again by Lemma 2, so that the function (9) has a closed graph; consequently, it is continuous, in view of the following lemma.

Lemma 4 *If M and N are topological spaces, N is compact and a mapping $f : M \to N$ has a closed graph, then f is continuous.* ∎

As was shown, if the case (i) of Theorem 1 does not take place, then the set Z_m^k (for some m and k) contains a non-empty open set U. The restriction of the continuous function (9) to U is continuous as well. This completes the proof of Theorem 1.[1] ∎

Remarks. 1. The closed set F in the formulation of Theorem 1 can be replaced by an arbitrary F_σ set, in particular, by any open set. This is an immediate consequence of the theorem.

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[1] The proof presented here is a simplified version of the original proof; the simplification is due to Artur Avila.

2. The set F in Theorem 1 does not need to be fixed. The following generalization of Theorem 1 is true.

Theorem 1*. *Let $\{A_s\}_{s\in S}$ be a family of self-adjoint operators satisfying the conditions of Theorem 1. Let $\mathcal{F}$ be a closed subset of the product $S \times \mathbb{R}$; denote by $\mathcal{F}_s$ $(s \in S)$ its (closed) cross-section*

$$\mathcal{F}_s := \{E \in \mathbb{R} \mid (s, E) \in \mathcal{F}\}$$

and by Z the set of all $s \in S$ for which the operator A_s has at least one eigenvalue in $\mathcal{F}_s$. Then either

(i) Z is a meager set, or

(ii) there exist a non-empty open set $U \subset Z$ and a continuous function

$$s \mapsto \mathcal{E}_s \ : \ U \to \mathbb{R}$$

such that for all $s \in U$ we have: $\mathcal{E}_s$ is an eigenvalue of A_s and $\mathcal{E}_s \in \mathcal{F}_s$.

The proof differs from that of Theorem 1 only in that now we define the topological space $\mathcal{T}$ by $\mathcal{T} := \mathcal{F} \times \mathcal{H}$ and then define $\mathcal{T}_m$ as follows: $\mathcal{T}_m := \mathcal{F}_m \times B_1$, where $\mathcal{F}_m = \{(s, E) \in \mathcal{F} \mid \mid E \mid \leq m\}$. ∎

Corollary 1 *Let $A_s = A + B_s$, where A is a self-adjoint operator in a separable Hilbert space $\mathcal{H}$ and B_s is a bounded self-adjoint operator in $\mathcal{H}$ depending on the point s of a complete metric space S continuously in the* norm *sense. Denote by $\sigma_{\mathrm{ess}}(A_s)$ the essential spectrum of A_s (which in general depends on s). Then either*

(i) for a generic $s \in S$ there are no eigenvalues of A_s in $\sigma_{\mathrm{ess}}(A_s)$, or

(ii) there is a non-empty open set $U \subset S$ and an eigenvalue $\mathcal{E}_s$ of A_s $(s \in U)$ that belongs to $\sigma_{\mathrm{ess}}(A_s)$ and depends continuously on $s \in U$. ∎

Proof. In order to apply Theorem 1^*, we need to verify that the set $\mathcal{F} := \{(s, E) \in S \times \mathbb{R} \mid E \in \sigma_{\mathrm{ess}}(A_s)\}$ is closed. This follows immediately from the following characterization of the essential spectrum [8]: $E \in \sigma_{\mathrm{ess}}(A_s)$ if and only if for any $\varepsilon > 0$ there exists an infinitely-dimensional linear set $D \subset D(A_s)$ such that $\|A_s x - Ex\| \leq \varepsilon \|x\|$ for all $x \in D$. ∎

4   The case of simple eigenvalues

Theorem 2 *Let $\{A_s\}_{s \in S}$ be a family of operators that satisfies the conditions of Theorem 1, and F a closed subset of $\mathbb{R}$. Assume, in addition, that all eigenvalues of the operators A_s are simple. Then an alternative takes place: either*

(i) for all s in some dense G_δ set, the operator A_s has no eigenvalues in F, or

(ii) there exist a non-empty open set $U \subset Z$ and two mappings $U \ni s \mapsto \mathcal{E}_s \in F$ and $U \ni s \mapsto \mathcal{Y}_s \in \mathcal{H} \setminus \{0\}$ such that

    (a) $A_s \mathcal{Y}_s = \mathcal{E}_s \mathcal{Y}_s$ for all $s \in U$;

    (b) the mapping $s \mapsto \mathcal{E}_s$ is continuous;

    (c) the mapping $s \mapsto \mathcal{Y}_s$ is weakly continuous;

    (d) there is a dense G_δ set Q in U such that for any $s^* \in Q$ we have $\|\mathcal{Y}_s - \mathcal{Y}_{s^*}\| \to 0$ as $U \ni s \to s^*$.

Proof. Assume that the case (ii) of Theorem 1 takes place. In what follows, we use the objects and notation introduced in the proof of that theorem. For each point s of $Z_m^k = \mathrm{pr}_S(\mathrm{Sol}_m^k)$, there is only one triple $(s, E, y) \in \mathrm{Sol}_m^k$ with the given s. (The uniqueness of $E = \mathcal{E}_s$ was established in the proof of Theorem 1; the uniqueness of y, denoted below by $\mathcal{Y}_s$, follows from the equality $(y, g_k) = \frac{3}{4}$ (see (5)) and the assumed simplicity of the eigenvalue $\mathcal{E}_s$ of A_s.)

Consequently, the set Sol_m^k is the graph of a mapping $Z_m^k \ni s \mapsto (\mathcal{E}_s, \mathcal{Y}_s) \in F_m \times B_1$, which is continuous by Lemmas 1 and 4. In particular, the mapping $Z_m^k \ni s \mapsto \mathcal{Y}_s \in B_1$ is weakly continuous, so that its restriction to the open subset U of Z_m^k is weakly continuous as well. Therefore, the mapping $U \ni s \mapsto (\mathcal{E}_s, \mathcal{Y}_s) \in F_m \times B_1$ has properties (a), (b) and (c). Obviously, $\mathcal{Y}_s \neq 0$. It remains to prove (d).

Fix an orthonormal basis $e_1, e_2, \ldots$ of the Hilbert space $\mathcal{H}$. The function $h(s) := \|\mathcal{Y}_s\|^2$, $s \in Z_m^k$, is the pointwise limit of a sequence of continuous

functions: $h(s) = \lim_{n \to \infty} h_n(s)$, where

$$h_n(s) = \sum_{j=1}^{n} |(\mathcal{Y}_s, e_j)|^2.$$

Consequently, there is a dense G_δ set in Z_m^k, say X, such that the function $h(\cdot)$ is continuous at all points of X [18, Theorem 7.3].

If $s^* \in X$, then, as $Z_m^k \ni s \to s^*$, we have $\mathcal{Y}_s \overset{w}{\to} \mathcal{Y}_{s^*}$ and $\|\mathcal{Y}_s\| \to \|\mathcal{Y}_{s^*}\|$, which implies $\|\mathcal{Y}_s - \mathcal{Y}_{s^*}\| \to 0$. Setting $Q := X \cap U$ completes the proof. ∎

5   A sufficient condition for genericity of purely continuous spectrum

Theorem 3 *Let $\{A_s\}_{s \in S}$ be an operator family satisfying the conditions of Theorem 1. Let F be a closed subset of $\mathbb{R}$. Denote by D the set of all eigenvalues of all the operators A_s: $D := \cup_{s \in S} \sigma_{\mathrm{p}}(A_s)$. Suppose that*
(a) S is locally connected;
(b) the subset $F \cap D$ of $\mathbb{R}$ has no interior points;
(c) for any $E \in F$, the set

$$N_E := \{s \in S : \ E \text{ is an eigenvalue of } A_s\} \tag{10}$$

has no interior points.
Then for all s in some dense G_δ set $S_0 \subset S$, the operator A_s has no eigenvalues in F.

Proof.   Assume the converse. Then the case (ii) of Theorem 1 takes place, so that there exist an open set $U \subset S$ and a continuous function $s \mapsto \mathcal{E}_s \colon U \to F$ such that $\mathcal{E}_s$ is an eigenvalue of A_s for all $s \in U$. Let U_0 be a connected component of U. By (a), it is open. Thus, we have a continuous function $U_0 \ni s \mapsto \mathcal{E}_s \in D \cap F$ and, since U_0 is connected, the set $\{\mathcal{E}_s \colon s \in U_0\} \subset D \cap F$ is connected as well, being, therefore, an interval. By the assumption (b),

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this interval consists of a single point, say E_0. This implies that the set (10) with $E = E_0$ contains an open set, U_0. However, this contradicts (c). ∎

6 Applications

6.1 Ergodic one-dimensional Schrödinger operators with Cantor spectrum

6.1.1 Continuum operators

Let S be a complete metric space; $\{G^t\}_{t\in\mathbb{R}}$ a group of its homeomorphisms such that $G^t(s)$ is continuous in t for any s; $V(\cdot)$ a bounded continuous real-valued function on S; and μ a Borel measure on S such that $\mu(S) = 1$. For $s \in S$, define a self-adjoint operator H_s on $L^2(\mathbb{R})$ by

$$H_s = -\frac{d^2}{dt^2} + v_s, \tag{11}$$

where v_s is the operator of multiplication by the function

$$v_s(t) = V(G^t(s)). \tag{12}$$

Suppose that

(A) the group $\{G^t\}_{t\in\mathbb{R}}$ preserves the measure μ and is μ-ergodic;

(B) the group is minimal, that is, the trajectory $\{G^t(s)\}_{t\in\mathbb{R}}$ of any point $s \in S$ is dense in S;

(C) $\mu(U) > 0$ for any non-empty open set $U \subset S$.

The operators H_s and $H_{G^\tau(s)}$ are unitarily equivalent: $H_{G^\tau(s)} = U_\tau^{-1} H_s U_\tau$, where $(U_\tau y)(t) = y(t - \tau)$; hence their spectra $\sigma(H_s)$ and $\sigma(H_{G^\tau(s)})$ are identical. Furthermore, the multiplication operator v_s depends on s strongly continuously, so that the operator family (11) satisfies the strong resolvent continuity condition: if $s_k \to s$, then H_{s_k} converges to H_s in the strong resolvent sense [20, Theorem VIII.25]. Consequently, for any $E \in \sigma(H_s)$ there exist $E_k \in \sigma(H_{s_k})$ such that $E_k \to E$ [20, Theorem VIII.24]. The

assumption (B), therefore, implies that the spectrum of H_s does not depend on s. Denote this closed set by Σ. It does not contain isolated points [19, Theorem 2.11].

Theorem 4 *Suppose that the group $\{G^t\}_{t\in\mathbb{R}}$, the measure μ and the function V satisfy the above assumptions. If S is locally connected and Σ is nowhere dense in $\mathbb{R}$, then there is a dense G_δ set $S_0 \in S$ such that for any $s \in S_0$ the operator H_s defined by the equations (11), (12) has no eigenvalues.*

Proof. We use Theorem 3, putting $F := \Sigma$. As was said, the operator family (11) satisfies the strong resolvent continuity condition. The conditions (a) and (b) of Theorem 3 are fulfilled due to the assumptions of Theorem 4. The condition (c) is satisfied, because the operators (11) cannot have eigenvalues of infinite multiplicity, so that the set (10) has μ-measure zero [19, Theorem 2.12] and, in view of (C), cannot contain an open set. ∎

Corollary 2 *Let H_s, where $s = (s_1, \ldots, s_d) \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ $(d \geq 2)$, be the operator (11) in $L^2(\mathbb{R})$ with the potential*

$$v_s(t) = V(s_1 + \alpha_1 t, \ldots, s_d + \alpha_d t), \quad t \in \mathbb{R};$$

here V is a continuous real-valued function on $\mathbb{T}^d$, and the numbers $\alpha_1, \ldots, \alpha_d$ are rationally independent. If the spectrum of the operator H_s is nowhere dense (for some s and hence for all $s \in \mathbb{T}^d$), then for all s in some dense G_δ set $S_0 \subset \mathbb{T}^d$ the operator H_s has no eigenvalues.

Proof. Apply Theorem 4, setting $S = \mathbb{T}^d$ and defining the homeomorphism $G^t\colon \mathbb{T}^d \to \mathbb{T}^d$ $(t \in \mathbb{R})$ by $G^t(s_1, \ldots, s_d) = (s_1 + \alpha_1 t, \ldots, s_d + \alpha_d t)$, μ being the Lebesgue measure on $\mathbb{T}^d$. ∎

6.1.2   Discrete operators

1. Let S be a complete metric space, G its homeomorphism, $V(\cdot)$ a bounded continuous real-valued function on S, and μ a Borel measure on S such that

$\mu(S) = 1$. For $s \in S$, define a self-adjoint operator H_s on $l^2(\mathbb{Z})$ by

$$(H_s y)(n) = y(n-1) + y(n+1) + v_s(n)y(n), \quad n \in \mathbb{Z}, \tag{13}$$

where

$$v_s(n) = V(G^n(s)). \tag{14}$$

Suppose that

(A) the homeomorphism G preserves the measure μ and is μ-ergodic;

(B) G is minimal, that is, the trajectory $\{G^n(s)\}_{n \in \mathbb{Z}}$ of any point $s \in S$ is dense in S;

(C) $\mu(U) > 0$ for any non-empty open set $U \subset S$.

All the operators H_s ($s \in S$) have the same spectrum, from now on denoted by Σ (with interior $\mathrm{Int}(\Sigma)$). Furthermore, this closed set does not have isolated points. These statements, as well as the following theorem, can be proved in the same way as in the continuum case.

Theorem 5 *Suppose that the homeomorphism G, the measure μ and the function V satisfy the above assumptions. If S is locally connected and Σ is nowhere dense in $\mathbb{R}$, then there is a dense G_δ set $S_0 \in S$ such that for all $s \in S_0$ the operator H_s defined by the equations (13), (14) has no eigenvalues.* ■

Corollary 3 *Let H_s, where $s = (s_1, \ldots, s_d) \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ ($d \geq 1$), be the operator (13), (14) in $l^2(\mathbb{Z})$, where G is a shift on $\mathbb{T}^d$:*

$$G(s_1, \ldots, s_d) = (s_1 + \alpha_1, \ldots, s_d + \alpha_d), \tag{15}$$

the numbers $1, \alpha_1, \ldots, \alpha_d$ being rationally independent. If $\mathrm{Int}(\Sigma) = \emptyset$, then for all s in some dense G_δ set $S_0 \subset \mathbb{T}^d$ the operator H_s has no eigenvalues. ■

Corollary 4 *Let H_s, where $s = (s_1, s_2) \in \mathbb{T}^2$, be the operator (13), (14) in $l^2(\mathbb{Z})$, where G is a skew-shift on $\mathbb{T}^2$:*

$$G(s_1, s_2) = (s_1 + \alpha, s_2 + s_1) \tag{16}$$

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with an irrational α. If $\mathrm{Int}(\Sigma) = \emptyset$, then for all s in a dense G_δ set $S_0 \subset \mathbb{T}^2$ the operator H_s has no eigenvalues. ∎

2. Some ergodic Schrödinger operators having Cantor spectrum for generic values of the functional parameter are studied in [1]. These are operators over a strictly ergodic homeomorphism that fibers over an almost periodic dynamical system.

Proposition 1 [1]   *Let S be a compact metric space and $G\colon S \to S$ a strictly ergodic homeomorphism (that is, G is minimal and uniquely ergodic). Suppose that there exist an infinite compact Abelian group K and an onto continuous mapping $p\colon S \to K$ such that $p(G(s)) = p(s) + \alpha$ for some $\alpha \in K$ and all $s \in S$. Then for a generic $V \in C(S, \mathbb{R})$ the operator H_s^V in $l^2(\mathbb{Z})$ defined by*

$$(H_s^V y)(n) = y(n-1) + y(n+1) + V(G^n(s))y(n), \quad n \in \mathbb{Z}, \qquad (17)$$

has Cantor spectrum.

Proposition 1 and Theorem 5 imply the following statement.

Corollary 5 *Let S, G, K, and p be such as in Proposition 1. Assume, in addition, that S is locally connected. Then for a generic $V \in C(S, \mathbb{R})$ there exists a dense G_δ set $S_V \subset S$ such that for all $s \in S_V$ the operator (17) does not have eigenvalues.* ∎

Examples of particular interest include those where G is a shift or a skew-shift on the torus (see the above Corollaries 3 and 4).

Corollary 6 *Let H_s^V, where $V \in C(\mathbb{T}^d, \mathbb{R})$ and $s \in \mathbb{T}^d$ $(d \geq 1)$, be the operator in $l^2(\mathbb{Z})$ given by*

$$(H_s^V y)(n) = y(n-1) + y(n+1) + V(s_1 + n\alpha_1, \ldots, s_d + n\alpha_d)\, y(n), \quad n \in \mathbb{Z}, \quad (18)$$

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the numbers $1, \alpha_1, \ldots, \alpha_d$ being rationally independent. Then for a generic $V \in C(\mathbb{T}^d, \mathbb{R})$ and all s in some dense G_δ set $S_V \subset \mathbb{T}^d$ (which may depend on V) the operator H_s^V has no eigenvalues. ∎

Remark. For a different way to obtain this result, see [13].[2]

Corollary 7 *Let H_s^V, where $V \in C(\mathbb{T}^2, \mathbb{R})$ and $s = (s_1, s_2) \in \mathbb{T}^2$, be the following operator in $l^2(\mathbb{Z})$:*

$$(H_s^V y)(n) = y(n-1) + y(n+1) + V(G^n(s))y(n), \quad n \in \mathbb{Z},$$

where $G(s_1, s_2) = (s_1 + \alpha, s_2 + s_1)$ with an irrational α. Then for a generic $V \in C(\mathbb{T}^2, \mathbb{R})$ and all s in some dense G_δ set $S_V \subset \mathbb{T}^2$ (which may depend on V) the operator H_s^V has no eigenvalues. ∎

3. Recently Goldstein and Schlag [11] proved that the spectrum is a Cantor set for quasi-periodic operators with one-frequency analytic potentials for almost all values of the frequency in the regime of positive Lyapunov exponent. Their main result (Theorem 1.1) implies the following statement.

Proposition 2 (Corollary to [11, Theorem 1.1]). *Suppose the operator family A_ω^α ($\omega \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$) is given by the equation*

$$(A_\omega^\alpha y)(j) = y(j-1) + y(j+1) + V(j\alpha + \omega)\, y(j), \quad j \in \mathbb{Z}, \qquad (19)$$

where $\alpha \in \mathbb{R}$ and $V(\cdot)$ is a 1-periodic real-analytic function on $\mathbb{R}$. Suppose $\alpha_0 \in \mathbb{R}$ satisfies the Diophantine condition

$$\|k\alpha_0\| \geq \frac{c}{k(1 + \log k)^a}, \quad k = 1, 2, \ldots, \qquad (20)$$

---

[2] Note that the proof of one of lemmas in [13] (Lemma 2) is incomplete: it contains a statement about sequences of Schrödinger operators with convergent potentials and weakly convergent eigenfunctions that is not actually proved. To correct that, the end of the proof of the lemma should be replaced by an argument similar to that used in the proof of Lemma 1 above.

with some $a > 1$, $c > 0$. Here $\| \cdot \|$ denotes the distance from the nearest integer. Suppose, furthermore, that the Lyapunov exponent $L(E, \alpha_0)$ of the operator $A_\omega^{\alpha_0}$ satisfies the inequality

$$L(E, \alpha_0) \geq \gamma_0 > 0, \quad E' \leq E \leq E''. \tag{21}$$

Then there exists $\delta > 0$ such that for almost all $\alpha \in (\alpha_0 - \delta, \alpha_0 + \delta)$ the spectrum of A_ω^α (which, for irrational α, does not depend on ω) is nowhere dense in $[E', E'']$.

Proposition 2 and Corollary 3 lead to the following statement.

Theorem 6 *Suppose the operator (19), where $V(\cdot)$ is a 1-periodic real-analytic function on $\mathbb{R}$, has strictly positive Lyapunov exponent $L(E, \alpha_0) > 0$ for some α_0 satisfying the Diophantine condition (20) and all E in a compact interval $\Delta = [E', E'']$.*

Then for almost all α in a small enough neighborhood of α_0, the operator A_ω^α has singular continuous spectrum in Δ for a dense G_δ set of $\omega \in \mathbb{T}$.

Proof. According to the result of Bourgain – Jitomirskaya [4], the real-analyticity of V and the irrationality of α_0 imply that the Lyapunov exponent $L(E, \alpha)$ is jointly continuous in E and α at all points (E, α_0), $E \in \mathbb{R}$. (Note that $L(E, \alpha)$, for rational α, is understood here as the Lyapunov exponent averaged over $\omega \in \mathbb{T}$.) It follows that (21) holds for some small $\gamma_0 > 0$; moreover, for all $\alpha \in (\alpha_0 - \beta, \alpha_0 + \beta)$ with small enough $\beta > 0$ we have

$$L(E, \alpha) \geq \gamma_0/2 > 0, \quad E' \leq E \leq E''. \tag{22}$$

By Proposition 2, the operator A_ω^α has Cantor spectrum for almost all $\alpha \in (\alpha_0 - \delta, \alpha_0 + \delta)$, where $\delta \in (0, \beta]$ is sufficiently small. By Corollary 3, for any such α and all ω in a dense G_δ set $T_\alpha \in \mathbb{T}$, the operator A_ω^α does not have eigenvalues in Δ.

The singularity of the spectrum of A_ω^α in the interval Δ is implied by inequality (22); this follows from the Ishii – Pastur – Kotani theorem (see, e.g., [6]) for almost all ω and from the result of Last and Simon [17] for all ω. ∎

Corollary 8 *Suppose the operator (19) with a 1-periodic real-analytic function V has positive Lyapunov exponent $L(E, \alpha) > 0$ for almost all $(\alpha, E) \in [\alpha', \alpha''] \times [E', E'']$. Then for almost every $\alpha \in [\alpha', \alpha'']$ the operator A_ω^α has singular continuous spectrum in $[E', E'']$ for a dense G_δ set of $\omega \in \mathbb{T}$.*

Proof. Let $I = [\alpha', \alpha'']$, $\Delta = [E', E'']$ and $R = I \times \Delta$. Let Y be the union of all rectangles $Q = [\alpha_Q', \alpha_Q''] \times [E_Q', E_Q''] = I_Q \times \Delta_Q$ with rational vertices, such that for all $\alpha \in I_Q \setminus N_Q$, where $|N_Q| = 0$ ($|\cdot|$ denotes the Lebesgue measure), the operator A_ω^α has Cantor spectrum on Δ_Q and a positive Lyapunov exponent a.e. on Δ_Q. It follows from the proof of Theorem 6 (and the fact that almost all α are Diophantine in the sense of (20)) that $|R \setminus Y| = 0$. In addition, $|\bigcup_Q N_Q| = 0$. This implies that for a.e. $\alpha \in I$ the interval Δ is almost entirely covered with countably many intervals on each of which the operator A_ω^α has Cantor spectrum and a.e. positive Lyapunov exponent; therefore, the same is true for the whole interval Δ. Consequently, for a.e. $\alpha \in I$ the spectrum of A_ω^α on Δ is singular for all ω and, for generic ω, singular continuous. ∎

Remark. Another sufficient condition for the operator (19) to have Cantor spectrum was given by Sinai [23].

6.2 Rank one perturbations

Consider an operator family

$$A_t = A + tP, \quad t \in \mathbb{R},$$

where A is a cyclic self-adjoint operator and P is an orthogonal projection onto its one-dimensional cyclic subspace $\mathbb{C}\varphi$ ($\|\varphi\| = 1$): $Py = (y, \varphi)\varphi$. All

the operators A_t have the same essential spectrum, which we denote by Σ_{ess}. It was proved in [12] and [7] (in different ways) that *for a generic t there are no eigenvalues of A_t in Σ_{ess}.*

The first and easiest step in both proofs (especially in [7]) was to show that the set

$$\{E \in \Sigma \mid E \text{ is an eigenvalue of } A_t \text{ for some } t\} \tag{23}$$

is meager. Now we will show how to replace the rest of the proof by a reference to Theorem 3.

We apply Theorem 3 with $S = \mathbb{R}$ and $F = \Sigma_{\mathrm{ess}}$. Assumptions (a) and (b) of Theorem 3 are fulfilled; to prove (c), assume first that E and $\widetilde{E}$ are eigenvalues of the operators A_t and $A_{\widetilde{t}}$, respectively, with nonzero eigenvectors y and $\widetilde{y}$. Since the vector φ is cyclic for A, it is easy to see that $(y, \varphi) \neq 0$ and $(\widetilde{y}, \varphi) \neq 0$, so we may assume that $(y, \varphi) = (\widetilde{y}, \varphi) = 1$. Then the obvious identity

$$(\widetilde{E} - E)(\widetilde{y}, y) = (\widetilde{t} - t)(\widetilde{y}, Py)$$

takes the form

$$(\widetilde{E} - E)(\widetilde{y}, y) = \widetilde{t} - t,$$

so that $\widetilde{t} = t$ if $\widetilde{E} = E$. Therefore, for any $E \in \mathbb{R}$ the set

$$N_E := \{t \in \mathbb{R} \colon E \text{ is an eigenvalue of } A_t\}$$

contains at most one point, which proves (c).

6.3   Even almost periodic potentials

Let $v(t)$ be a real-valued almost periodic function on $\mathbb{R}$, S its Bohr compact, and $A_s = -d^2/dt^2 + v_s(t)$ $(s \in S)$ the corresponding family of Schrödinger operators in $L^2(\mathbb{R})$. It was proved in [15] that *if $v(t)$ is even, then for a generic $s \in S$ the operator A_s has no eigenvalues.* (The $l^2(\mathbb{Z})$ version of this result was also proved in [15].)

Now we will show how a somewhat more general statement can be derived from the results of the present work.

Let S be a complete metric space; $\{G^t\}_{t\in\mathbb{R}}$ a group of its homeomorphisms such that $G^t(s)$ is continuous in t for any s; and $V(\cdot)$ a bounded continuous real-valued function on S. For $s \in S$, define a self-adjoint operator A_s on $L^2(\mathbb{R})$ by

$$A_s = -\frac{d^2}{dt^2} + v_s, \tag{24}$$

where v_s is the operator of multiplication by the function

$$v_s(t) = V(G^t(s)). \tag{25}$$

Theorem 7 *Suppose that for some $s_0 \in S$, setting $\gamma(t) := G^t(s_0)$, we have:*
(a) $V(\gamma(-t)) = V(\gamma(t))$ for all $t \in \mathbb{R}$;
(b) the trajectory $\{\gamma(t)\}_{t\in\mathbb{R}}$ of s_0 is dense in S;
(c) there exists a sequence t_k ($|t_k| \to \infty$) for which $\gamma(t_k) \to s_0$.
Then there is a dense G_δ set $S_0 \subset S$ such that for all $s \in S_0$ the operator (24), (25) has no eigenvalues.

Proof. Apply Theorem 2 with $F = \mathbb{R}$. Assume that the case (ii) of the theorem takes place. Pick a point $s^* \in Q$; denoting $\mathcal{Y}_{s^*}$ by y^*, we have: for any $\delta > 0$ there exists a neighborhood U' of s^* ($U' \subset U$) such that $\|\mathcal{Y}_s - y^*\| \le \delta$ for any $s \in U'$.

In view of (b) and (c), we can choose a sequence $u_k \in \mathbb{R}$ so that $|u_k| \to \infty$ and $s_k := \gamma(u_k) \in U'$ for all k. Then we have

$$\|\mathcal{Y}_{s_k} - y^*\| \le \delta, \quad k = 1, 2, \ldots \tag{26}$$

The function $\mathcal{Y}_{s_k}(t)$ is a solution of the equation $-y'' + v_{s_k}(t)\, y = \mathcal{E}_{s_k} y$, or equivalently, since $v_{s_k}(t) = V(G^t(s_k)) = V(\gamma(t + u_k))$, the equation

$$-y'' + v(t + u_k)\, y = \mathcal{E}_{s_k} y, \tag{27}$$

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where $v(t) = V(\gamma(t))$. It follows that the function $\mathcal{Y}_{s_k}(-t)$ solves the "reflected" equation $-y'' + v(-t + u_k)y = \mathcal{E}_{s_k}y$, or

$$-y'' + v(t - u_k)\, y = \mathcal{E}_{s_k}y \tag{28}$$

(here we use the evenness of the function $v(\cdot)$, which follows from (a)).

Since the equation (28) is the $(-2u_k)$-shift of (27) and the L^2-solution of each of these equations is uniquely determined up to a constant factor, we have

$$\mathcal{Y}_{s_k}(-t) = c_k \mathcal{Y}_{s_k}(t - 2u_k), \tag{29}$$

where $|c_k| = 1$.

It follows from (29) and (26) that

$$\|y^*(-t) - c_k\, y^*(t - 2u_k)\| \le 2\delta, \quad k = 1, 2, \ldots$$

But for large k, $|2u_k|$ is large, so the functions $y^*(t - 2u_k)$ and $y^*(-t)$ are almost orthogonal, and hence passing to the limit we have $2\|y^*\|^2 \le 4\delta^2$. Since $\delta > 0$ can be chosen arbitrarily small, we obtain: $y^* = 0$. We have arrived at a contradiction, which shows that the case (ii) of Theorem 2 is impossible. ∎

Corollary 9 *Let $v\colon \mathbb{R} \to \mathbb{R}$ be a bounded uniformly continuous function, and S be the set of all pointwise limits of its shifts $v_u(t) = v(t + u)$ $(u \in \mathbb{R})$ endowed with the topology of pointwise convergence (so that S is a compact metrizable topological space). Suppose that $v(-t) = v(t)$ for all $t \in \mathbb{R}$ and $v(t + u_n) \to v(t)$ for some sequence $u_n \in \mathbb{R}$ $(|u_n| \to \infty)$ and all $t \in \mathbb{R}$. Then for a generic $s \in S$ the operator $-d^2/dt^2 + s(t)$ in $L^2(\mathbb{R})$ does not have eigenvalues.*

Proof. Define homeomorphisms G^t $(t \in \mathbb{R})$ of S by $(G^t(s))(u) = s(u + t)$ and the function V on S by $V(s) = s(0)$. Then the function (25) is $v_s(t) = V(G^t(s)) = (G^t(s))(0) = s(t)$, and it remains to apply Theorem 7. ∎

In the particular case, where $v(t)$ is an even almost periodic function, we obtain the result [15].

The $l^2(\mathbb{Z})$ versions of these results can be proved similarly.

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