Nilpotence, Localization, and Dualizability

Connor Meredith (*University of Colorado Boulder*)

It is currently unknown what types of solvable interval may occur in the congruence lattice of a finite, dualizable, algebra in a congruence modular variety. Recently, we have investigated the dualizability of nilpotent nonabelian algebras that belong to congruence modular varieties. A classic result of Quackenbush and Szabó and a more recent result of Nickodemus shows that a finite group is dualizable if and only if it does not have a nonabelian Sylow subgroup. A finite group is nilpotent if and only if it is a direct product of groups of prime power order, but the connection between nilpotence and prime power direct decomposition seen in groups does not extend to the general setting. Instead, there is a second, distinct, form of nilpotence called supernilpotence. Supernilpotence was defined by Aichinger and Mudrinski for algebras in congruence permutable varieties and makes use of the higher commutator defined by Bulatov. There is no general implication between nilpotence and supernilpotence, but in the case of finite algebras, supernilpotence implies nilpotence. It is known that if A is a finite nilpotent algebra in a congruence modular variety, then the presence of a supernilpotent nonabelian congruence in any algebra in the prevariety generated by A prevents A from being dualizable.

In this talk, we will discuss several new results concerning the relationship between nilpotence and dualizability. One such result is:

Theorem. *Let $N > 0$. There exists a dualizable nilpotent nonabelian algebra of size N (of finite type in a congruence modular variety) if and only if N is not a prime power.*

The forward direction of this equivalence is a special case of a Theorem of Bentz and Mayr. We will present algebras that witness the converse. The exact role that supernilpotence plays in preventing the dualizability of finite nilpotent algebras has not been fully determined. Let A be a finite nilpotent algebra. On one hand, if $\mathrm{ISP}(A)$ contains an algebra with a nonabelian supernilpotent congruence, then A is nondualizable. Bentz and Mayr ask if the absence of such an algebra is enough to guarantee the dualizability A. We provide a negative answer:

Theorem. *There exists a finite nilpotent algebra A such that*

1. *for each B in $\mathrm{ISP}(A)$, every k-supernilpotent congruence of B with $k \geq 2$ is Abelian and*
2. *A is (inherently) nondualizable.*

Our construction of such an algebra relies on the existence of a supernilpotent nonabelian localization and an appeal to the following theorem.

Theorem. *Let A be a finite algebra and let e be an idempotent unary term operation of A. If the localization of A to the neighborhood $e(A)$ is nondualizable, then A is nondualizable.*

We will finish the talk by discussing the limitations of our technique that are brought about by the interaction between localization and the higher commutator.

Theorem. *Let A be a Mal'cev algebra with an idempotent unary term operation e. Suppose $e(A)$ is a generating set for A. For each binary relation θ of A, let θ^* denote the congruence of A generated by θ and let θ_* denote the restriction of θ to $e(A)$. Then for any congruences $\alpha_1, \ldots, \alpha_n$ of $e(A)$,*

$$[\alpha_1^*, \ldots, \alpha_n^*]_* = [\alpha_1, \ldots, \alpha_n].$$

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