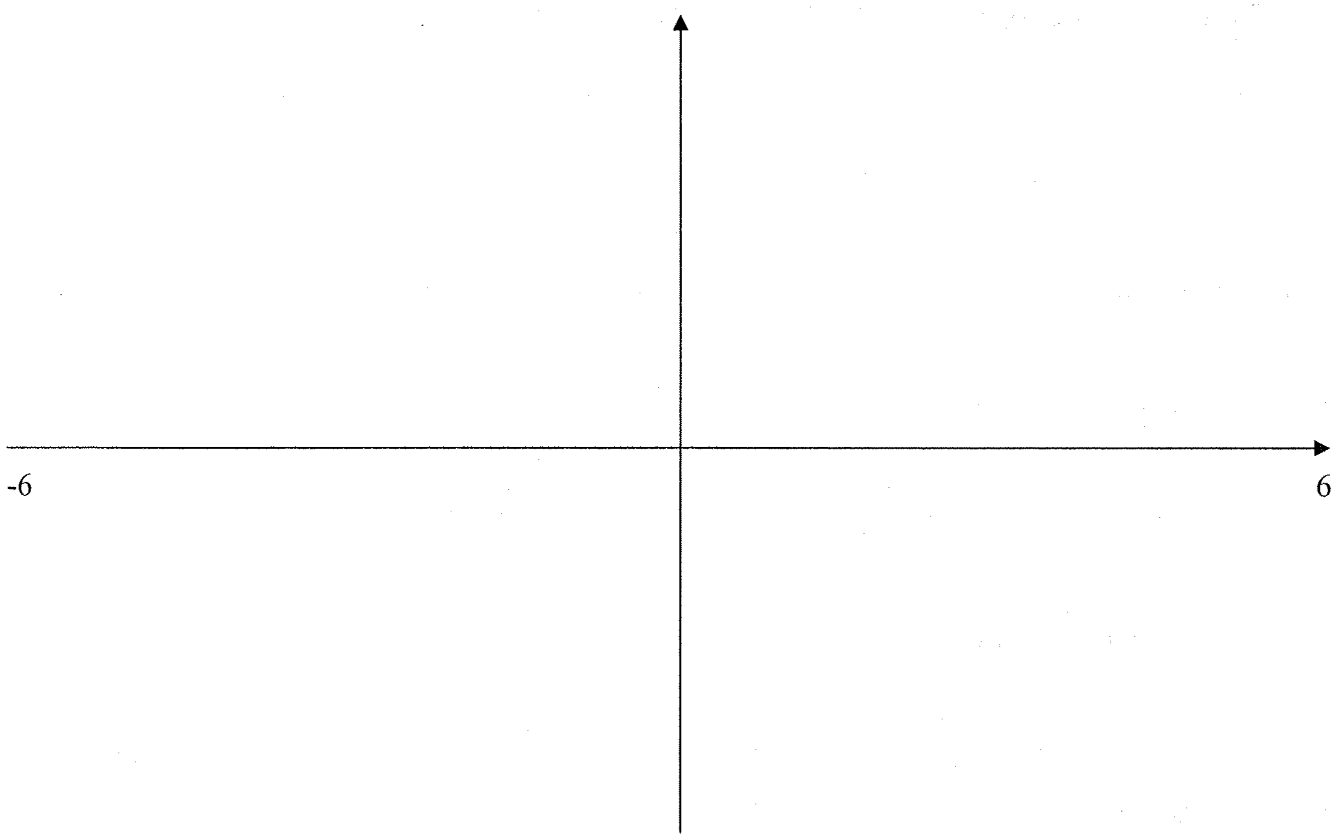
Math 1241 Project 3 Name:

Derivatives, Zeros, and the Newton's Method

In the first part of this project, we will explore (graphically and numerically) the relation between the derivative of a function and the function itself. In the second part, we will use Newton's method to find zeros of functions through calculus.

Let $f(x) = x^4 - 17x^2 + 18$.

1. Enter this function as Y_1 and set your window to $-6 < x < 6$ and $-60 < y < 60$. Copy the graph below.



2. Using your calculator, find the zeros of $f(x)$. Press 2nd TRACE to get to the CALC menu. Choose 2:Zero. The calculator will ask for a lower bound. Move the cursor to the left of the zero (the x-intercept). Press enter. Then move the cursor to the right of the zero to get the upper bound. Then give a guess. The approximate root is returned on the screen. Find all the zeros and record them with an accuracy of 3 decimal places.

3. Back to the graph to find all the local extrema: local maximum and local minimum. Move the cursor close to a local maximum. Press 2nd TRACE , pick 4: maximum, and follow the above procedure. Record the x- and y-coordinate of each of these points, accurate to 3 decimal places, and label which is a local min and which is a local max.

4. Graph the numerical derivative Y_2 of Y_1 by pressing

[Y=] [▽] [MATH] [8] [VARS] [▷] [1:Function] [1: Y_1] [,] [X,T,θ] [,] [X,T,θ] [)]

You will see Y_2 = nDeriv (Y_1 , X , X). Press graph.

Copy the graph of the derivative on the axes on the previous page (use a different color if possible).

5. Use this graph to find the zeros of $Y_2 = f'(x)$.

What is the relationship between the roots of $f'(x)$ and the graph of $f(x)$?

What is the relationship between the sign of $f'(x)$ and the graph of $f(x)$?

What do you observe about the relative locations of the roots of $f'(x)$ and $f(x)$?

6. Find graphically the x-coordinates of all the local extrema of $Y_2 = f'(x)$.

7. Set Y_3 = nDeriv (Y_2 , X , X). This the second derivative $f''(x)$ of $f(x)$. With a different color , copy the graph of $Y_3 = f''(x)$ on the first page.

8. Find the zeros of $f''(x)$.

What is the relationship between the roots of $f''(x)$ and the graph of $f'(x)$?

What is the relationship between the roots of $f''(x)$ and the graph of $f(x)$?

What is the relationship between the sign of $f''(x)$ and the graph of $f(x)$?

9. The function $f(x) = x^4 - 17x^2 + 18 = (x-4)(x-1)(x+1)(x+4) + 2$. So it is natural to guess that $f(x)$ has roots near $x = \pm 1$ and $x = \pm 4$. An approximation was found graphically in problem 2 and the approximate values were given correct to 6 decimal places. Suppose that we require more accuracy or just we want to approximate the zeros of a function with calculus not graphically. One of the techniques is *Newton's Method.*

To solve the equation $f(x) = 0$, we start with a first approximation of the solution, and use the formula $x_{n+1} = x_n - \dfrac{f(x)}{f'(x)}$ to obtain a sequence of approximations $x_2, x_3, x_4, \ldots$ that converges to the desired root. If the Newton's method fails, a better first approximation is needed.

In our case $f(x) = x^4 - 17x^2 + 18 = 0$ and a convenient first approximation is $x_1 = 1$ for example.

Enter $Y_1 = X^4 - 17X^2 + 18$, then put $Y_2 = X - Y_1 / nDeriv(Y_1, X, X)$.
The second formula is for the approximations.

Now on the HOME screen, type [VARS] [▷] [1] [2] [(] [1] [)] [ENTER]

Then type [2nd] [ENTRY] [◁] [◁] [ANS] [ENTER]

Then type [2nd] [ENTRY] [ENTER] as many times as needed.

(To increase the number of digits after the decimal point, go to mode and highlight 9 on the 2nd line.)

You see:

$Y_2(1)$	1.066666676
$Y_2(\text{Ans})$	1.065148657
$Y_2(\text{Ans})$	1.065147909
$Y_2(\text{Ans})$	1.065147909
$Y_2(\text{Ans})$	1.065147909

Thus $x_2 = 1.066666676$, $x_3 = 1.065148657$, $x_4 = 1.065147909$, ... so one of the zeros is 1.065147909 and it required 4 approximations.

Now repeat the same procedure to find the zero near 4: $x_1 =$

$x_2 =$ $\qquad$ $x_3 =$ $\qquad$ $x_4 =$ $\qquad$ $x_5 =$

So another zero is:

If you start with the approximation $x_1 = 3$, how many steps are needed?

10. Let's check algebraically some of the answers that we got graphically.

Note that $f(x) = x^4 - 17x^2 + 18 = X^2 - 17X + 18$ is a quadratic function.

Using the quadratic formula, the zeros X = and then $x =$

The first derivative $f'(x) =$

$f'(x) = 0$ if $x =$ and these are the x-coordinates of the local _____and _____of $f(x)$.

The second derivative $f''(x) =$

$f''(x) = 0$ if $x =$ and these are the x-coordinates of the points of________of $f(x)$.

11. The function $f(x) = \sin x$ has infinitely many zeros. Use Newton's Method to find the first 2 positive zeros showing the first few approximations in each case.

x_1	
x_2	
x_3	
x_4	

x_1	
x_2	
x_3	
x_4	

The first positive zero is : ________ The second positive zero: __________

What happens if you took $\pi/2$ as a first approximation? Explain.