Riemann Sums and Definite Integrals

The area under the graph of a positive function is given by the definite integral of the function. The definite integral can be approximated by the following sums:

Left Riemann Sum: $\int_a^b f(x)dx \approx L_n = \Delta x[f(x_0) + f(x_1) + ... + f(x_{n-1})]$

Right Riemann Sum: $\int_a^b f(x)dx \approx R_n = \Delta x[f(x_1) + f(x_2) + ... + f(x_n)]$

Midpoint Rule: $\int_a^b f(x)dx \approx M_n = \Delta x[f(\bar{x}_1) + f(\bar{x}_2) + ... + f(\bar{x}_{n-1})]$

Trapezoidal Rule: $\int_a^b f(x)dx \approx T_n = \dfrac{\Delta x}{2}[f(x_0) + 2f(x_1) + ... + 2f(x_{n-1}) + f(x_n)]$

Simpson's Rule: $\int_a^b f(x)dx \approx S_n = \dfrac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + ... + 4f(x_{n-1}) + f(x_n)]$

Where $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$, $\bar{x}_i$ the midpoint of the ith subinterval, and n is even for Simpson's Rule.

It turns out that the **Trapezoidal** approximation $T_n = \dfrac{L_n + R_n}{2}$ and **Simpson's** approximation

$S_n = \dfrac{T_{n/2} + 2M_{n/2}}{3}$

In all of these methods we get more accurate approximations when we increase the value of n. The **Error** in using an approximation is:

Error = Actual value of the integral - Approximation = $\int_a^b f(x)dx$ - Approximation

Error Bounds for Midpoint and Trapezoidal Rules:

Suppose that $|f''(x)| \le K_1$ for $a \le x \le b$. Then $|E_M| \le \dfrac{K_1(b-a)^3}{24n^2}$ and $|E_T| \le \dfrac{K_1(b-a)^3}{12n^2}$

Error Bounds for Simpson's Rules:

Suppose that $|f^{(4)}(x)| \le K_2$ for $a \le x \le b$. Then $|E_S| \le \dfrac{K_2(b-a)^5}{180n^4}$

These Error Bounds are very useful to estimate the errors and the accuracy of the approximations without having to find the value of these approximations, especially for large n's.
These Error Bounds are also helpful in estimating the number of partitions required to guarantee a specific accuracy when approximating an integral.

The above Approximating Sums can be found using a simple program on the calculator called **RIEMANN**.

PROGRAM : RIEMANN	To create this program on your calculator:
: Float	• Press $\boxed{\text{PRGM}}$ $\boxed{\blacktriangleright}$ $\boxed{\blacktriangleright}$ to display the **PRGM NEW** menu
: ClrHome	
: Prompt A, B, N	• Press $\boxed{\text{ENTER}}$ to select **1: Create New**. The **Name =**
: (B-A)/N $\rightarrow$ H	prompt is displayed (and the A-LOCK is on with the TI 83
: 0 $\rightarrow$ S	Plus), type in **RIEMANN** the name of the program
: For (I, 1, N-1)	
: A+I*H $\rightarrow$ X	• The commands **Float** and **ClrHome** are in the **CATALOG**
: S+Y_1(X) $\rightarrow$ S	press $\boxed{\text{2nd}}$ $\boxed{0}$ and scroll down.
: End	
	• The $\boxed{\text{ALPHA}}$ key lets you enter the alphabetic characters .
: (Y_1(A)+S)*H $\rightarrow$ L	To enter R, for example, press $\boxed{\text{ALPHA}}$, release it, and then
: (S+Y_1(B))*H $\rightarrow$ R	press $\boxed{\times}$. If you have to enter several alphabetic characters,
: (L+R)/2 $\rightarrow$ T	press $\boxed{\text{2nd}}$ $\boxed{\text{ALPHA}}$ to get the **A-lock** so that you avoid
: Output(4, 1, ″ L ″)	pressing the $\boxed{\text{ALPHA}}$ key many times.
: Output(4, 7, L)	
: Output(5, 1, ″ R ″)	• After typing the program's name, press $\boxed{\text{ENTER}}$. The
: Output(5, 7, R)	program editor is displayed.
: Output(6, 1, ″ T ″)	
: Output(6, 7, T)	• The commands **Disp** , **Prompt** and **Output** are in the
	PRGM I/O(input/output) menu. Press $\boxed{\text{PRGM}}$ $\boxed{\blacktriangleright}$.
: 0 $\rightarrow$ M	
: A+H/2 $\rightarrow$ X	• The commands **For** and **End** are in the **PRGM CTL**
: For(I, 1, N)	(control) menu. Press $\boxed{\text{PRGM}}$
: M+ Y_1(X) $\rightarrow$ M	
: X+H $\rightarrow$ X	• To get the **Y_1** , press $\boxed{\text{VARS}}$ $\boxed{\blacktriangleright}$ to display the **Y-VARS**
: End	menu, select **1: Function** to display the **Y_n** functions.
: M*H $\rightarrow$ M	
: Output(7, 1, ″ M ″)	• ″ is $\boxed{\text{ALPHA}}$ $\boxed{+}$, : is $\boxed{\text{ALPHA}}$ $\boxed{.}$, $\rightarrow$ is the $\boxed{\text{STO}}$
: Output(7, 7, M)	$\boxed{\blacktriangleright}$ key.
: Output(8, 1, ″ S(2N) ″)	
: Output(8, 7, (T+2*M)/3)	• When done typing the program, Press $\boxed{\text{2nd}}$ $\boxed{\text{QUIT}}$ to
	leave the program editor and return to the home screen.

We can use the **RIEMANN** program to approximate the following integral: $\int_0^2 3x^2 dx$

- Press $\boxed{\text{Y} =}$ and set **$Y_1 = 3X^2$** . Press $\boxed{2^{\text{nd}}}$ $\boxed{\text{QUIT}}$ to go to the home screen.
- Press $\boxed{\text{PRGM}}$ to display the **PRGM EXEC** menu. Select the program's name and press $\boxed{\text{ENTER}}$ to execute the program.
- The program will prompt you for **A**, the left endpoint of the interval, **B**, the right endpoint of the interval, and **N**, the number of partitions. It gives you the Left Riemann Sum **L**, the right Riemann Sum **R**, the midpoint approximation **M**, the Trapezoidal approximation **T**, and Simpson's approximation **S**.
- To change A, B, or N, just press $\boxed{\text{ENTER}}$. The program executes another time and prompts you to enter new values for A, B, and N.

Problem 1

Use the **RIEMANN** program to approximate $\int_0^2 3x^2 dx$. Set **Y₁ = 3X²**, **A = 0, B = 2,** and **N** = 10. Get the

following: $L_{10} = 6.84$, $R_{10} = 9.24$, $T_{10} = 8.04$, $M_{10} = 7.98$, $S_{20} = 8$. Note that to find S_{10}, you take **N = 5**.
To get a better approximation of the integral, you increase the number of partitions **N**.

Fill in the following table to approximate the integral $\int_0^2 3x^2 dx$: (answers correct to 6 decimal places)

N	L_N	R_N	T_N	M_N	S_N
10	6.84	9.24	8.04	7.98	8
20					
40					
80					

Given that $\int_0^2 3x^2 dx = 8$ and using the table above,

Which method gave the best approximations?   _____
Which value of N gave the best approximations?   _____
Which method(s) gave an underestimate of the integral? _____
Which method(s) gave an overestimate of the integral?   _____

The error in the above approximations is | ERROR = Actual value of integral – Approximation |
With **N = 10**, the error in the left Riemann sum is **E$_L$** = 8 - 6.84 = 1.16

 Fill in the table below to find the errors in the above approximations (use 6 decimal places)

N	E_L	E_R	E_T	E_M	E_S
10	1.16	-1.24	-0.04	0.02	0
20					
40					
80					

Which method gave the least errors?   _____
Which value of N gave the least errors?   _____
Note that the Trapezoidal and Midpoint Rules are much more accurate than the endpoint
approximations. The size of the error in the Midpoint Rule is about half the size of the error in the
Trapezoidal Rule. Simpson's Approximations are the most accurate.

Problem 2

In this problem we will use the **RIEMANN** program to approximate the value of $\pi = \int_0^1 \frac{4}{1+x^2}dx$

Use your calculator with $Y_1 = 4/(1+X^2)$ to fill in the following table. (Answers to 6 decimal places)

N	L_N	R_N	T_N	M_N	S_N
4					
8					
16					
32					

Using the table, approximately, how many partitions are needed to approximate π to within 0.0005:
when using the Midpoint Rule? _____
when using the Trapezoidal Rule? _____
when using the Simpson's Rule? _____

Note that your answers might not be the smallest number of partitions that will give you such precision. In problem 3, we learn how to find better estimates of the number of partitions by using the Error Bound Formulas.

Problem 3

In this problem we deal with the **Actual Errors** = Actual value of integral − Approximations, and the **Estimates of Errors** using the Error Bounds given on the first page of this project.

Consider the function $f(x) = \frac{1}{x}$ and the integral $\int_1^4 \frac{1}{x} dx$. (Give answers with 6 decimal places)

A) In this part we find the actual value of the errors when approximating $\int_1^4 \frac{1}{x} dx$.

   (i) Find $M_{10} =$ _____ , $T_{10} =$ _____ , and $S_{10} =$ _____

   (ii) You can evaluate the integral $\int_1^4 \frac{1}{x} dx$ using MATH 9 in your calculator

   $fnInt(1/X, X, 1, 4) =$ _____   or by hand $\int_1^4 \frac{1}{x} dx = \ln 4 =$ _____

   (iii) For $n = 10$, find the actual error $E_M = \int_1^4 \frac{1}{x} dx - M_{10} =$ _____

   the actual error $E_T =$ _____ , and the actual error $E_S =$ _____

B)   It is possible to estimate these Errors without finding the approximations M_{10}, T_{10}, and S_{10}. In this part we find an estimate of the errors using the Error Bounds formulas.

Error Bounds for Midpoint and Trapezoidal Rules:

Suppose that $|f''(x)| \le K_1$ for $a \le x \le b$. Then $|E_M| \le \dfrac{K_1(b-a)^3}{24n^2}$ and $|E_T| \le \dfrac{K_1(b-a)^3}{12n^2}$
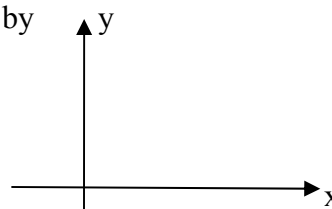
Error Bounds for Simpson's Rules:

Suppose that $|f^{(4)}(x)| \le K_2$ for $a \le x \le b$. Then $|E_S| \le \dfrac{K_2(b-a)^5}{180n^4}$

(i) Find the following derivatives of $f(x) = \dfrac{1}{x}$:

$f'(x) = $ _____ , $f''(x) = \dfrac{2}{x^3}$, $f^{(3)}(x) = $ _____ , $f^{(4)}(x) = $ _____

(ii) To find K_1, sketch the graph of $y = |f''(x)|$ on the interval $[1, 4]$ by pressing $\boxed{Y=}$ $\boxed{MATH}$ $\blacktriangleright$ $\boxed{1}$ 2 / x^3 to get Y_1 = abs(2 / x³)

The *maximum* value of $|f''(x)|$ is $K_1 = $ _____.

Or use the following inequalities: $1 \le x \le 4 \Rightarrow 1 \le x^3 \le 64 \Rightarrow \dfrac{2}{64} \le \dfrac{2}{x^3} \le \dfrac{2}{1}$.

So $|f''(x)| \le 2 = K_1$

(iii) With $n = 10$ partitions and using the above formulas for Error Bounds, find (Show your work)

$|E_M| \le \dfrac{K_1(b-a)^3}{24n^2} = \dfrac{2(4-1)^3}{24(10)^2} = $ _____ , and $|E_T| \le $ _____

(iv) Sketch the graph of $y = |f^{(4)}(x)|$ on the interval $[1, 4]$ to find K_2 an Upper Bound

(or Maximum) of $|f^{(4)}(x)|$, $K_2 = $ _____

and $|E_S| \le $ _____

(v) Are the Actual Errors found in part A) compatible with the Error Bounds in part B)?

C) (i) Use the Error Bound formulas to find the maximum possible error (i.e. an upper bound for the

error) in approximating $\int_1^4 \frac{1}{x} dx$ with $n = 50$ and using the Trapezoidal rule.

$|E_T| \le$ _____

(ii) Use the Error Bound formulas to find the maximum possible error in approximating $\int_1^4 \frac{1}{x} dx$

with $n = 10$ using the Simpson's rule. $|E_S| \le$ _____

(iii) Using your answers to part (i) and (ii), the number of partitions needed to approximate

$\int_1^4 \frac{1}{x} dx$ correct to 2 decimal places is approximately:

$n =$ _____ with the Trapezoidal rule, and $n =$ _____ with the Simpson's rule.

D) Use the Error Bound formulas to find how large do we have to choose n so that the

approximations T_n, M_n, and S_n to the integral $\int_1^4 \frac{1}{x} dx$ are accurate to within 0.00001:

Trapezoidal rule: $|E_T| \le \dfrac{K_1(b-a)^3}{12n^2} < 0.00001$ ➔ $\dfrac{2(4-1)^3}{12n^2} < 0.00001$

➔ $n^2 > \dfrac{2(3)^3}{12(0.00001)}$ ➔ $n >$ _____   ➔ $n =$ _____

Midpoint rule: $n =$ _____ (show work)

Simpson's rule: $n =$ _____ (show work)