Zernike Expansions for Non-Kolmogorov Turbulence

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ABSTRACT

We investigate the expression of non-Kolmogorov turbulence in terms of Zernike polynomials. Increasing the power-law exponent of the three-dimensional phase power spectrum from 2 to 4 results in a higher proportion of wavefront energy being contained in the tilt components. Closed-form expressions are given for the variances of the Zernike coefficients in this range. For exponents greater than 4, a von Karman spectrum is used to numerically compute the variances as a function of exponent for different outer-scale lengths. We find in this range that the Zernike-coefficient variances depend more strongly on outer scale than on exponent, and that longer outer-scale lengths lead to more energy in the tilt terms. The scaling of Zernike-coefficient variances with pupil diameter is an explicit function of the exponent.

Keywords: atmospheric turbulence, adaptive optics.

1. INTRODUCTION

While the Kolmogorov formulation[1] has been widely used to successfully describe atmospheric turbulence, some turbulence conditions exist where experimental data does not support it. For Kolmogorov turbulence, the three-dimensional power spectral density of phase fluctuations has the form

$$\Phi_\varphi(k) = \frac{0.023\ k^{-11/3}}{r_0^{5/3}} \qquad (1)$$

where k is spatial frequency (cy/length) and r_0 is a normalization factor with units of length that gives the correct dimensionality of the power spectrum[2]. We interpret r_0 as the pupil diameter over which the piston-subtracted wavefront variance is equal to 1 rad^2 for the case of Kolmogorov turbulence. We find this definition more convenient for our purposes than the original[3] definition of r_0 in terms of the integral of the modulation transfer function. These two definitions agree to within a few percent.

The exponent for the inverse spatial-frequency dependence has been experimentally observed to be both larger and smaller than the value of 11/3 that derives from the Kolmogorov theory. Exponent values around 5 are encountered in high-altitude (statospheric) stellar-scintillation studies[4,5], while measurements affected by turbulence nearer to the ground[6-8] yield exponents in the range of 3 to 3.65.

It is thus of interest to investigate the behavior of non-Kolmogorov turbulence[9-10] having a range

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of exponents. We will consider a wavefront expansion for the general-exponent case in terms of variances of Zernike coefficients. This will allow us to investigate the behavior of the turbulent wavefront and also to assess our ability to correct for that turbulence by use of an adaptive-optical system.

The generalized form of Eq. (1) for the phase spectrum is

$$\Phi_\varphi(k) = \frac{A_\beta\, k^{-\beta}}{\mathcal{r}_0^{\beta-2}} \tag{2}$$

where $\mathcal{r}_0$ is an analogous quantity to r_0, which reduces to r_0 for the case of β=11/3. The constant A_β has a value such that, for any power law β chosen for the spectrum, the piston-subtracted wavefront variance (which we denote Δ_1) is normalized to 1 rad^2 over a pupil diameter $D = \mathcal{r}_0$. The numerical values of the Zernike-coefficient variances (that we denote as $< | a_j^2 | >$) are then the relative wavefront energies contained in each Zernike term. Because the wavefront variance directly impacts image quality, this normalization provides an equivalent-image-quality basis for comparison of the amounts of different aberrations in the turbulent wavefront as a function of β.

2. ZERNIKE EXPANSION AND RESIDUAL WAVEFRONT ERROR

We use the definitions of Noll[11] for the Zernike polynomials and their Fourier transforms.[12] We perform all calculations with the transform-domain Zernike polynomials

$$Q_j(k,\phi) = \sqrt{n+1}\,\frac{J_{n+1}(2\pi k)}{\pi k}\begin{Bmatrix} (-1)^{n/2}\sqrt{2}\cos m\phi,\ j\ \text{even} \\ (-1)^{n/2}\sqrt{2}\sin m\phi,\ j\ \text{odd} \\ (-1)^{n/2},\ m=0 \end{Bmatrix} \tag{3}$$

where ϕ is the azimuthal angle in the transform domain, j is the mode number, n is the radial degree, and m is the azimuthal frequency.

We are interested in the residual mean-squared wavefront error Δ_J after terms from j = 1 to J are corrected (as with an adaptive-optical system), defined as

$$\Delta_J = \left\langle \varphi^2 \right\rangle - \sum_{j=1}^{J} \left\langle \left| a_j \right|^2 \right\rangle \tag{4}$$

where $< \varphi^2 >$ is the total phase variance of the random wavefront

$$\left\langle \varphi^2 \right\rangle = \sum_{j=1}^{\infty} \left\langle \left| a_j \right|^2 \right\rangle , \tag{5}$$

and the $< | a_j |^2 >$ are the Zernike-coefficient variances. The piston variance $< | a_1 |^2 >$ is infinite, and the phase variance is also infinite if outer-scale effects are neglected. These infinities cancel when the two terms are subtracted. The resulting piston-subtracted wavefront variance Δ_1 is normalized to 1 which determines the value of A_β.

The Zernike-coefficient variances are defined[11] as

$$\left\langle \left| a_j^2 \right| \right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_j^*(k)\, Q_j(k')\, \Phi_\varphi(k/R, k'/R)\, d\bar{k}\, d\bar{k}' \tag{6}$$

where R is the pupil radius, and the general form of the cross-phase spectrum is

$$\Phi_\varphi(k/R, k'/R) = A_\beta \left(\frac{R}{r_0}\right)^{\beta-2} k^{-\beta}\, \delta(k - k') \tag{7}$$

Substituting Eq. (7) into Eq. (6), the expression for the Zernike-coefficient variances becomes

$$\left\langle \left| a_j \right|^2 \right\rangle = A_\beta \left(\frac{R}{r_0}\right)^{\beta-2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_j^*\, Q_j\, k^{-\beta}\, \delta(k - k')\, d\bar{k}\, d\bar{k}' \tag{8}$$

Using Eq. (3) and 2R = D, Eq. (8) can be written (with the change of variables $2\pi k = \kappa$) as

$$\left\langle \left| a_j \right|^2 \right\rangle = 8\, A_\beta \left(\frac{D}{r_0}\right)^{\beta-2} (n+1)\, \pi^{\beta-1} \int_0^{\infty} \kappa^{-(\beta+1)}\, J_{n+1}^2(\kappa)\, d\kappa \quad , \tag{9}$$

which is valid for any m, and any n not equal to zero. We find that the Zernike-coefficient variances have a $(D/r_0)^{\beta-2}$ dependence, and thus the scaling of the Zernike-coefficient variances with pupil diameter is explicitly a function of β.

The integral in Eq. (9) can be expressed in closed form[13] as

$$\int_0^{\infty} \kappa^{-(\beta+1)}\, J_{n+1}^2(\kappa)\, d\kappa = \frac{\Gamma(\beta+1)\, \Gamma\left(\frac{2n+2-\beta}{2}\right)}{2^{\beta+1} \left\{\Gamma\left(\frac{\beta+2}{2}\right)\right\}^2 \Gamma\left(\frac{2n+4+\beta}{2}\right)} \tag{10}$$

for β and n satisfying $(2n + 2) > \beta > -1$. In that range, the Zernike-coefficient variances are expressed as

$$\left\langle \left| a_j \right|^2 \right\rangle = 8\, A_\beta \left(\frac{D}{r_0}\right)^{\beta-2} (n+1)\, \pi^{\beta-1} \frac{\Gamma(\beta+1)\, \Gamma\left(\frac{2n+2-\beta}{2}\right)}{2^{\beta+1} \left\{\Gamma\left(\frac{\beta+2}{2}\right)\right\}^2 \Gamma\left(\frac{2n+4+\beta}{2}\right)} \tag{11}$$

For the piston case (j=1, n=0), the integral of Eq. (10) diverges for $\beta > 2$, which includes all cases of interest to us. Using the normalization condition $\Delta_1 = 1$, we develop an expression for A_β, which when combined with Eq. (11) yields

$$\left\langle \left| a_j \right|^2 \right\rangle = \left(\frac{D}{r_0} \right)^{\beta-2} \frac{(n+1)}{\pi} \frac{\Gamma\left(\frac{2n+2-\beta}{2} \right) \Gamma\left(\frac{\beta+4}{2} \right) \Gamma\left(\frac{\beta}{2} \right) \sin\left(\pi \frac{\beta-2}{2} \right)}{\Gamma\left(\frac{2n+4+\beta}{2} \right)} , \tag{12}$$

valid for $n \geq 1$ and $2 < \beta < 4$. The Zernike-coefficient variances are equal for any j having the same value of n. We denote $< | a_{2\text{-}3} |^2 > \equiv < | a_2 |^2 > = < | a_3 |^2 >$; $< | a_{4\text{-}6} |^2 > \equiv < | a_4 |^2 > = < | a_5 |^2 > = < | a_6 |^2 >$; $< | a_{7\text{-}10} |^2 > \equiv < | a_7 |^2 > = < | a_8 |^2 > = < | a_9 |^2 > = < | a_{10} |^2 >$.

3. ZERNIKE-VARIANCE COEFFICIENTS FOR $2 < \beta < 4$

We evaluate Eq. (12) with $2 \leq j \leq 11$ and $2 < \beta < 4$, for the case of $D = r_0$. The Zernike-coefficient variances scale with pupil diameter as $(D/r_0)^{\beta-2}$, consistent with Eq. (13). Figure 1 shows the two tilt terms $< | a_{2\text{-}3} |^2 >$ as a function of β, while Fig. 2 shows three curves for the next higher-order terms: $< | a_{4\text{-}6} |^2 >$, $< | a_{7\text{-}10} |^2 >$, and $< | a_{11} |^2 >$, as functions of β.

All of the variances obey the normalization condition $\Delta_1 = 1$ for any given value of β, and thus the sum over all the coefficients for $j \geq 2$ must be unity. All of the coefficients approach zero as β approaches 2. As β increases toward 4, the two tilt coefficients of Fig. (1) increase toward 0.5. Although Eq. (12) has a singularity at $\beta = 4$, the interpretation is that an increasing amount of the energy in the piston-subtracted wavefront are contained in the tilt terms as β approaches 4. Figure 2 shows that all of the coefficients for $j > 3$ approach zero as β approaches 4, as required by the normalization, because an increasing amount of energy is contained in the tilt terms. Figures 1 and 2 (or Eq. (12)) can be used to evaluate the residual mean-squared error Δ_J for any $4 > \beta > 2$.

4. ZERNIKE-VARIANCE COEFFICIENTS FOR $\beta > 4$

Equation (12) for the Zernike-coefficient variances is not valid in the range $\beta > 4$, so we use numerical techniques to investigate the variances as a function of β for different outer-scale lengths. Combining Eq. (9) and the expression for A_β yields the Zernike-coefficient variances in terms of spatial-frequency integrals

$$\left\langle \left| a_j \right|^2 \right\rangle = 4\,(n+1) \left(\frac{D}{r_0} \right)^{\beta-2} \frac{\int_0^\infty \kappa^{-(\beta+1)} J_{n+1}^2(\kappa)\, d\kappa}{\int_0^\infty \kappa^{-(\beta-1)} \left\{ 1 - \frac{4\, J_1^2(\kappa)}{\kappa^2} \right\} d\kappa} \tag{13}$$

The integrals involved in Eq. (13) are divergent for small κ, which requires the use of an explicit outer scale, L_0, in the calculation. We modify Eq. (2) to account for outer-scale effects, resulting in the von Karman[1] form of the spectrum

$$\Phi_\varphi(\kappa) = \frac{A_\beta\left(\kappa^2 + \left(\pi\frac{D}{L_0}\right)^2\right)^{-\beta/2}}{r_0^{\beta-2}} \quad . \tag{14}$$

If Eq. (14) is used in a development similar to that which produced Eq. (13), the analogous expression is

$$\left\langle \left| a_j \right|^2 \right\rangle = 4\,(n+1)\left(\frac{D}{r_0}\right)^{\beta-2} \frac{\displaystyle\int_0^\infty \left(\kappa^2 + \left(\pi\frac{D}{L_0}\right)^2\right)^{-\beta/2} \kappa^{-1}\, J_{n+1}^2(\kappa)\, d\kappa}{\displaystyle\int_0^\infty \left(\kappa^2 + \left(\pi\frac{D}{L_0}\right)^2\right)^{-\beta/2} \kappa \left\{1 - \frac{4\,J_1^2(\kappa)}{\kappa^2}\right\} d\kappa} \quad . \tag{15}$$

We choose the following outer scales for the computation of the $<|a_j|^2>$ from Eq. (15): $L_0 = 10$ D, $L_0 = 100$ D, and $L_0 = 1000$ D. Figure 3 shows $<|a_{2\text{-}3}|^2>$, and Fig. 4 shows $<|a_{4\text{-}6}|^2>$, as functions of β and L_0/D, for the case of $D = r_0$.

We note from Figs. 3 and 4 that the coefficients have a stronger dependence on L_0 than on β. For any given β, a decreasing outer scale takes energy away from the tilt terms. The normalization $\Delta_1 = 1$ then requires that the higher-order terms gain energy as L_0 decreases. Conversely, the tilt terms dominate the expansion for for large L_0. Also, increased values of β lead to larger amounts of tilt.

6. CONCLUSIONS

We investigated the behavior of the Zernike-coefficient variances of a turbulent wavefront having a general exponent β. Our normalization of unity piston-subtracted wavefront variance provides an equal-image-quality comparison of the relative energy content of the various Zernike components in the turbulent wavefront as β varies. The scaling of the Zernike-coefficient variances with pupil diameter is proportional to $(D/r_0)^{\beta-2}$.

For $2 < \beta < 4$, the amount of energy in the tilt terms increases nearly linearly with β, with the limit that the tilt terms contain all of the energy at $\beta \to 4$. The higher-order terms increase in importance for smaller values of β. The case of $\beta = 2$ corresponds to all of the variance of the turbulent wavefront being contained in the piston term.

For $\beta > 4$, an outer scale L_0 must be assumed in the calculation. The tilt terms dominate the expansion with increasing outer scale or increasing β. The higher-order terms correspondingly decrease with increasing L_0 and β. Consistent with Ref. 9, the Zernike-coefficient variances show a stronger dependence on L_0 than on β, for $\beta > 4$.

REFERENCES

1. V. I. Tatarski, *Wave Propagation in a Turbulent Medium* (McGraw Hill, New York, 1961).
2. D. L. Fried, JOSA **55**, 1427-1435 (1965).
3. D. L. Fried, JOSA **56**, 1372-1379 (1966).
4. F. Dalaudier, A. S. Gurvich, V. Kan, and C. Sidi," Adv. Space Res. **14**, 61-64 (1994).
5. M. S. Belen'kii and A. S. Gurvich, Proc. SPIE **2471**, J. C. Dainty, ed. (1995).
6. R. G. Buser, JOSA **61**, 488-491 (1971).
7. M. Bester, W. Danchi, C. Degiacomi, L. Greenhill, and C. Townes, Astrophys. J. **392**, 357-374 (1992).
8. D. Dayton, B. Pierson, B. Spielbusch, and J. Gonglewski, Opt. Lett. **17**, 1737-1739 (1992).
9. A. I. Kon, Waves in Random Media **4**, 297-306 (1994).
10. B. Stribling, B. Welsh, and M. Roggemann, Proc. SPIE **2471**, J. C. Dainty, ed. (1995).
11. R. J. Noll, JOSA **66**, 207-211 (1976).
12. J. Y. Wang and J. K. Markey, JOSA **68**, 78-87 (1978).
13. I. S. Gradshteyn & I. M Ryzhik, *Table of Intergrals, Series, and Products (Fifth Edition)*, (Academic Press, New York, 1994), formula 6.574.2.

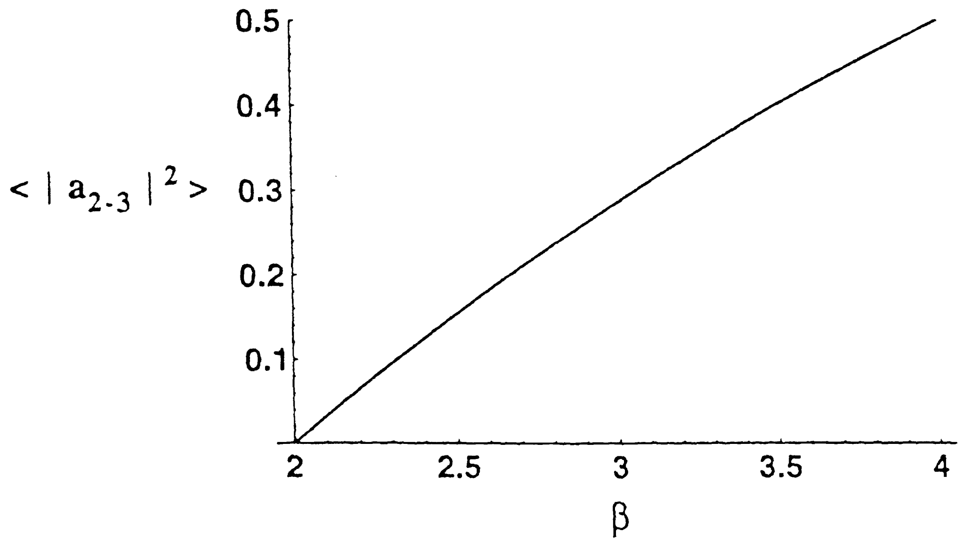

Fig. 1. Zernike-tilt-coefficient variances (for the case of $D = r_0$) $< | a_{2\text{-}3} |^2 >$ as a function of β, for $2 < \beta < 4$.

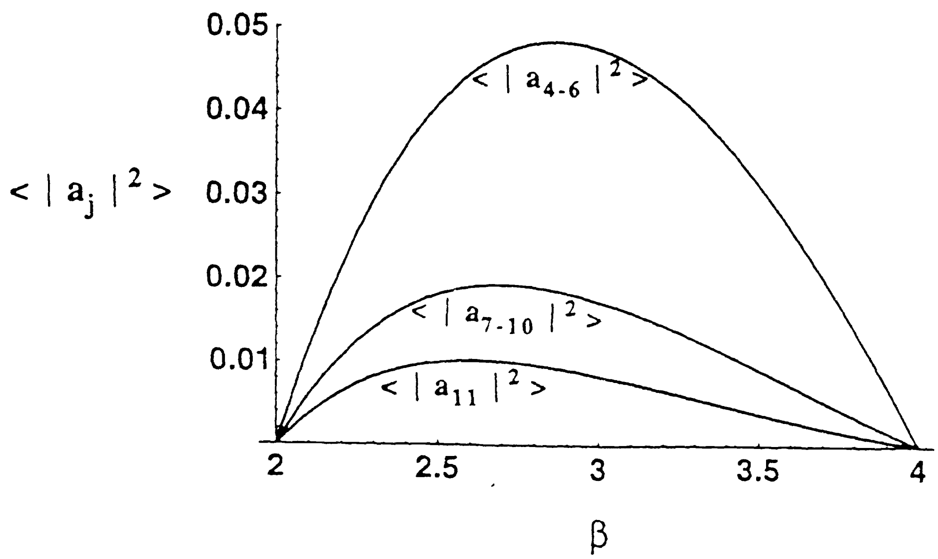

Fig. 2. Higher-order Zernike-coefficient variances (for the case of $D = r_0$) $< | a_{4\text{-}6} |^2 >$, $< | a_{7\text{-}10} |^2 >$, and $< | a_{11} |^2 >$, as functions of β, for $2 < \beta < 4$.

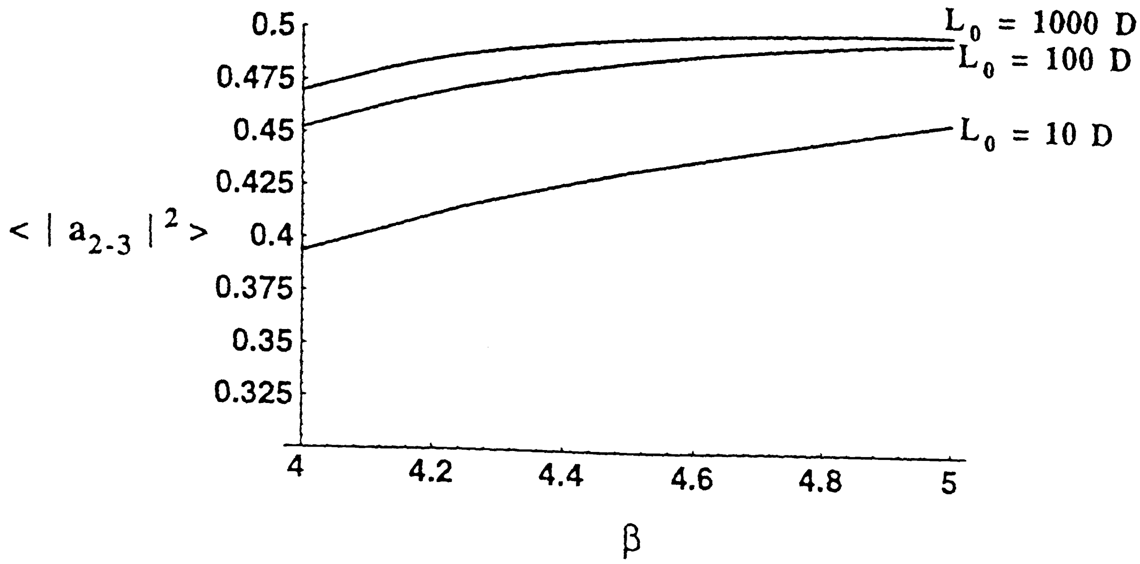

Fig. 3. Zernike-tilt-coefficient variances (for the case of $D = r_0$) $< | a_{2\text{-}3} |^2 >$ as a function of β, for $\beta > 4$, with L_0/D = 10, 100, and 1000.

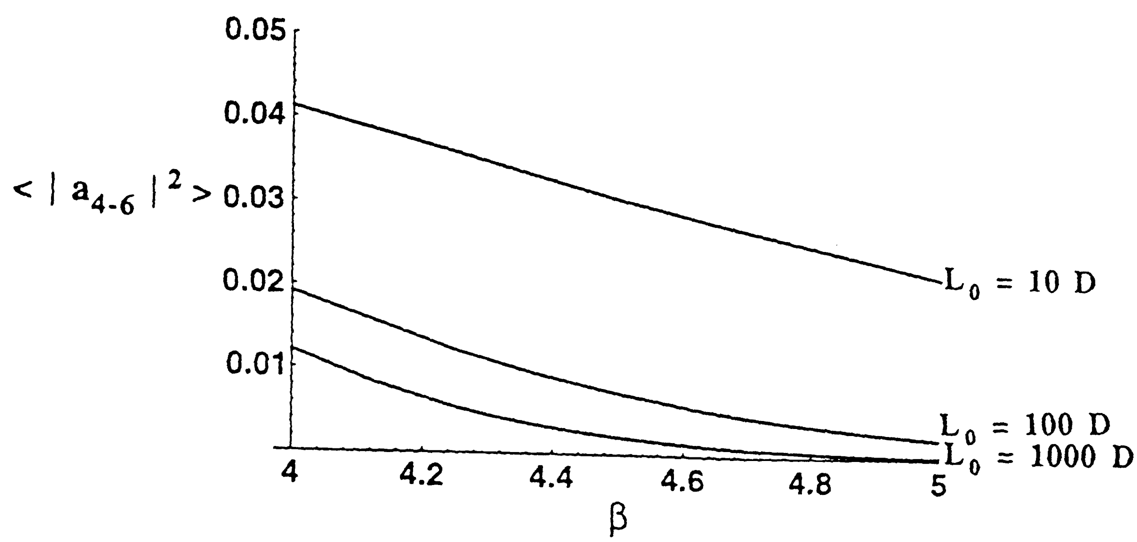

Fig. 4. Zernike-coefficient variances (for the case of $D = r_0$) $< | a_{4\text{-}6} |^2 >$ as a function of β, for $\beta > 4$, with L_0/D = 10, 100, and 1000.