Modulation transfer functions of fractal layers

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Abstract

Transmission characteristics of layers that contain fractal aggregates are investigated. In particular, the modulation transfer functions of these layers are studied in comparison with the case of uniformly dispersed particles. The effects of aggregation type (fractal dimension) and aggregate size are discussed.

Keywords: transfer functions, fractal layer, remote sensing

1 Introduction

Light propagation through a random medium is a problem of great practical interest. In optical communications, light scattering and absorption attributable to particulates along the optical path degrade the system's performance which is usually described in terms of optical transfer functions. Significant difference exist between the case of a uniform dispersion of particulates between transmitter and receiver and the case when the same particulates are agglomerated to form aggregates of various sizes. Aggregates of atmospheric aerosols, dust particles, or smokes are successfully described as fractal clusters. Depending on the aggregates size and type, the optical properties of fractal clusters can differ significantly from those of the primary particles and those of compact agglomerates. Accordingly, the photon propagation is affected by the degree of aggregation that occurs along the optical path. This is of interest not only from the viewpoint of a structural description but also for the evaluation of observable quantities in a long-range propagation [1,2]. Knowledge about the effects of aggregation on the optical transfer functions can lead to improvements of classical techniques in active and passive remote sensing.

2 Fractal random layers

Fractal layers have been successfully used to describe various wave propagation and scattering phenomena. Band-limited fractal screens have been used to model the atmospheric refractivity fluctuations [3] and fractal surface models have been considered in explaining diffraction effects produced by thin layers [4].

This paper focuses on layers containing agglomerates of particles that can be described as fractal aggregates. Various models characterize the complex process of aggregation [5] but their results are most pregnantly represented in terms of the fractal dimension D. The fractal dimension is determined from the relation $N \propto (R/a)^D$, where N is the number of constituent particles of size a and R is the aggregate radius of gyration. Depending on the type of aggregation, the fractal dimension D of a cluster takes a value between 1 and 3 describing the relative compactness of the aggregate.

The practical problem of the structural description of a random phase screen made of fractal aggregates should answer questions referring to both the value of the fractal dimension D and the extension of the scaling range R. In addition, the electromagnetic wave propagation through such a screen depends on the number density ρ of individual scattering centers.

2.1 Light scattering and absorption by fractal layers

In the regime of weak scattering, when multiple scattering is negligible, the attenuation length of a layer containing a uniform dispersion of particles is $l_0 = (\rho\sigma_0^s)^{-1}$, where σ_0^s is the scattering cross-section of an individual particle. For sparse distribution of aggregates, a similar relation describes the attenuation length of a fractal layer: $l_{cl} = (\rho_{cl}\sigma_{cl}^s)^{-1}$. In this case, ρ_{cl}^s is the cluster number density and σ_{cl}^s is the scattering cross-section of the whole cluster.

We start with the scattering cross-section of a cluster considered to be an independent scatterer. For this large scatterer, the differential scattering cross-section into the solid angle element $d\Omega$ at the angle θ is given by

$$\frac{d\sigma_{cl}^s}{d\Omega} = N^2 \frac{d\sigma_0^s}{d\Omega} S(\theta), \tag{1}$$

where $d\sigma_0^s/d\Omega$ is the differential scattering cross-section at the angle θ for an individual particle, while $S(\theta)$ is the structure function that describes the long-range fractal-like correlation between the components of a cluster. Equation (1) disregards the possible contributions from the inside-cluster scattering. However, intensive studies [6,7] have revealed that the multiple scattering inside the fractal cluster renormalizes the mean scattering field but does not affect the angular profile of scattering pattern, i.e., the scaling form of $S(\theta)$.

Introducing the modulus of the scattering wave vector as $q = 2k\sin(\theta/2)$ with $k = 2\pi/\lambda$, the total scattering cross-section of a cluster is obtained by integrating Eq. (1) over all the scattering angles:

$$\sigma_{cl}^{s} \propto N^2 k^{-2} \int_0^{2k} \frac{d\sigma_0^s}{d\Omega} S(q) q dq. \tag{2}$$

The scattering mean free path l_{cl} can be directly measured, for instance, in a ballistic transmission experiment. In this regime, the structure function $S(\theta)$ can also be measured directly.

For an average cluster size R larger than the wavelength, one expects a proper description of scattering phenomena in terms of the geometrical optics. For opaque clusters with $D > 2$, many particles are practically shadowed even if their size is small $a < \lambda$. However, $N^{\frac{2}{D}}$ particles are active in the scattering process and, being spatially correlated, scatter the light in an amplitude basis. Therefore, in the definitions of scattering cross-section of Eqs. (1) and (2), the factor of N^2 must be replaced by $N^{\frac{2}{D}}$. As expected, this approach leads to the geometrical optics limit of $\sigma_{cl}^{s} \propto R^2$ for $D > 2$ and $\sigma_{cl}^{s} \propto N\sigma_0^s$ for $D < 2$.

By neglecting the inside-cluster multiple interaction effects for $D < 2$, the absorption cross-section σ_{cl}^{a} of a cluster of N particles can be estimated to be N times the absorption cross-section σ_o^a of the constituent particle. For large clusters in the limit of geometrical optics, the assumption of aggregation in compact spheres of radius R leads to $\sigma_{cl}^{a} \sim R^2 = N^{\frac{2}{3}}$ and, therefore, strongly underestimates the absorption in comparison to the case of a fractal aggregate with the same number of particles $\sigma_{cl}^{a} \sim N$. When $D > 2$ and for large clusters, the absorption approaches the geometrical optics limit of $\sigma_{cl}^{a} \sim N^{\frac{2}{D}}$.

2.2 Structure functions

To evaluate the scattering coefficients, we need appropriate forms for the structure function of clusters and the cross-section of individual particles. A simple form for the fractal structure function is the Fisher-Burford formula [8]

$$S(q,N) = S(0) \left[\frac{3D}{3D + 2q^2 N^{\frac{2}{D}}} \right]^{\frac{D}{2}}. \tag{3}$$

Note that the more familiar Heyney-Greenstein scattering form factor [9] for large particles $H(\mu, g) = (1 - \overline{\mu}^2)(1 + \overline{\mu}^2 - 2\overline{\mu}\mu)^{-3/2}$ given in terms of $\mu = \cos\theta$ and the asymmetry parameter $\overline{\mu} = \langle \cos\theta \rangle$ is the particular case of $S(q)$ in Eq. (3) for $D = 3$.

Real systems are polydisperse and a probability distribution of cluster sizes $p(N)$ must be considered. In this case an average structure function

$$S(q) = \frac{\int_0^\infty p(N) N^2 S(q,N) dN}{\int_0^\infty p(N) N^2 dN} \tag{4}$$

is to be considered.

As for $d\sigma_0^s / d\Omega$, different scattering approximations may be used, depending on the size a of particles as compared with the wavelength λ.

2.3 Optical depth of fractal layers

To assess the strength of the wave attenuation when it propagates through a layer of thickness L, one must evaluate the optical depth $\tau_{cl} = \rho_{cl}(\sigma^s_{cl} + \sigma^a_{cl})L$.

By using Eqs. (2) and (3), the scattering cross-section can be written as

$$\sigma^s_{cl} \propto \sigma^s_0 (ka)^{-2} N^{1-\frac{2}{D}} \left\{ \left[1 + \frac{8}{3D}(ka)^2 N^{\frac{2}{D}}\right]^{\frac{2-D}{2}} - 1 \right\} \tag{5}$$

and has, for large clusters, different behaviors depending on the value of the fractal dimension

$$\sigma^s_{cl} \propto \begin{cases} \sigma^s_0 (ka)^{-D} N, & D < 2 \\ \sigma^s_0 (ka)^{-2} N^{2-\frac{2}{D}}, & D > 2. \end{cases} \tag{6}$$

This represents a scattering cross-section per particle which is independent of the cluster size when $D < 2$ but increases with N when $D > 2$.

For the evaluation of optical transfer function, it is of interest to estimate how, for a given number density ρ of particles and a given pathlength L, the optical depth τ_{cl} is modified from the value τ_0 corresponding to the case of a uniform dispersion of particles. Using Eqs. (1) and (5), it can be shown that

$$\tau_{cl} = \tau_0 \frac{\sigma^s_0 (ka)^{-2} N^{-\frac{2}{D}} \left\{ \left[1 + \frac{8}{3D}(ka)^2 N^{\frac{2}{D}}\right]^{\frac{2-D}{2}} - 1 \right\} + \sigma^a_0}{\sigma^s_0 + \sigma^a_0} \tag{7}$$

3 Modulation Transfer Function

The radiative transfer equation [9] can be solved in the case of mostly forward scattering when many scattering events are considered. In the so-called small angle approximation (SAA), a closed solution can be found for a Gaussian type of scattering function

$$G(q) = \frac{k^2 \alpha^2}{\pi} \exp\left(-\alpha^2 q^2\right). \tag{8}$$

The parameter α relates to the particle size and determines the forward peaking of the scattering function. In SAA, photons are propagating essentially parallel to the beam axis, the full details of the scattering function are irrelevant, and only the average scattering angle $< q^2 >= 2k^2 < 1 - \cos\theta >$ is required

$$< q^2 >= k^{-2} \int_0^{2k} G(q) q^3 dq = \alpha^{-2}. \tag{9}$$

The requirements of this approximation are usually fulfilled for atmospheric propagation. However, the validity of SAA becomes questionable for very large scattering particles or for particles with appreciable backscattering.

The modulation transfer function in SAA approximation for a scattering function such as the one in Eq. (10) is [9]

$$M(f) = \exp\left[-\tau + \frac{\alpha\sigma_0^s L}{2\sqrt{\pi} f} erf\left(\frac{\pi f}{\alpha}\right)\right]. \tag{10}$$

A series expansion of Eq. (10) results in

$$M(f) = \begin{cases} \exp\left(-\sigma_0^a L - \sigma_0^s L \frac{\pi^2 f^2}{3\alpha^2}\right), & f < F_0 \\ \exp\left(-\tau_0\right), & f > F_0 \end{cases} \tag{11}$$

where F_0 is a cutoff frequency equal to the ratio between particle size and wavelength.

4 MTF of fractal layers

In the case of aggregated medium, the values of σ_0^a, σ_0^s, and τ_0 are replaced with the corresponding values given in Eqs. (2) and (7). An equivalent average cosine of the scattering angle is found by evaluating the integral of Eq. (9) for the structure functions given in Eq. (3) or (4) corresponding to a monosized and, respectively, polydisperse collection of fractal aggregates. We can further attribute

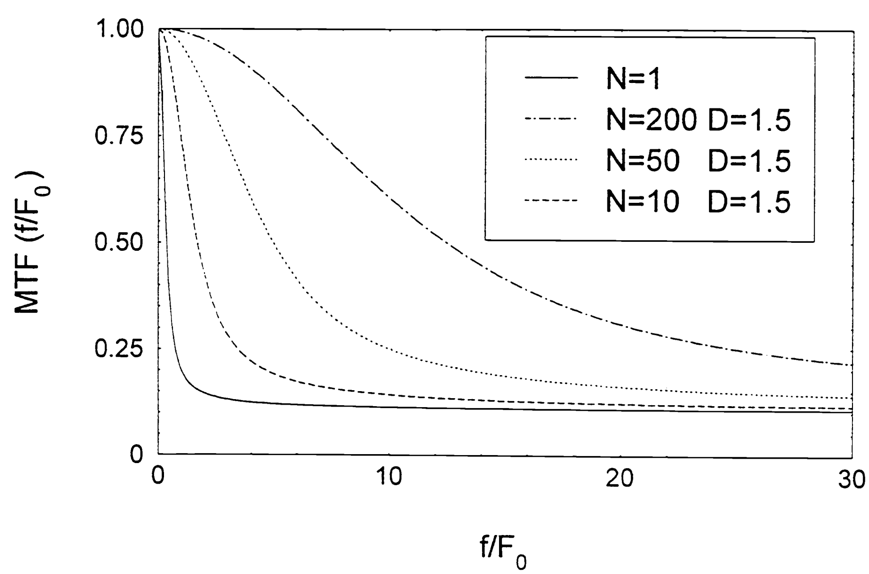

Figure 1: Comparison between MTF curves corresponding to fractal layers with different aggregate size and the same fractal dimension D=1.5.

this value to an equivalent particle with a corresponding α_{cl} given by

$$\alpha_{cl}^{-2} = k \int_0^{2k} S(q) q^3 dq \tag{12}$$

and use the closed form of Eq. (10) (or the asymptotic form of Eq. (11)) to calculate the MTF corresponding to the fractal layer

$$M_{cl}(f) = \exp\left[-\tau_{cl} + \frac{\alpha_{cl}\sigma_{cl}^{s}L}{2\sqrt{\pi f}}erf\left(\frac{\pi f}{\alpha_{cl}}\right)\right]. \tag{13}$$

In this case the MTF cutoff frequency will be

$$F_{cl} = \frac{aN^{\frac{1}{D}}}{\lambda} = F_0 N^{\frac{1}{D}}. \tag{14}$$

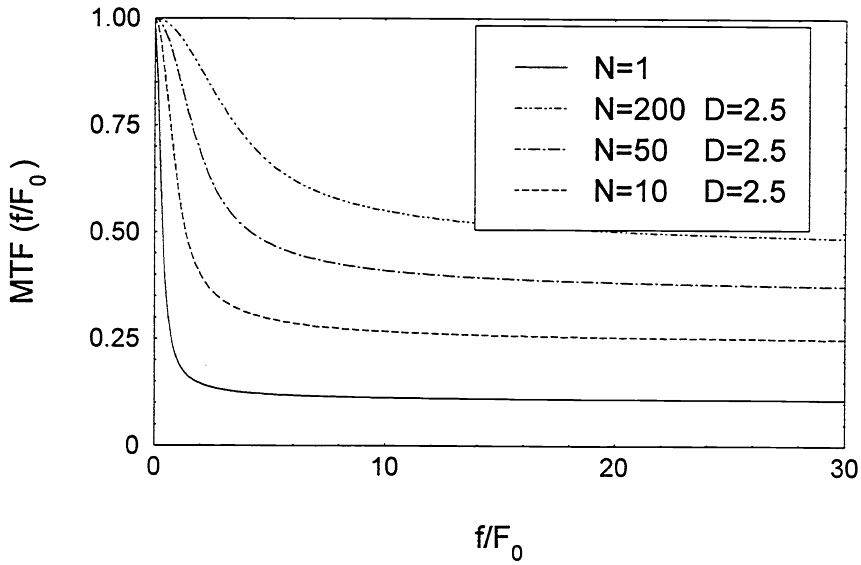

Figure 2: Comparison between MTF curves corresponding to fractal layers with different aggregate size and the same fractal dimension D=2.5.

We present in Fig. 1 results of MTF evaluation based on Eq. (13) for the case of a fractal layer with fractal dimension $D = 2.5$ and various extensions of the scaling range. Also shown is the MTF corresponding to the case of a uniform dispersion of particles. Note that, for all the four cases shown in Fig. 1, the total number of particles in the scattering volume is the same. The differences visible in Fig. 1 are all induced by the presence and extent of the aggregation. A similar evaluation is presented in Fig. 2 for the case of a fractal dimension $D = 1.5$.

In both cases, as the aggregation process evolves, the cutoff frequencies are increased as expected from Eq. (14). However, for transparent clusters when $D < 2$, the MTF curves tend to saturate at the same level and, according to Eq. 11, this means a similar optical depth for all of the aggregation stages. Indeed, this can be obtained in the limit of large clusters from Eq. (7). A similar conclusion can be drawn in the geometrical optics limit when $\sigma_{cl}^{s} \propto N\sigma_0^{s}$. In the other case, when the clusters are optically opaque $D > 2$, the optical depth decreases when increases N and therefore the MTF curves saturate at higher levels.

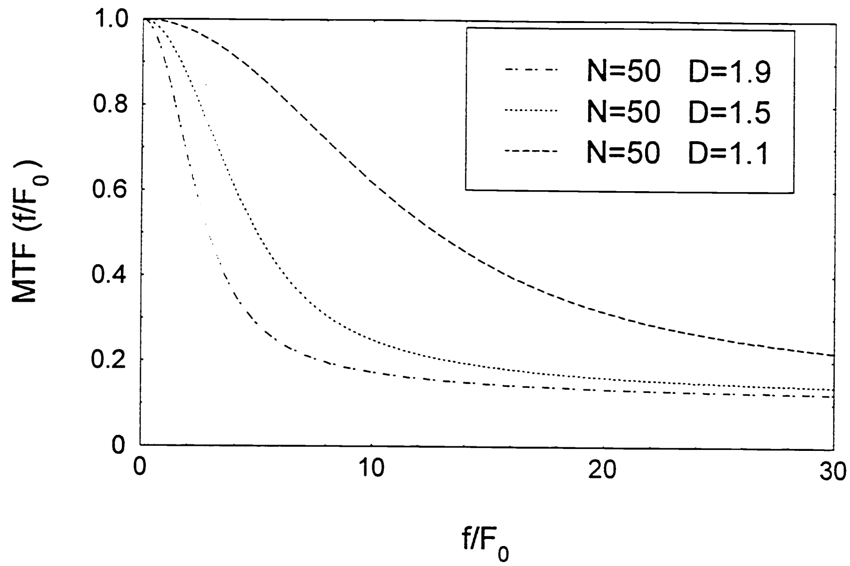

Figure 3: Comparison between MTF curves corresponding to fractal layers with the same aggregate size and different fractal dimensions.

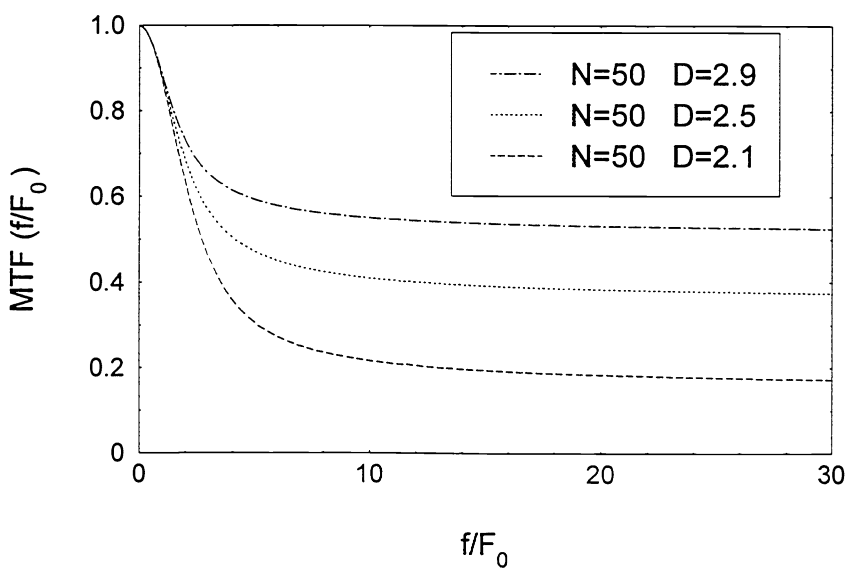

Figure 4: Comparison between MTF curves corresponding to fractal layers with the same aggregate size and different fractal dimensions

This behavior occurs because, when $D > 2$, according to Eq. (6) the scattering cross-section per particle increases with N at a lower rate than the decrease of the cluster's number density. As for the cutoff frequency, this increases in both cases for larger clusters. However, stronger dependence is found for transparent clusters with $D < 2$ because their overall size, for a given number of monomers, is larger than the size corresponding to a cluster with the same number of constituents but with $D > 2$.

The growth process of structures made of similar particles is governed by the potential energy of interaction between particles and the field distribution of external influences. As a result of this complex interaction, different types of aggregation can occur and lead to the formation of structures with different fractal dimensions.

The influence of aggregation type on MTF behavior is exemplified in Figs. 3 and 4. As can be seen, different characteristics are found again for the cases with $D < 2$ and, respectively, $D > 2$. For transparent aggregates, as the fractal dimension decreases, the spatial frequency cutoff increases because, for a constant number of particles, smaller D determines a larger cluster. In the case of opaque clusters, there is not such a large size difference between clusters with different fractal dimension. Accordingly, an almost constant cutoff frequency is evident. However, the number density of clusters is strongly reduced for large D leading to a smaller optical depth and, accordingly, a higher saturation level the MTF curve.

5 Conclusions

When aggregation occurs in a scattering medium, the result is a distribution of clusters and the propagation of light can be thought as being developed through a medium with a rescaled number density ρ_{cl}, a modified optical depth τ_{cl}, and with correspondingly increased scattering σ_{cl}^s and absorption σ_{cl}^a cross-sections. The modulation transfer function of such a layer is directly dependent on the conditions of aggregation.

Distinction should be made between clusters with fractal dimensions smaller than, and greater than, 2. In the case of $D < 2$, the clusters are partially transparent and all the particles belonging to a greater cluster are subject to single-scattering interactions. Therefore, the optical depth of the scattering layer is almost not affected by aggregation. However, because photons are practically scattered by larger particles determines an increase of the spatial frequencies that can be transferred through the layer.

When $D > 2$, the cluster is said to be optically opaque, some of its constituents being shadowed and incapable of single-scattering interactions. In this case, the more particles are shadowed the lower the optical depth becomes and better information can be transfered through the scattering layer. Also, because photons are scattered again by a larger particle, the cutoff frequency increases as the aggregation process evolves. However, this increase is at a lower rate than in the case of transparent clusters.

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