Analysis of Oversampling Requirements in Infrared Scene Projectors

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ABSTRACT

This paper deals with quantification of a two-dimensional (2-D) sampling process by pixel array. The idea is based on transformation of the Wigner-Seitz cell, which defines the sampling lattice in the spatial domain, into a "bandwidth cell" in the spatial frequency domain. The area of the bandwidth cell is a quantitative measure of the sampling process. On this basis a description of the oversampling process is developed. We compare different configurations of the sampling pixel array.

Keywords : pixel array, two dimensional sampling, modulation transfer function, oversampling.

1. INTRODUCTION

One of the most important influence parameters in imaging systems is the spatial sampling rate of the imaging sensor. Recently these have been enormous improvements in developing sensors and scene projectors for the infrared (IR) Focal Plane Arrays (FPA). However the number of pixels in the two dimensional (2-D) FPA is lower than the number of pixels in the display. The number of pixels in the FPA is much lower than the number of pixels in the visible sensor (CCD). Therefore the phenomena of aliasing is much more noticeable in the IR than in the visible. The spatial sampling rate is the most important resolution limitation of these sensors. One of the solutions that had been suggested to this problem is to perform oversampling with the projector pixel array. However this solution is expensive and very difficult to implement. We developed a procedure to know how much oversampling is needed and what are the improvements that we expect to obtain in the image quality as result of oversampling.

In this paper the oversampling process will be quantified in terms of image quality. The bandwidth area around a point in the spatial frequency domain is identified, defining a quantitative measure for the sampling process. Oversampling simulation for different lattice configurations of sampling grids is presented, including the rectangular lattice and the hexagonal lattice.

2. AVERAGE SAMPLING MTF

In the case of sampling by two-dimensional array of finite-sized pixels there are two distinct MTF contributions involved : one for the sampling process associated with the finite spacing between samples, and one for the spatial-averaging process associated with the finite size of the pixels. We assume that these two MTFs multiply to yield an aggregate MTF for the sampling - and - averaging process. The multiplication of MTFs is dependent upon assumptions of linearity and shift invariance. The definition [1] of the sampling MTF in terms of an average over all possible positions of the scene with respect to the sampling locations essentially defines a shift-invariant sampling MTF.

The MTF contribution of finite-sized pixels is already well known. For a one-dimensional rectangular pixel, the pixel MTF is given by the sinc - function formula

$$\mathrm{MTF_{pixel}} = \mathrm{sinc}(\xi\ p) \tag{1}$$

where $\mathrm{sinc}(x) = (\sin \pi x)/(\pi x)$, ξ is the image spatial frequency, and p is the pixel size. The square pixel is the most common shape for imaging-array applications, although other shapes are possible such as circular, hexagonal [2] or tapered [3] . As shown in refs. 2 and 3, the pixel MTF is in general two dimensional.

The MTF in Eq. 1 does not take into account the spatial distance between the samples or the sampling rate. The exact mathematical process for driving the MTF in terms of the sampling rate is given elsewhere [4,5,6]. The derivation of the sampling MTF is based on statistical treatment of the intensity sampled by the array of pixels. The final result is given as the sinc-function,

$$MTF_{samp} = sinc(\xi \cdot \Delta) \tag{2}$$

where Δ is the distance between the samples. The overall MTF is given by multiplication of these two functions,

$$MTF_{total} = MTF_{pixel} \cdot MTF_{samp} = sinc(\xi \cdot p) \cdot sinc(\xi \cdot \Delta) \tag{3}$$

Equation 3 shows that two parameters determines the quality of the sampling image, the pixel size p , and the distance between the pixel in the array, which is the pixel pitch - Δ . Decreasing both of these parameters will improve the total MTF where in the limit when p and Δ approach to zero, the MTF approaches unity and the sampled signal approach to a continuous signal.

A similar statistical analysis can be applied in the case of 2-D array. The analytical derivation for this case is given in details in Ref. 2. The two parameters for the 1-D case p and Δ replaced by two other parameters which suitable for the 2-D case. Instead of p , the normalized pixel function $p(x)$ is used and it defines with respect to a pixel centered at the origin of the lattice:

$$p(\boldsymbol{x}) = \begin{cases} 1/A_p & \text{for } \boldsymbol{x} \text{ inside pixel,} \\ 0 & \text{for } \boldsymbol{x} \text{ outside.} \end{cases} \tag{4}$$

where **x** is defined as a 2-D position vector.

Instead of using the distance Δ between samples as in the case of 1-D sampling, we use a different definition which is taken from crystallography , [7] the Wigner-Seitz cell. "A Wigner-Seitz cell about a lattice point is the region of space that is closer to that point than to any other lattice point" [7]. The normalized Wigner-Seitz function is defined with respect to the lattice point at the origin:

$$w(\boldsymbol{x}) = \begin{cases} 1/A_w & \text{for } \boldsymbol{x} \text{ inside Wigner - Seitz cell,} \\ 0 & \text{for } \boldsymbol{x} \text{ outside.} \end{cases} \tag{5}$$

where A_W is defined as the Wigner-Seitz cell area.

In Figs 1a and 1b the Wigner-Seitz cell is shown for the rectangular and the hexagonal lattice respectively.

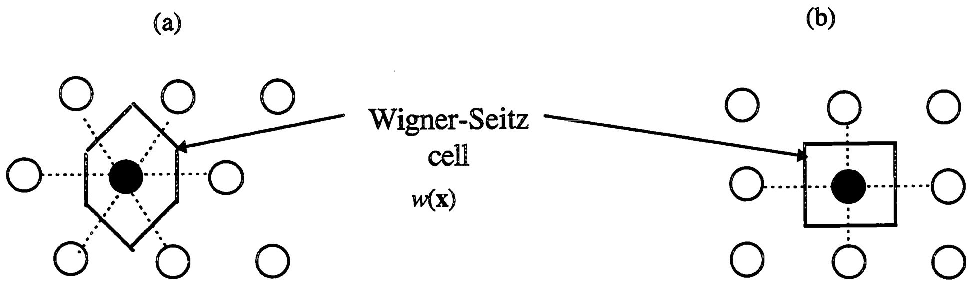


Fig. 1 (a) The Wigner-Seitz cell for a hexagonal lattice (b) The Wigner-Seitz cell for a rectangular lattice .

The six sides of the cell bisect the lines joining the central points (the black point) to its six nearest neighboring points (shown as dashed lines). In two dimensions the Wigner-Seitz cell is always a hexagon (Fig. 1a) unless the lattice is rectangular (Fig. 1b) [7] .

The final result in the spatial frequency domain for the MTF in 2-D is given by,

$$\mathrm{MTF} = P(\mathbf{f})W(\mathbf{f}) . \tag{6}$$

where **f** is a 2-D spatial frequency vector, $P(\mathbf{f})$ and $W(\mathbf{f})$ are the Fourier transforms of $p(\mathbf{x})$ $w(\mathbf{x})$, respectively. The MTF depends not only on the pixel shape, but also on the lattice structure. As for the 1-D case, we find that as the pixel footprint decreases and the distance between samples decreases, $P(\mathbf{f})$ and $W(\mathbf{f})$ increases respectively. From Fig. 1 we find that the area of w(x) in the hexagonal case is smaller than the area of w(x) in the rectangular case by factor of $\sqrt{3}/2$ for the same distance between samples. The meaning of this result is that the MTF of the rectangular lattice is poorer than the MTF of the hexagonal lattice. The equivalent result has been derived in Ref. 2 , where it has been shown that hexagonal pixel shapes are slightly better than rectangular ones in terms of MTF.

3. DEFINITION FOR BANDWIDTH CELL IN THE SPATIAL FREQUENCY DOMAIN

In this section a new measure in the spatial frequency domain for quantifying the sampling process is suggested. The idea is to transform the "Wigner-Seitz" cell in the spatial domain into an equivalent "bandwidth cell" in the spatial frequency domain. Fig. 2 describes the method of deriving the "bandwidth cell" for a rectangular lattice according to the distance between the center point of the "Wigner-Seitz" cell to its edges points. This distance Δ_θ represents the possible resolution in each direction, and it depends on the angle θ relative to the horizontal sampling distance Δ according to,

$$\Delta_\theta = \Delta / \left(2\cos(\theta)\right). \tag{7}$$

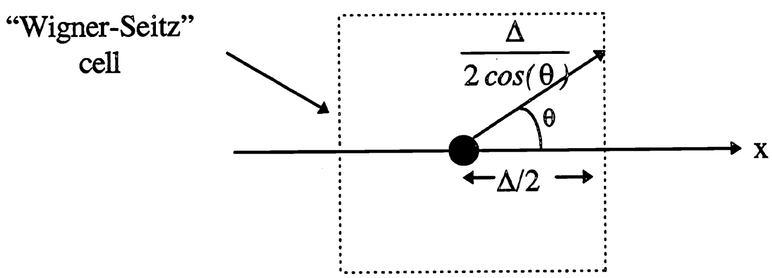
The parameter Δ_θ can also be treated as the sampling distance at angle θ relative to the horizontal axis. By the definition of Eq. 7 it possible to denote the sampling MTF at a given angle θ by replacing the parameter Δ in Eq. 2 by Δ_θ so the corresponding MTF is given by,

$$MTF_{samp\_\theta} = sinc(\xi_\theta \cdot \Delta_\theta) \tag{8}$$

The reciprocal of Δ_θ is the location of the first zero of the *sinc* function, which represents the spatial frequency bandwidth BW_θ of the sampling process in each direction,

$$BW_\theta = \frac{1}{\Delta_\theta} = 2\cos(\theta)/\Delta \tag{9}$$

The function in Eq. 9 is valid in the angle sector between -45° to 45° and it repeats itself in a 90^0 sector as it shown in Fig. 2. BW_θ is a radial function that represents the spatial frequency bandwidth resulting from the sampling process. As seen in this figure the highest - resolution directions are the horizontal and vertical, while poorer resolution is obtained along the diagonal axis (45^0 and 135^0) .These results can be explained by referring to Fig. 3 . The number of pixels in a rectangular array are the same in horizontal axis and the diagonal axis (equal to 6 for this example). However the length of the array in the horizontal direction is shorter than the diagonal distance by factor of $\sqrt{2}$, with the result that the sampling interval Δ_{45^o} is longer by the same factor than the sampling interval Δ_{0^o} . Therefore the bandwidth in the horizontal direction BW_{0^o} expected to be higher by $\sqrt{2}$ than the bandwidth in the diagonal direction BW_{45^o} .The same analysis can be applied on the "Wigner-Seitz" cell of the hexagonal lattice and the result for the "bandwidth cell" is shown in Fig. 4. The result here is
similar to that of the rectangular lattice. However, the symmetric range of BW_θ is within a 60^0 sector instead of a 90^0 sector.



$$BW_\theta = \frac{1}{2 \cdot \frac{\Delta}{\cos(\theta)}} = \frac{\cos(\theta)}{2 \cdot \Delta}$$

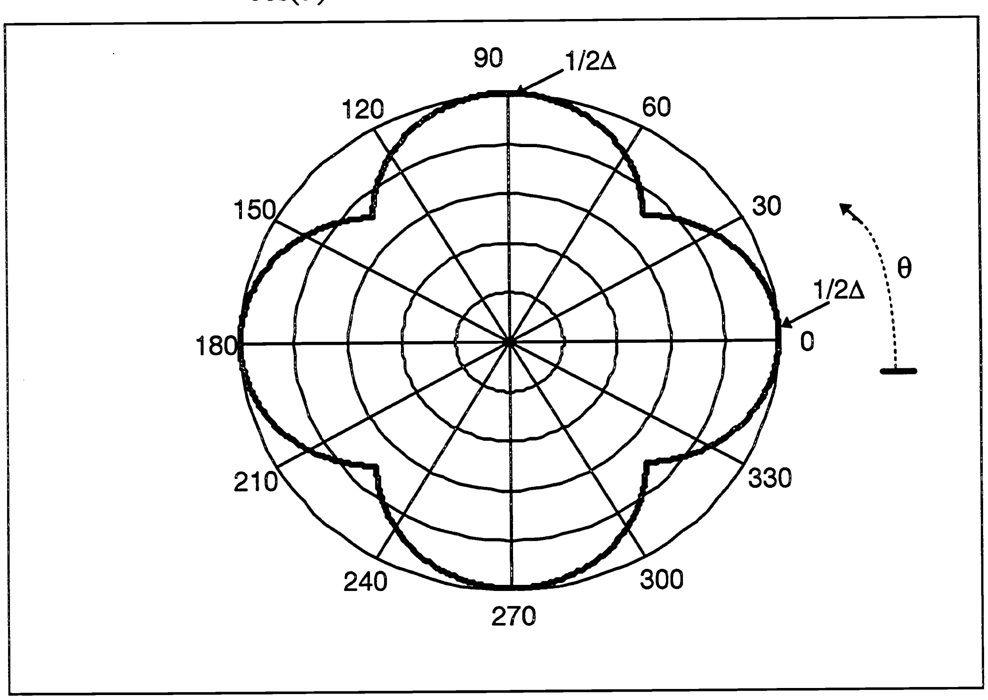


Fig. 2 : Derivation of the "Bandwidth cell" BW_θ from a Wigner-Seitz cell of a rectangular sampling lattice .

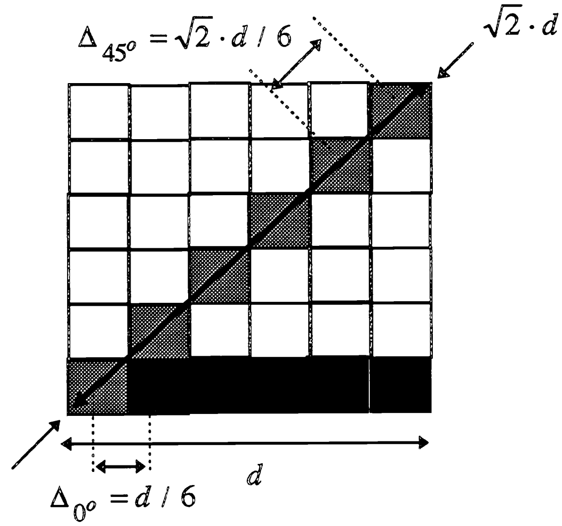


Fig. 3 : Comparison between the sampling intervals Δ_{0^o} and Δ_{45^o} for a rectangular array of 6 by 6 pixels .

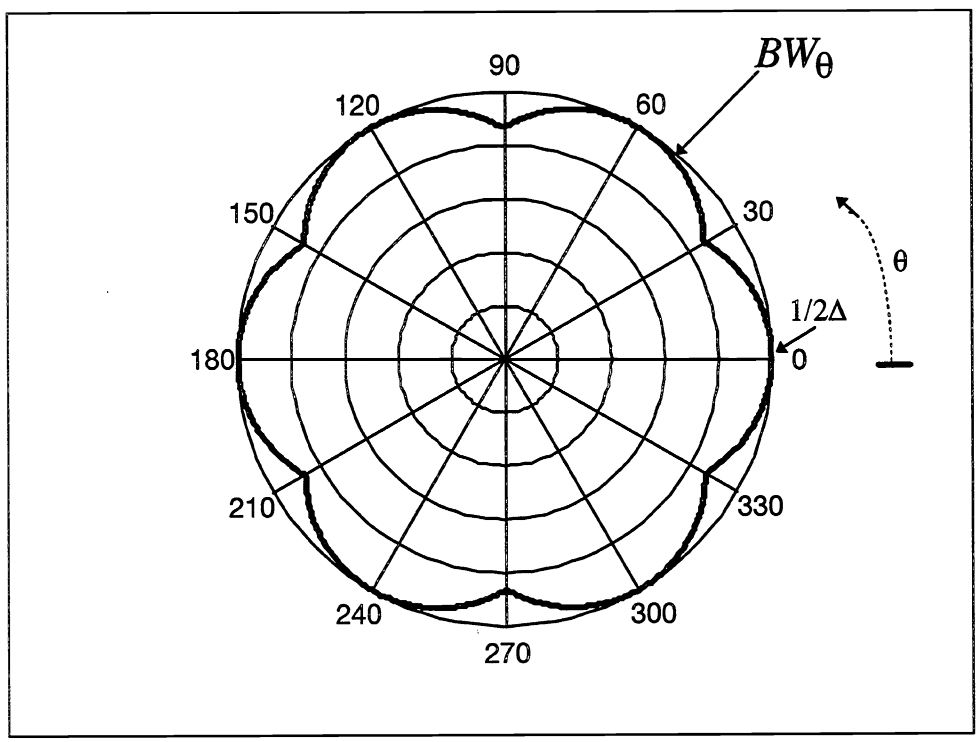


Fig. 4 : The "Bandwidth cell" BW_θ of a Hexagonal sampling lattice .

4. AREA MEASUREMENT OF THE BW_α FUNCTION

In order to compare between the two lattices it useful to compare the areas that enclosed by the functions BW_θ of the two lattices. This area represents the total bandwidth that can be obtained after the sampling process. The area that is bound by the function BW_θ as a function of the maximum value of θ for which Eq. 9 is valid (θ_{max}), is given by ,

$$S = \left[\frac{\pi}{2} + \frac{\pi}{4} \cdot \frac{sin(2\theta_{max})}{\theta_{max}}\right] \tag{10}$$

for the rectangular lattice $\theta_{max} = 45°$ and the enclosed area is equal to,

$$S_{rec} = \left[\frac{\pi}{2} + 1\right] \tag{11}$$

and for the hexagonal lattice $\theta_{max} = 30°$ and the enclosed area is equal to,

$$S_{hex} = \left[\frac{\pi}{2} + \frac{3\sqrt{3}}{4}\right] \tag{12}$$

The ratio between these two areas is given by $\frac{S_{hex}}{S_{rec}} = 1.116$, this result gives an advantage of 11% for the hexagonal lattice on the rectangular lattice. A similar result was obtained in Ref. 6 , but with a different approach to the analysis.

5. QUANTITATIVE MEASURE FOR OVERSAMPLING

The oversampling process can be described as the following. An image falls on and is sampled by a pixel array. This sampled information is first recorded. The image is then shifted by a fraction of the pixel pitch and sampled again. The information from the second sample is superimposed on that from the first sample to produce an oversampled image. This process can be done several times, with the number of extra samples identified as the degree of the oversampling process.

The amount of improvement that can be obtained by increasing the order of oversampling is obtained for a 2-D contiguous pixel array with pixel size p and sampling interval $\Delta = p$ in the horizontal and the vertical directions. The analysis is based on a method that was applied for a 1-D contiguous pixel array with pixel size p as in Ref. 4 . The order of oversampling, n, is defined as the number of extra samples taken. For instance, an order of 4 means that the array is shifted consecutively four times by a distance of $p/5$ to obtain four extra samples, giving a total of five samples. In Ref. 4 the investigated parameter was the value of MTF_{total} (Eq. 3) at the Nyquist spatial frequency, i.e., $\xi = 1/(2\Delta) = 1/(2p)$. Equation (3) then becomes,

$$MTF_{total} = sinc(1/2) \cdot sinc(1/2n) \tag{13}$$

A graph of MTF_{total} at Nyquist frequency vs. the order of oversampling is shown in Fig. 5 [4]. It is clear from the figure that the greatest improvement in performance is obtained when n is going from zero to first order. The amount of improvement that can be expected from the oversampling process is reduced rapidly in subsequent increments of the order.

A similar analysis can be applied to the 2-D pixel array. However, the analysis should include the angle dependence of the Wigner-Seitz cell (Eq. 8) because of the sampling lattice and the square pixel size. We calculate the value of MTF_{total} at Nyquist frequency as a function of the angle θ. This value depends on the sampling interval Δ_θ and the pixel size p_θ which is measured from the center point to the edge points of the pixel as a function of θ (see Fig. 6) . The result of the analysis is the average of MTF_{total} at Nyquist frequency over an angle of 2π. In this analysis we investigate three types of oversampling process as described in Fig. 7 : (a) oversampling in the horizontal direction, (b) oversampling in the horizontal and vertical directions and (c) oversampling along the diagonal axis. A graph of the average MTF_{total} at Nyquist frequency against the order of oversampling is shown in Fig. 8.

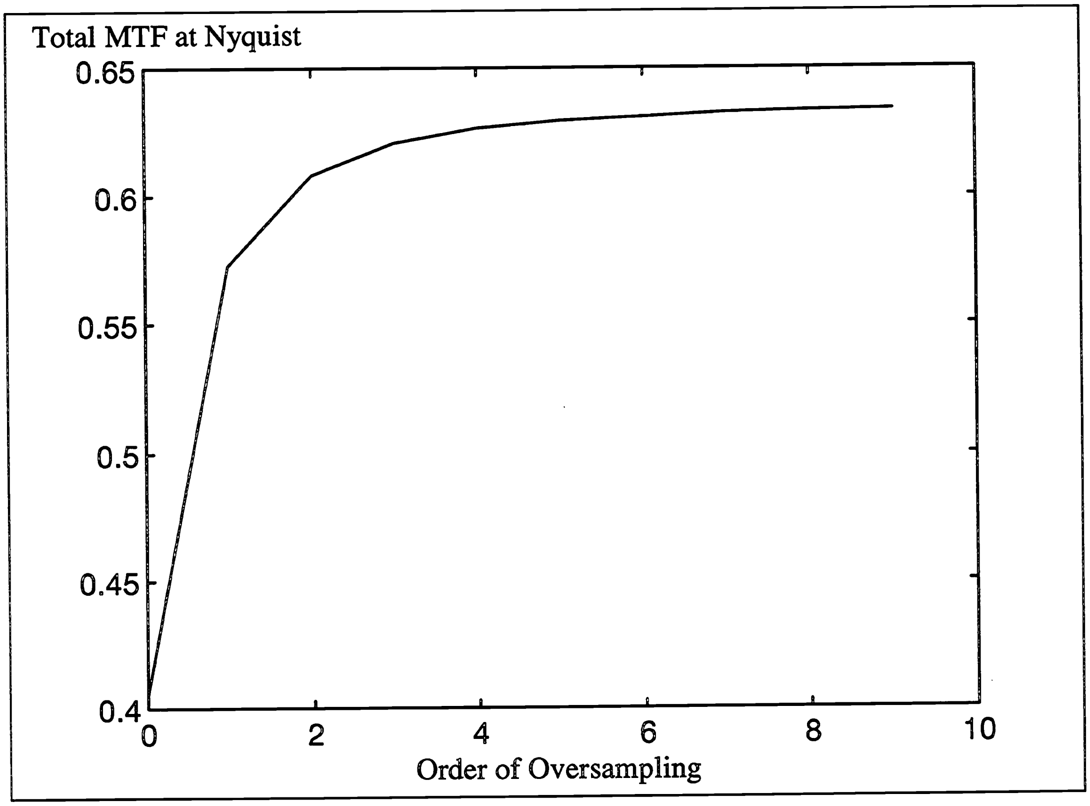


Fig. 5. A graph at the Nyquist frequency is plotted against the order of oversampling for a 1-D contiguous pixel array.

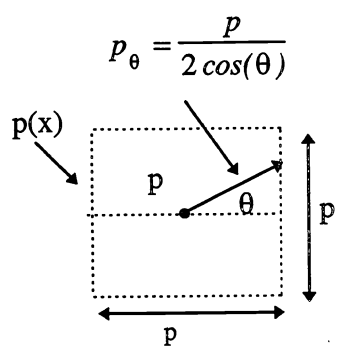


Fig. 6 : Deriving p_θ as function of the pixel size p and θ .

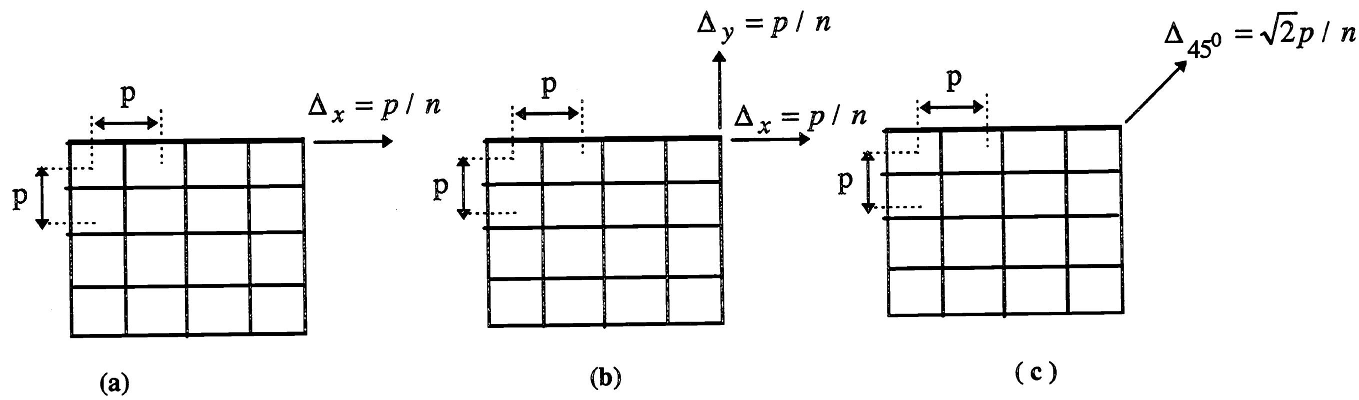


Fig. 7 Different types of oversampling : (a) horizontal direction (b) horizontal and vertical directions (c) diagonal direction.

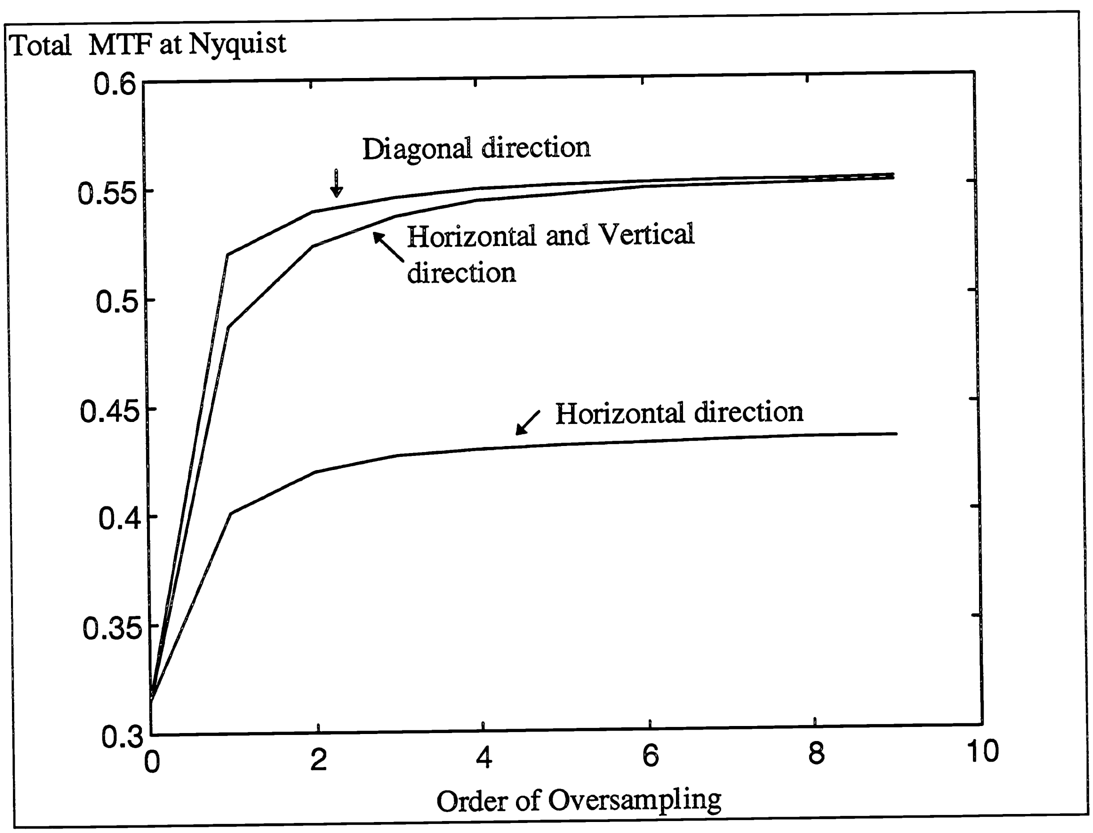


Fig. 8. A graph at the Nyquist frequency is plotted against the order of oversampling for three types of 2-D contiguous pixel array.

6. RESULTS

From Fig. 8 it is possible to compare the oversampling process for the three different configurations. The most successful way to perform the oversampling is by moving the sensor at small steps in the diagonal axis $\theta = 45^o$,(Fig. 7c). Oversampling at both horizontal and vertical directions (Fig. 7b) yields lower performance. However, as the order of the oversampling process increases the performances of both methods become similar. The third method of sampling in the horizontal direction (Fig. 7a), yields the least improvement of resolution as increasing the order of oversampling. At all three configurations it is clear from the figure that the greatest improvement in performance is obtained when n is going from zero to first order, as in the case of oversampling in 1-D. These results can be explained by using our definition for "Bandwidth cell" at section 3. It can be shown that the area bound by the function BW_α for each of the sampling intervals obtains its maximum value for sampling along the diagonal axis and the minimum value for sampling at the horizontal direction.

7. CONCLUSIONS

A new quantitative measure for the sampling process by 2-D pixel array was defined. This measure, the bandwidth cell in the spatial frequency domain is related to the Fourier transform of the Wigner -Seitz cell in the spatial domain. The area enclosed by the bandwidth cell is the quantitative measure which allows comparison between different lattices. According to this measure the hexagonal lattice performance in the sense of average bandwidth is superior to the rectangular lattice by 11%. Three different configurations of the sampling process are considered. The quantitative measure for this analysis is the average value of MTF_{total} at Nyquist frequency. The conclusion from this analysis is that the most efficient way to perform the sampling process is by shifting the sensor array along the 45° diagonal axis.

8. ACKNOWLEDGMENT

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9. REFERENCES

1. S. K. Park, R. Schowengerdt, and M. A. Kaczynski, "Modulation-transfer-function analysis for sampled image systems," *Appl. Opt.* **23**, 2572-2582 (1984) .
2. K. J. Barnard and G. D. Boreman, "Modulation transfer function of hexagonal staring focal plane arrays," *Opt. Eng.* **30**, 1915-1919 (1991).
3. G. Boreman and A. Plogstedt, "Spatial filtering by a line-scanned nonrectangular detector application to SPRITE readout MTF," *Appl. Opt.* 28, 1165-1168 (1989).
4. K. M. Hock, "Effect of oversampling in pixel arrays," *Opt. Eng.* **34**, 1281-1288 (1995) .
5. L. de Luca and G. Cardone, "Modulation transfer function cascade model for a sampled IR imaging system," Appl. Opt. **13**, 1659-1664 (1991) .
6. O. Hadar, A. Dogariu and G. D. Boreman, "Angular dependence of sampling MTF," accepted for publication in *Applied Optics*.
7. N. W. Ashcroft, and N. D. Mermin, *Solid State Physics*, pp. 63-83, Saunders College, Philadelphia (1976).