On-axis and off-axis propagation of Gaussian beams in gradient index media

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The object–image formula and an expression for the lateral magnification are obtained for a gradient radial index medium used with Gaussian beams. Angular acceptance conditions for on-axis and off-axis incidence are discussed. Meridional and sagittal sections of a Gaussian beam are analyzed, and numerical methods are used to find the evolution of the beam. The eccentricity parameter of the elliptical spot and the astigmatism of the beam are obtained as a function of propagation distance.

I. Introduction

Among the several types of nonuniform index media,[1–4] the most commonly used are the gradient radial index distribution (GRIN) media with square law dependence, $n = n_0(1 - \frac{1}{2}Ar^2)$. When these systems are used with laser sources it is necessary to take into account the special properties of the propagation of Gaussian laser beams in these media.

Several approximations to this problem can be done. The most usual is to model the laser beam, assumed in the Gaussian mode, like a bundle of rays.[5,6] Thus, for the GRIN system, ray tracing is possible using a computer algorithm.[7,8]

We present a different approach to this problem. We use the complex radius of curvature q defined by Kogelnik and Li[9] to represent the Gaussian beam. For the GRIN we use the ABCD matrix characteristic of the paraxial approximation. Although restricting our analysis somewhat, useful formulas are obtained for a number of cases of interest.

In Sec. II the ABCD matrix of a GRIN is introduced, and the object–image relation and magnification formula are obtained from a general analysis of the ABCD law for Gaussian beams. In Sec. III we analyze the problem of on-axis propagation using the model proposed above. Several properties of the evolution of the beam inside the GRIN are shown, and the angular acceptance condition for a Gaussian beam is discussed. The off-axis case is treated in Sec. IV by the method of defining an ABCD matrix for each infinitesimal sagittal section of the GRIN. These are multiplied to obtain the matrix of the entire GRIN. Also, the eccentricity parameter of the ellipsoidal transverse distribution in amplitude is plotted for several cases.

In the practical calculations used to plot the figures of this paper we assume a real GRIN from NSG, Inc. for which the parameters are $n_0 = 1.5637$ and $\sqrt{A} = 0.499\ \mathrm{mm}^{-1}$ (for a wavelength of 630 nm) with a GRIN lens diameter of 1 mm. For the Gaussian beam we assume a waist radius of 10 μm.

II. ABCD Matrix of a GRIN

The principal characteristic that matches the behavior of a GRIN is its index distribution function defined by the following expression:

$$n(r) = n_0(1 - \tfrac{1}{2}Ar^2), \tag{1}$$

where the constant n_0 and A are the two parameters involved in the ray tracing along the GRIN. Substituting this index in the differential equation of rays[3] $d(ndr/ds)/ds = \nabla n$, a relation between the height and slopes (x,x') of the geometrical rays at each point z along the GRIN is obtained:

$$x = x_0 \cos(\sqrt{A}z) + \frac{x_0'}{n_0\sqrt{A}} \sin(\sqrt{A}z), \tag{2a}$$

$$x' = -x_0 n_0 \sqrt{A} \sin(\sqrt{A}z) + x_0' \cos(\sqrt{A}z), \tag{2b}$$

where x_0 and x_0' are the height and slope of the ray at the input plane.

The intrinsic spatial period is related only with the A parameter of the GRIN and can be written as

$$P = \frac{2\pi}{\sqrt{A}}. \tag{3}$$

The linear relations in Eq. (2) are able to be written

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in matrix form to obtain the ABCD matrix of the system:

$$\begin{bmatrix} \cos(2\pi x) & \dfrac{\sin(2\pi x)}{\sqrt{A}n_0} \\ -n_0\sqrt{A}\sin(2\pi x) & \cos(2\pi x) \end{bmatrix}, \tag{4}$$

where we use x as the normalized length, $x = z/P$, where z is the actual distance along the axis. The usual values of x for real GRINs are $x = 0.23, 0.25, 0.29, 0.50$, but we allow x to vary continuously to find the evolution of the Gaussian beam as a function of distance.

At this point we can begin to treat Gaussian beams. We use the complex radius of curvature q defined by Kogelnik and Li,[9] ($1/q = 1/R - i\lambda/\pi\omega^2$). Then the evolution of the beam is given by the well known ABCD law $q_2 = (Aq_1 + B)/(Cq_1 + D)$.

An analytical solution to the problem of object and image beam waist matching can be obtained using the general expression of Bernabeu and Alda[10] in the following way:

$$x_2 = \frac{1}{2\pi n_0 \tan(2\pi x)} + \frac{x_1 - \dfrac{1}{2\pi n_0 \tan(2\pi x)}}{\cos^2(2\pi x) + 4\pi^2 n_0^2 \sin^2(2\pi x)\left[x_1^2 - \dfrac{1}{b_{01}^2 P^2}\right] - 2\pi n_0 x_1 \sin(2\pi x)}, \tag{5}$$

where x_1 and x_2 are the object and image normalized positions (with respect to the period P) of the beam waists $x_1 = l_1/P$, $x_2 = l_2/P$ (l_1 and l_2 are the actual object and image distances, respectively); b_{01} is the inverse of the Rayleigh range z_r and is related to the Gaussian width of the beam $b_{01} = \lambda/\pi\omega_{01}^2$.

The lateral magnification for this case, i.e., the relation between the image and object beam waists, is

$$\beta = \frac{\omega_{02}}{\omega_{01}} = \frac{1}{\left[\cos^2(2\pi x) + 4\pi^2 n_0^2 \sin^2(2\pi x)\left(x_1^2 - \dfrac{1}{b_{01}^2 P^2}\right) - 2\pi n_0 x_1 \sin(2\pi x)\right]^{1/2}}. \tag{6}$$

With these two relations [Eqs. (5) and (6)] the location and width of the output beam waist can be found given the characteristics of the GRIN and the location ($l_1 = x_1 P$) and size of the waist (b_{01}) of the incoming beam.

III. Evolution of an On-Axis and Gaussian Beam Inside a GRIN

From the ABCD law for Gaussian beams and the ABCD matrix defined in Eq. (4) it is possible to know the complex radius of curvature at every position along the GRIN.

An analytical solution can be obtained substituting the elements of the ABCD matrix in the general expressions[10,11]

$$\mathrm{Re}\left(\frac{1}{q_2}\right) = \frac{\mathrm{AC} + \mathrm{BD}b_{01}^2}{\mathrm{A}^2 + \mathrm{B}^2 b_{01}^2}, \tag{7}$$

$$\mathrm{Im}\left(\frac{1}{q_2}\right) = -\frac{b_{01}^2}{\mathrm{A}^2 + \mathrm{B}^2 b_{01}^2}. \tag{8}$$

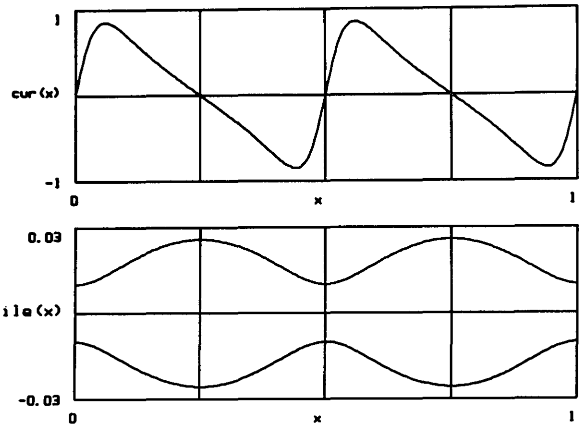

Fig. 1. Curvature and profile of the beam ($1/e$ in amplitude). Note the coincidence of the plane wave positions, $1/R = 0$, and the maximum and minimum of the beam inside the GRIN. The functions are plotted using the normalized distance x as the variable. The vertical axes are mm^{-1} and mm, respectively, and $l_1 = 0$ mm.

The graphic representation of these equations, after substituting the A, B, C, and D elements, is plotted in Fig. 1. We can see from this figure that the radius of curvature is infinite at both the minimum beam waist and the position where the width is maximum. From Fig. 1 and the analytical behavior of the denominator of Eq. (7) it is found that a plane wave appears each $0.25P$ inside the GRIN, corresponding to the alternative maximum and minimum spot sizes. The situation with the zero curvature and maximum spot size inside the GRIN is analogous to the point at infinity in free Gaussian beam propagation.

Another problem with the propagation of a Gaussian beam inside the GRIN is that of the acceptance angle that defines the GRIN for on-axis propagation. This angle is given by the expression[12]

$$\theta_{\max\,\mathrm{GRIN}} = \sin^{-1}(n_0\sqrt{A}r_0), \tag{9}$$

where r_0 is the radius of the GRIN. This parameter is easily applied to a classical ray trace where the field angle is defined. But, when dealing with Gaussian beams, it is necessary to relate it to the parameters of the beam and also to the position of the beam waist.

In this way we propose a method for knowing how much of the beam will be accepted by the GRIN; in this sense we propose the Gaussian width (e^{-1} decay in amplitude) as the limit of the beam. To accomplish this we assume a general case where the beam waist is located a distance $l_1 = x_1 P$ from the input plane of the

GRIN. Then the ABCD matrix from the input beam waist position becomes

$$\begin{bmatrix} \cos(2\pi x) & x_1 P\cos(2\pi x) + \dfrac{P}{2\pi n_0}\sin(2\pi x) \\ -\dfrac{2\pi n_0}{P}\sin(2\pi x) & -2\pi n_0 x_1 \sin(2\pi x) + \cos(2\pi x) \end{bmatrix}. \quad (10)$$

Now we need to know how wide the beam is inside the GRIN, because this width must never be greater than r_0 for the beam to remain inside the GRIN. The lateral magnification that relates the transverse dimension of the beam will be used to calculate the width.

When we relate the Gaussian width of the outcoming beam and the width of the input beam waist, the lateral magnification is defined in terms of the ABCD matrix, i.e.,[10,11]

$$\beta^2 = A^2 + B^2 b_{01}^2. \quad (11)$$

In this case

$$\beta^2 = \cos^2(2\pi x) + \left[x_1 P\cos(2\pi x) + \frac{P}{2\pi n_0}\sin(2\pi x)\right]^2 b_{01}^2. \quad (12)$$

To know where the maximum size of the beam will appear, we differentiate this expression with respect to x and set it equal to zero. This gives the following relation:

$$\tan(4\pi x_{ext}) = \frac{\dfrac{4x_1P^2b_{01}^2}{n_0}}{4\pi(1\,P\,x_1^2P^2b_{01}^2) - \dfrac{P^2b_{01}^2}{\pi n_0^2}}. \quad (13)$$

From here it is possible to find the position of the extremal points. But now we need to know if these extremal points are maxima or minima. To discriminate we can use a phenomenological criteria. When the incoming beam is divergent at the input plane of the GRIN, i.e., $x_1 > 0$ (real waist), the first extremal that will appear is a maximum, the second is minimum, and so on. On the other hand, where the beam is converging (virtual waist), the minimum appears first and the maximum after. We notice also that the period of the function $\tan(4\pi x)$ is $x = 0.25$. This means that the extremal points appear with this period alternating the maximum and minimum spot size. These observations are in agreement with the radius of curvature and beam width functions (see Fig. 2).

Using the value of x_{ext}, the position of the maximum spot size, given by Eq. (13), the calculation of the real width of the beam uses the general expression for the lateral magnification. The condition of acceptance is now

$$\omega_{max} < r_0, \quad (14)$$

which becomes

$$\omega_{01}^2\cos^2(2\pi x_{max}) + \left[x_1P\cos(2\pi x_{max}) + \frac{P}{2\pi n_0}\sin(2\pi x_{max})\right]^2\theta_0^2 < r_0^2, \quad (15)$$

where $\theta_0 = \lambda/\pi\omega_{01}$ is the divergence of the Gaussian beam in the far field and x_{max} is the position of the maximum extremal point.

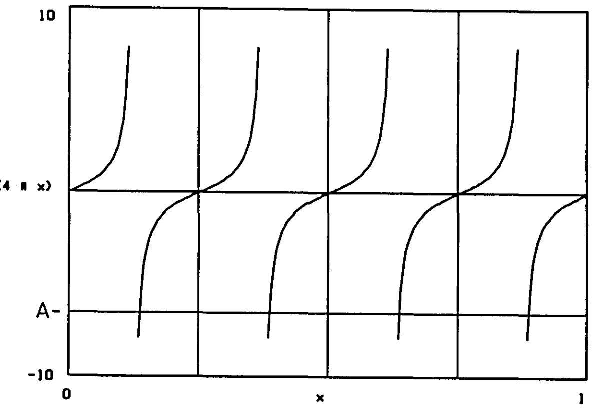

Fig. 2. Points where the function $\tan(4\pi x)$ reaches the value given in Eq. (13) where the beam has its maximum and minimum spot sizes. In this case $\omega_{01} = 0.01$ mm, $l_1 = 1$ mm, and the right-hand side of Eq. (13) gives a value $A = -6.509$.

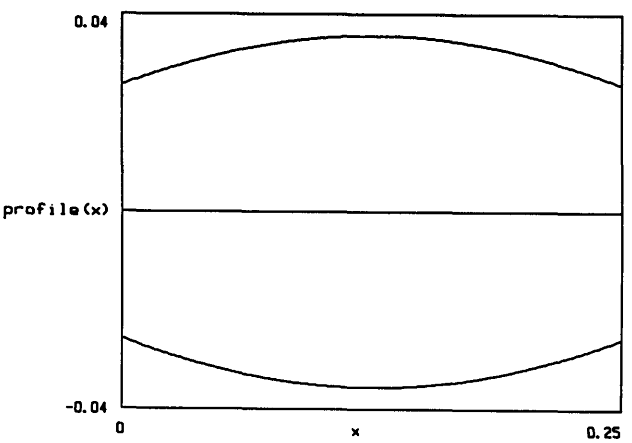

Fig. 3. Symmetric behavior of the beam in a 0.25P length. The profile of the beam (1/e in amplitude) is plotted. The vertical axis is in millimeters.

An interesting case appears from Eq. (13). When the denominator is zero the solution to the extremal points is given by $x = 1/8 + k\,1/4$ (k is an integer). This condition is obtained when the beam waist is located at

$$x_1 = \sqrt{\frac{1}{4\pi^2 n_0^2} - \frac{1}{P^2 b_{01}^2}}. \quad (16)$$

If the evolution of the beam profile is plotted (Fig. 3) we can see that an object–image symmetry appears for a GRIN with $x = 0.25$. This allows the GRIN to be used as a coupling system instead of a collimating system. For this case there exists an added constraint due to the square root. For it to remain real valued, it is required that

$$b_{01} > 2\pi n_0/P, \quad (17)$$

which is the same condition obtained in this system to define the Gaussian principal planes,[13] i.e., a pair of conjugate planes with lateral magnification equal to one. This is because a symmetrical behavior implies a lateral magnification equal to one.

IV. Evolution on an Off-Axis Gaussian Beam Inside a GRIN

In the previous section we assumed a circular Gaussian beam incident on-axis to a GRIN with rotational symmetry around the propagation axis. With these symmetries it has been possible to use a meridional plane to find the complete behavior of the beam. In this section we assume a circular Gaussian beam at the input plane of the GRIN, but now the center of the spot is displaced from the symmetry axis of the GRIN. Besides it could be possible that the beam was tilted on a meridional plane (no skew beams).

To solve this problem we assume the following:

(1) The propagation axis of the beam acts like the chief ray in the sense that the beam propagates and shows its transverse dependence around this chief ray inside the GRIN.

(2) The meridional section, i.e., that which lies on the plane formed by the propagation axis of the beam and the symmetry axis of the GRIN, shows behavior described by the matrix of Eq. (4).

(3) The sagittal section, perpendicular to the meridional, is described by a variable GRIN along the trajectory of the chief ray of the beam.

In the meridional plane, the beam tracing is related to the ray tracing of the chief ray whose trajectory (R is the height and R' is the slope of this ray) is given by

$$R = \epsilon \cos(\sqrt{A}z) + \frac{\epsilon'}{n_0\sqrt{A}} \sin(\sqrt{A}z), \tag{18a}$$

$$R' = -\epsilon n_0\sqrt{A} \sin(\sqrt{A}z) + \epsilon' \cos(\sqrt{A}z), \tag{18b}$$

where ϵ,ϵ' are the height and slope, respectively, of the chief ray at the input plane of the GRIN. Figure 4 shows the beam profile in the meridional plane. It has been obtained by superimposing the Gaussian transversal distribution to the trajectory of the chief ray.

For the sagittal section, one needs to know what the refractive index distribution is that appears along this section. This index function is obtained by calculating Eq. (1) along the minimum chord of the transverse section at the point of the chief ray. The following equation is the solution:

$$n(y) = n_0\left(1 - \frac{1}{2}AR^2\right)\left(1 - \frac{1}{2}\frac{A}{1 - \frac{1}{2}AR^2}y^2\right), \tag{19}$$

where y is the coordinate along the sagittal direction (see Fig. 5).

We see that R [Eq. (18a)], the position of the trajectory of the chief ray, has a dependence with z along the GRIN. Therefore, the index seen by the sagittal section of the beam is a function of the axial position too. The principal parameters of the GRIN are the index at the center n_0 and the quadratic coefficient A. From the previous equation we find that for this case these coefficients are

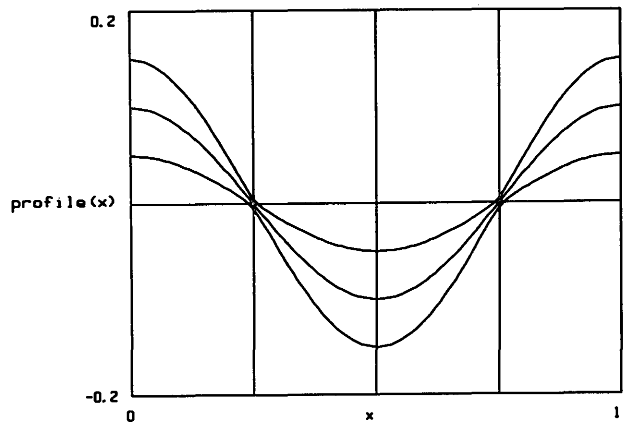

Fig. 4. Profile of the beam in the meridional section for an off-axis incidence. In this case ϵ = 0.1 mm, and the input beam waist is 50 μm, located at the input plane of the GRIN (x_1 = 0). The vertical axis is in millimeters.

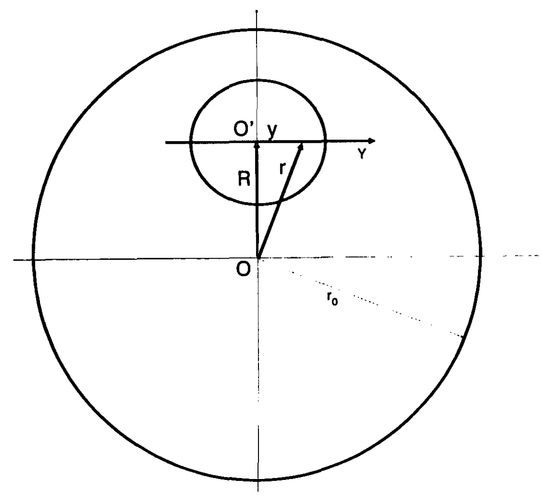

Fig. 5. Coordinate system for calculation of the refractive index distribution seen in the sagittal section of the GRIN. The distance between the center of the spot of the beam and the center of the GRIN is R, as in Eq. (18a).

$$n_0' = n_0(1 - \tfrac{1}{2}AR^2), \tag{20}$$

$$A' = \frac{A}{1 - \frac{1}{2}AR^2}. \tag{21}$$

Thus a different index distribution appears at every point along the chief ray trajectory.

In the proposed model, we need to know the ABCD matrix of the optical system used with the Gaussian beam. In this case the matrix for the sagittal section is obtained by multiplying the matrices for every infinitesimal section of the GRIN, where the index distribution is given by Eq. (19). We slice the GRIN into finite but small strips. We suppose that in each strip the index distribution is constant and given by Eq. (19) with R calculated at the middle point of the slice using Eq. (18a). The elementary matrix of each strip is given by

$$\begin{bmatrix} \cos\left(\dfrac{\sqrt{A}\Delta z}{\sqrt{1-\frac{1}{2}AR^2}}\right) & \dfrac{1}{n_0\sqrt{A}\sqrt{1-\frac{1}{2}AR^2}}\sin\left(\dfrac{\sqrt{A}\Delta z}{\sqrt{1-\frac{1}{2}AR^2}}\right) \\ -n_0\sqrt{A}\sqrt{1-\frac{1}{2}AR^2}\sin\left(\dfrac{\sqrt{A}\Delta z}{\sqrt{1-\frac{1}{2}AR^2}}\right) & \cos\left(\dfrac{\sqrt{A}\Delta z}{\sqrt{1-\frac{1}{2}AR^2}}\right) \end{bmatrix}, \tag{22}$$

where Δz is the length of the slice. Next the matrix multiplication is done to give the ABCD matrix for each position along the GRIN.

If we plot the evolution of the radius of curvature of the wavefront and the Gaussian width, behavior similar to the meridional section is obtained with minimum and maximum widths located at the position where the wavefront is plane (see Fig. 6). The parameters used to plot this figure and the following ones (Figs. 6–11) are $\omega_{01} = 0.01$ mm, $l_1 = 0$ mm, and those belonging to the SLS10 gradient lens, except that the diameter of the GRIN has been increased to 2 mm to enhance the effect of the off-axis propagation.

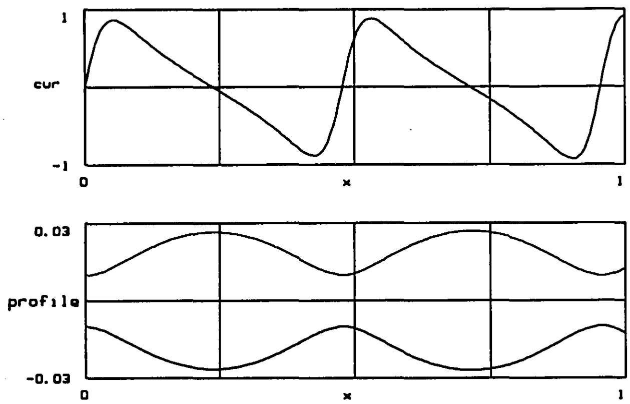

Fig. 6. Evolution of the curvature and Gaussian width of the sagittal section for off-axis propagation. Note the similar behavior with Fig. 1. $\epsilon = 0.95$ mm, $\epsilon' = 0$, and the vertical axes are in mm^{-1} and mm, respectively.

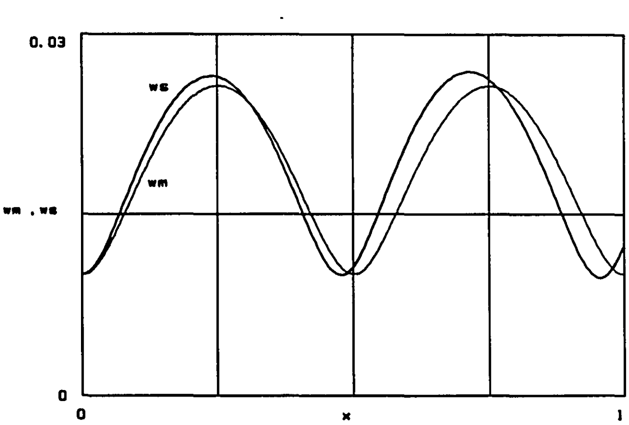

Fig. 7. Evolution of the Gaussian beamwidth for the meridional (ωm) and sagittal section (ωs). The intersection points represent the circular section of the spot ($\epsilon = 0.95$ mm, $\epsilon' = 0$). The vertical axis is in millimeters.

The differences between the two perpendicular sections, meridional and sagittal, are shown in Fig. 7. We can see from this figure that the Gaussian widths intersect at several points where a circular spot appears. But, in general, an orthogonal astigmatic spot is obtained along the GRIN. In this way one of the most interesting parameters of the system will be the eccentricity of the elliptical spot. In Fig. 8 we draw this

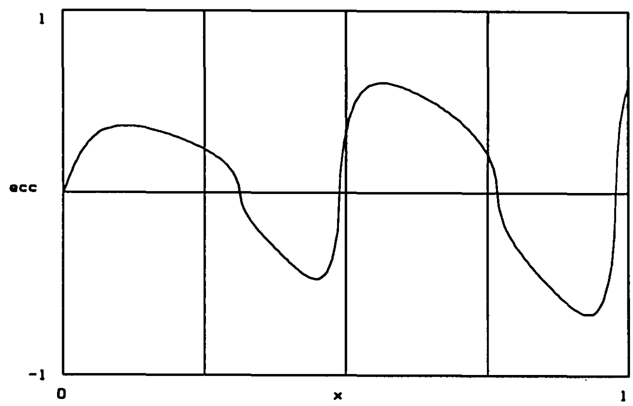

Fig. 8. Eccentricity of the spot (ecc). The eccentricity is usually positively valued. Here we use the negative sign to denote the orientation of the ellipse along the sagittal or meridional sections. If eccentricity < 0, the meridional section is greater and the ellipse is oriented in this direction. The entering parameters used for this plot are $\epsilon = 0.95$ mm, $\epsilon' = 0$.

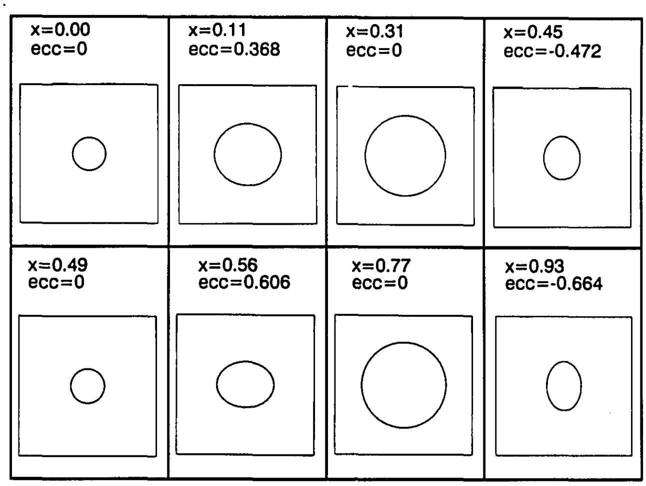

Fig. 9. Various sections of the beam along the GRIN, corresponding to maximum, minimum, and zeros of the eccentricity function for the same case as in the previous figures. The horizontal direction is sagittal.

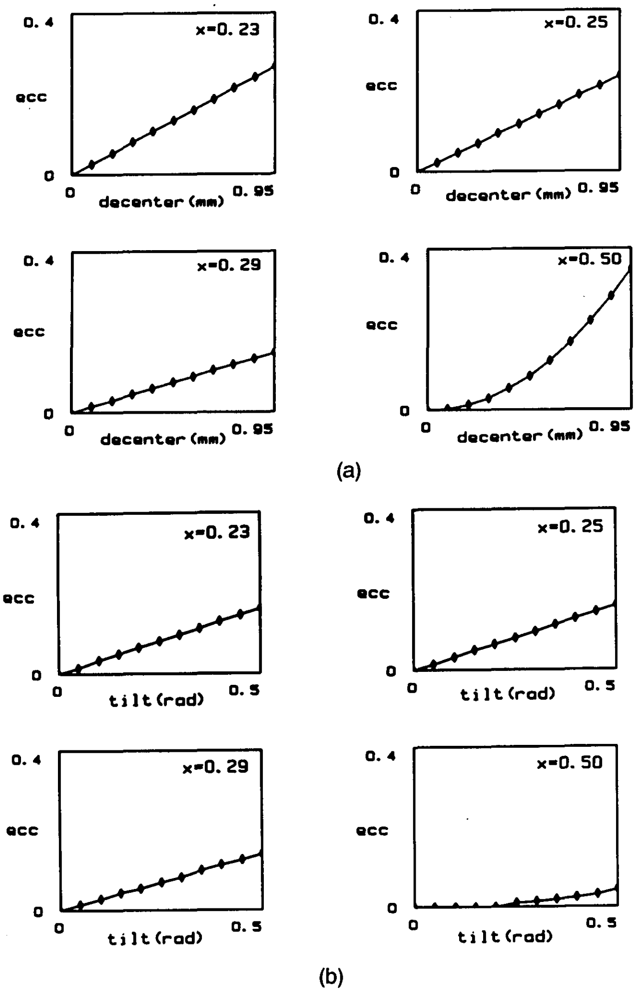

Fig. 10. Eccentricity of the spot as a function of the displacement distance ϵ [Fig. 10(a)] and tilt ϵ' [Fig. 10(b)] for the most common GRIN systems (0.23P, 0.25P, 0.29P, 0.50P).

parameter showing circular sections between regions of alternating sign of the eccentricity (see Fig. 9).

In Fig. 10 the eccentricity parameter is obtained in the most common GRIN systems, those with lengths of 0.23P, 0.25P, 0.29P, and 0.5P, as a function of the displacement of the beam and tilt.

Another interesting parameter plotted in Fig. 11 is the separation along the axis of propagation of the beam between the position of the beam waist for the sagittal and meridional planes. This distance is usually called astigmatism of the beam and must be taken into account when the beam is coupled with another optical system.

In off-axis propagation the acceptance condition has a simple analytic solution for the following case. We assume that the beam waist is at the input plane of the GRIN and consider the case in which $\epsilon = 0$ and there is only a tilt of the beam given by ϵ'. In this case the chief ray reaches its maximum distance from the axis of the GRIN at $x = 0.25$, 0.75, and the value is [from Eq. (18a)]

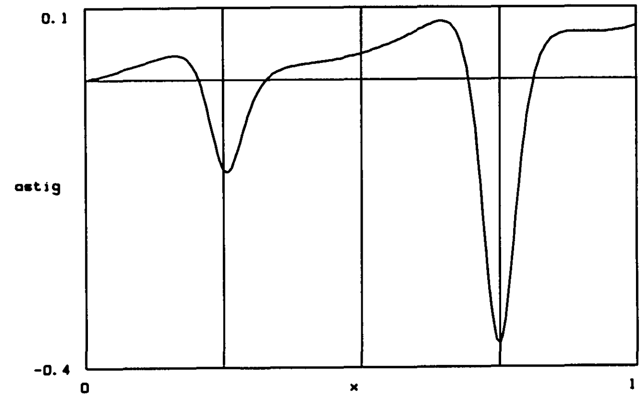

Fig. 11. Plot of the astigmatism of the beam, i.e., distance between the beam waist position in sagittal and meridional sections. The vertical axis is in millimeters.

$$R_{max} = \frac{P\epsilon'}{2\pi n_0}. \quad (23)$$

Besides in these points ($x = 0.25$) the beamwidth is maximum and ω_{max} is [from Eq. (12)]

$$\omega_{max} = \frac{P\theta_0}{2\pi n_0}. \quad (24)$$

Then the condition of acceptance will be

$$\frac{P}{2\pi n_0}(\theta_0 + \epsilon') < r_0, \quad (25)$$

which can be written as

$$\theta_0 + \epsilon' < \theta_{max\,GRIN}, \quad (26)$$

where $\theta_{max\,GRIN}$ is the angle given by Eq. (9).

V. Conclusion

A complete analysis of the Gaussian beam propagation along a GRIN has been developed using the complex radius of curvature and the ABCD matrix formulation. It allows simplification of the predesign and analysis of the systems using these elements. The astigmatic properties of a decentered and tilted beam have been found, and a practical method for calculating the parameters of the elliptical spot of the beam is provided.

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