Use of a Shack–Hartmann wave-front sensor to measure deviations from a Kolmogorov phase spectrum

T. W. Nicholls

Applied Optics Group, Blackett Laboratory, Imperial College, London SW7 2BZ, UK

G. D. Boreman

Department of Electrical Engineering, Center for Research and Education in Optics and Lasers, University of Central Florida, Orlando, Florida 32816

J. C. Dainty

Applied Optics Group, Blackett Laboratory, Imperial College, London SW7 2BZ, UK

Received August 17, 1995

Experimental results indicate that the statistics of phase measured across a telescope aperture do not always obey the power laws associated with the Kolmogorov model of atmospheric turbulence. We show that the statistical relations between a wave front and its aperture-averaged first derivative previously derived for a Kolmogorov spectrum can be easily generalized for any power law. We also show that a Shack–Hartmann sensor can be used to measure the form of the structure function of phase fluctuations, and experimental data are presented. © 1995 Optical Society of America

Optical propagation through turbulence has traditionally been described in terms of the power spectral density of phase (φ) fluctuations derived from the Kolmogorov model of turbulence[1]:

$$\Phi_\varphi(k) = \frac{0.023k^{-11/3}}{{r_0}^{5/3}}\,. \tag{1}$$

r_0 is the so-called Fried parameter[2] that conveniently describes the statistics of the wave front in a single variable.

However, detailed investigations into the statistics of the phase fluctuations have indicated significant departures from this behavior.[3–5] These observations have led to a more general prediction for the form of the phase spectrum:

$$\Phi_\varphi(k) = \frac{A_\beta k^{-\beta}}{{r_0}^{\beta-2}}\,. \tag{2}$$

The phase structure function, defined as

$$D_\varphi(\mathbf{r}) = \langle|\varphi(\mathbf{r}') - \varphi(\mathbf{r}' + \mathbf{r})|^2\rangle, \tag{3}$$

can easily be shown to be proportional to $r^{\beta-2}$. Although, in its original form, the parameter r_0 has no meaning unless the statistics are Kolmogorov, an analogous quantity ρ_0 can be defined so that the wave front variance over an aperture of diameter ρ_0 averages 1 rad^2:

$$D_\varphi(\mathbf{r}) = \gamma_\beta\left(\frac{r}{\rho_0}\right)^{\beta-2}\,. \tag{4}$$

Boreman and Dainty[6] recently applied this formulation to a modal wave-front expansion. The relative weighting of the different modes is found to be affected by changes in β, a result that has implications for the design of an adaptive-optics system. The purpose here is to suggest a method of measuring β with a standard Shack–Hartmann wave-front sensor and to use the results to analyze atmospheric data.

There are at least two known methods of reducing Shack–Hartmann centroid measurements to obtain the statistics of the phase fluctuations. One can estimate r_0 by calculating either the covariance or the differential variance of pairs of centroids. The centroid measurements provide an estimation of the aperture-averaged wave-front angle of arrival. The methods rely on the fact that the angle-of-arrival statistics may be directly related to the wave-front phase statistics as long as the form of the structure function is known. In the past, this form has been assumed to be Kolmogorov. However, any non-Kolmogorov behavior immediately renders such analysis invalid. The analysis methods will now be extended to non-Kolmogorov structure functions.

The relative motions of individual pairs of subimages produced by the Shack–Hartmann can be analyzed in terms of their covariances. Roddier[7] derived the dependence of the angle-of-arrival covariance on the Fried parameter, r_0. If the angle of arrival is arbitarily defined along the x axis,

$$\alpha(x, y) = -\left(\frac{\lambda}{2\pi}\right)\frac{\partial}{\partial x}\varphi(x, y), \tag{5}$$

then the angle-of-arrival covariance is defined as

$$B_\alpha(\mu, \eta) = \langle\alpha(x, y)\alpha(x + \mu, y + \eta)\rangle. \tag{6}$$

The covariance can easily be related to the structure function of phase by simple Fourier theory[7]:

$$B_\alpha(\mu, \eta) = -\frac{\lambda^2}{8\pi^2}\frac{\partial^2}{\partial\mu^2}D_\varphi(\mu, \eta). \tag{7}$$

0146-9592/95/242460-03$6.00/0

The non-Kolmogorov phase structure function of Eq. (4) can be written as

$$D_\varphi(\mu, \eta) = \gamma_\beta \rho_0{}^{-\beta+2}(\mu^2 + \eta^2)^{(\beta-2)/2}. \qquad (8)$$

By substituting Eq. (8) into Eq. (7), we obtain the following results for longitudinal and transverse covariance:

$$B_\alpha(d, 0) = 0.0127\gamma_\beta \lambda^2 \rho_0{}^{-\beta+2}(\beta - 2)(\beta - 3)d^{\beta-4}, \qquad (9)$$

$$B_\alpha(0, d) = 0.0127\gamma_\beta \lambda^2 \rho_0{}^{-\beta+2}(\beta - 2)d^{\beta-4}. \qquad (10)$$

Hence the ratio of these two quantities is given by

$$\frac{B_\alpha(d, 0)}{B_\alpha(0, d)} = \beta - 3. \qquad (11)$$

This analysis produces a very elegant result; however, for β values much less than 4, aperture averaging effects become significant even for large subaperture separations. Furthermore, covariance measurments are extremely sensitive to telescope tracking errors, and a differential method is preferable.

The method of Sarazin and Roddier[8] uses differential angles of arrival to estimate the statistics of the wave front. In this case, aperture averaging effects must be taken into account, and for this reason it is very difficult to make an analytic calculation. Sarazin and Roddier provide an approximate formula for the Kolmogorov case, but Fried[9] gives a general-case formula for a differential angle of arrival for any form of structure function and position of (circular) apertures:

$$\langle(\alpha_1 - \alpha_2)^2\rangle = \left(\frac{4}{\pi}\right)^4 \left(\frac{\lambda}{D}\right)^2 \int_0^{2\pi} \mathrm{d}\theta \int_0^1 u\,\mathrm{d}u \left\{\frac{1}{8}\cos^{-1} u + (1 - u^2)^{1/2}\left[\left(\frac{u^3}{12} - \frac{5u}{24}\right) + \left(\frac{u^3}{3} - \frac{u}{3}\right)\cos^2\theta\right]\right\} \times \left(\frac{1}{2}D_\varphi\left\{D[S^2 + 2Su\cos(\theta + \psi) + u^2]^{1/2}\right\} + \frac{1}{2}D_\varphi\{D[S^2 - 2Su\cos(\theta + \psi) + u^2]^{1/2}\} - D_\varphi(Du)\right). \qquad (12)$$

Here D is the diameter of a single aperture, S is the ratio of the subaperture separation to D, and ψ is the angle between the x axis and the line joining the centers of the two apertures. This method avoids any effects that are due to tracking errors.

The non-Kolmogorov structure functions introduced earlier to this equation can now be applied to Eq. (12). The transverse and longitudinal differential angle-of-arrival variances ($\sigma_t{}^2$ and $\sigma_l{}^2$, respectively) are represented by values of 0 and $\pi/2$ for ψ. It is useful to obtain the ratio between these two quantities, since several factors cancel and the result depends only on β and S. In particular, the ratio does not depend on the actual value of the seeing at the time (or on the factor γ_β). A numerical integration has been

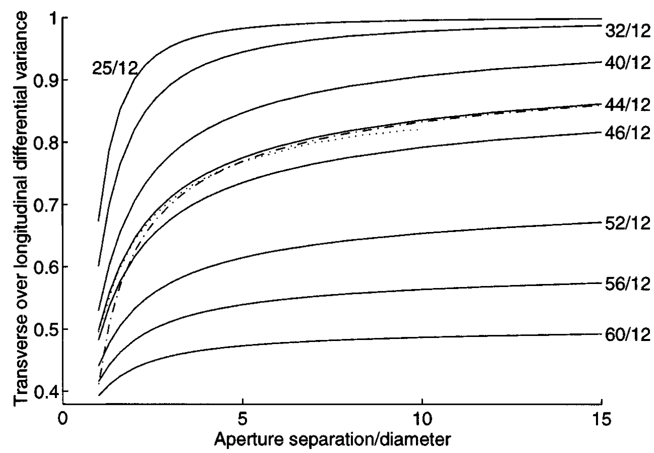

Fig. 1. Dependence of the ratio of transverse-to-longitudinal differential angle-of-arrival variance on the exponent (β) of the power spectrum of the phase. Fried's calculated values for a Kolmogorov spectrum are shown by the dotted curve; Sarazin and Roddier's approximate formula is plotted as the dashed curve.

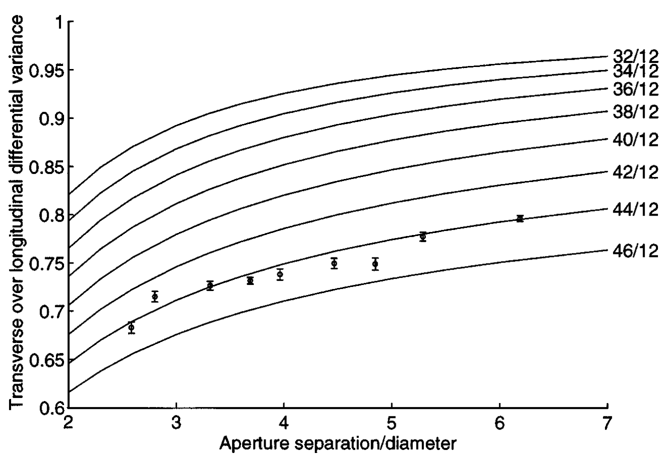

Fig. 2. Comparison of simulated centroid motions (symbols) with theory (solid curves).

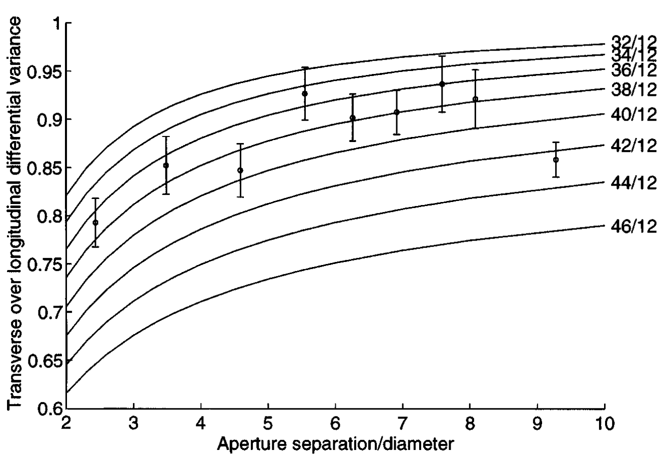

Fig. 3. Ratio of differential variances found in experimental data taken at La Palma (symbols) compared with theoretical curves for varying β.

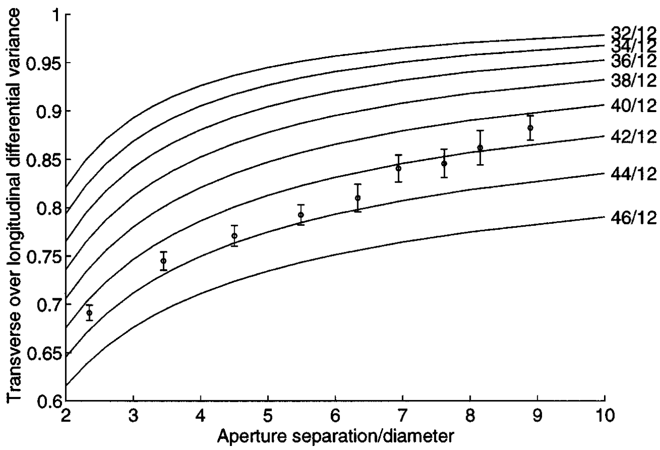

Fig. 4. Ratio of differential variances found in experimental data taken at Calar Alto (symbols) compared with theoretical curves for varying β.

performed, and curves of σ_t^2/σ_l^2 against S for a range of β values are shown in Fig. 1. Fried's calculated values for Kolmogorov turbulence ($\beta = 11/3$, dotted curve) are shown, together with a plot of Sarazin and Roddier's approximate formula (dashed curve).

We have calculated the differential variance ratio for 1000 simulated Kolmogorov wave fronts using centoids calculated by a Shack–Hartmann simulator with square lenslets. It can be seen from Fig. 2 that the points lie close to the curve predicted for a Kolmogorov spectrum (i.e., the 44/12 line).

The theoretical curves have been compared with experimental data taken by the 1-m Jacobus Kapteyn telescope in La Palma (see Fig. 3). These data were taken in December 1993 with square lenslets. Each point represents the mean value over 1000 frames of the variance ratio for a number of spot pairs with similar separations. The graph suggests that β is much closer to 3 than to the Kolmogorov prediction of 11/3 for these data. It should be recognized that this graph is by no means intended to be representative of the properties of the spectra associated with atmospheric turbulence in general. Figure 4 shows an equivalent plot from 20,000 frames taken at the 1.23-m telescope at Calar Alto. In this case, the behavior is much closer to that expected from Kolmogorov turbulence.

The authors thank Pedro Negrete-Regagnon for providing the Kolmogorov wave-front simulations and the Shack–Hartmann simulator. This research was supported by the UK Particle Physics and Astronomy Research Council under grant GR/H64156 and by The Royal Observatory, Edinburgh.

References

1. V. I. Tatarski, *Wave Propagation in a Turbulent Medium* (McGraw-Hill, New York, 1961) (translated from Russian by R. A. Silverman).
2. D. L. Fried, J. Opt. Soc. Am. **55,** 1427 (1965).
3. D. Dayton, B. Pierson, B. Spielbusch, and J. Gonglewski, Opt. Lett. **17,** 1737 (1992).
4. M. Bester, W. C. Danchi, L. J. Degiacomi, and C. H. Townes, Astrophys. J. **392,** 357 (1992).
5. R. G. Buser, J. Opt. Soc. Am. **61,** 488 (1971).
6. G. D. Boreman and J. C. Dainty, "Zernike expansions for non-Kolmogorov turbulence," J. Opt. Soc. Am. A (to be published).
7. F. Roddier, Prog. Opt. **19,** 281 (1981).
8. M. Sarazin and F. Roddier, Astron. Astrophys. **227,** 294 (1990).
9. D. L. Fried, Radio Sci. **10**(1), 71 (1975).