Sources of arbitrary states of coherence that generate completely coherent fields outside the source

Greg Gbur and Emil Wolf

Department of Physics and Astronomy and the Rochester Theory Center for Optical Science and Engineering, University of Rochester, Rochester, New York 14627

A new theorem concerning nonradiating stochastic sources is derived. The theorem is used to describe sources that can have a variety of different states of coherence and yet each of them generating a field that is completely spatially coherent outside the source domain. The bearing of this result on a basic unsolved problem is spectroscopy of partially coherent sources is noted. © 1997 Optical Society of America

Consider a fluctuating source distribution $Q(\mathbf{r}, t)$ that occupies, for all time, a finite domain D. Here $\mathbf{r}$ denotes a position vector of a point in space and t denotes the time. We assume that the source fluctuations are statistically stationary, at least in the wide sense (Ref. 1, p. 47).

Let $W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega)$ be the cross-spectral density function, at frequency ω, of the source distribution. It is known that under very general conditions W_Q may be expanded in a Mercer-type series, viz. (Ref. 1, Sec. 4.7.1, and Ref. 2)

$$W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) = \sum_n \lambda_n(\omega)\phi_n^*(\mathbf{r}_1, \omega)\phi_n(\mathbf{r}_2, \omega), \quad (1)$$

where the λ_n's and the ϕ_n's (called the coherent modes of the source) are the eigenvalues and the eigenfunctions, respectively, of the integral equation

$$\int_D W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega)\phi_n(\mathbf{r}_1, \omega)d^3r_1 = \lambda_n(\omega)\phi_n(\mathbf{r}_2, \omega), \quad (2)$$

$$\lambda_n(\omega) > 0 \quad \text{for all } n. \quad (3)$$

The radiant intensity $J(\mathbf{s}, \omega)$, i.e., the rate at which the source radiates energy at frequency ω, per unit solid angle around a direction specified by a unit vector $\mathbf{s}$ is given by the expression (Ref. 1, p. 232, with a slightly different definition of the Fourier transform)

$$J(\mathbf{s}, \omega) = \tilde{W}_Q(-k\mathbf{s}, k\mathbf{s}, \omega), \quad (4)$$

where

$$\tilde{W}_Q(\mathbf{K}_1, \mathbf{K}_2, \omega) = \int_D \int_D W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega)$$
$$\times \exp[-i(\mathbf{K}_1 \cdot \mathbf{r}_1 + \mathbf{K}_2 \cdot \mathbf{r}_2)]d^3r_1 d^3r_2 \quad (5)$$

is the six-dimensional spatial Fourier transform of the cross-spectral density function and

$$k = \omega/c, \quad (6)$$

c being the speed of light in vacuum.

On substituting from Eq. (1) into Eq. (4) it readily follows that in terms of the source modes ϕ_n the radiant intensity may be expressed in the form

$$J(\mathbf{s}, \omega) = \sum_n \lambda_n(\omega)|\tilde{\phi}_n(k\mathbf{s}, \omega)|^2, \quad (7)$$

where

$$\tilde{\phi}_n(\mathbf{K}, \omega) = \int_D \phi_n(\mathbf{r}, \omega)\exp(-i\mathbf{K} \cdot \mathbf{r})d^3r \quad (8)$$

is the three-dimensional spatial Fourier transform of the source mode $\phi_n(\mathbf{r}, \omega)$.

Because according to relation (3) the λ_n's are positive it is clear from Eq. (7) that if the source does not radiate at frequency ω, i.e., if

$$J(\mathbf{s}, \omega) = 0 \quad (9)$$

for all directions $\mathbf{s}$, then

$$\tilde{\phi}_n(\mathbf{K}, \omega) = 0 \quad (10)$$

for all real vectors $\mathbf{K}$ or magnitude $|\mathbf{K}| = k = \omega/c$ and for all n. This requirement implies that the monochromatic source

$$Q_n(\mathbf{r}, t) = \phi_n(\mathbf{r}, \omega)\exp(-i\omega t) \quad (11)$$

itself does not radiate.[3,4] Hence we have established the following theorem: *If a statistically stationary stochastic source does not radiate at frequency ω, all its coherent modes $\phi_n(\mathbf{r}, \omega)$ are nonradiating modes.*

It is also known that the field $\psi(\mathbf{r}, \omega)$ generated by a nonradiating monochromatic source of frequency ω [i.e., by a monochromatic source that generates a field whose radiant intensity $J(\mathbf{s}, \omega) = 0$ for all directions $\mathbf{s}$] vanishes at every point outside the source. Expressed differently if Eq. (10) holds for all $|\mathbf{K}|$ of magnitude ω/c, then[5]

$$\psi(\mathbf{r}, \omega) = 0 \quad \text{for all } \mathbf{r} \notin D. \quad (12)$$

We now show that the theorem that we just established can be used to describe sources of arbitrary states of spatial coherence that produce fields that are spatially completely coherent outside the source domain.

Consider a stochastic, statistically stationary source occupying a finite domain D, whose cross-spectral density has the mode expansion

$$W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) = \lambda_0(\omega)\phi_0^{*R}(\mathbf{r}_1, \omega)\phi_0^R(\mathbf{r}_2, \omega)$$
$$+ \sum_{n=1}^{N} \lambda_n(\omega)\phi_n^{*NR}(\mathbf{r}_1, \omega)\phi_n^{NR}(\mathbf{r}_2, \omega)$$
$$(13)$$

$(\mathbf{r}_1 \in D, \mathbf{r}_2 \in D)$, where N is an arbitrary positive integer. In this expression $\phi_0^R(\mathbf{r}, \omega)$ is a radiating mode and $\phi_n^{NR}(\mathbf{r}, \omega)$ $(n = 1, 2, \ldots)$ are nonradiating modes.[6] Obviously only the mode $\phi_0^R(\mathbf{r}, \omega)$ will generate a field outside D. Consequently the cross-spectral density of the field throughout the exterior of the source domain, i.e., for $\mathbf{r}_1 \notin D$, $\mathbf{r}_2 \notin D$, will be given by the equation (Ref. 1, Sec. 4.7.3, and Ref. 7)

$$W_\psi(\mathbf{r}_1, \mathbf{r}_2, \omega) = \lambda_0(\omega)\psi_0^{*R}(\mathbf{r}_1, \omega)\psi_0^R(\mathbf{r}_2, \omega), \quad (14)$$

where

$$\psi_0^R(\mathbf{r}, \omega) = \int_D \phi_0^R(\mathbf{r}', \omega)\frac{\exp(ik|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} d^3r'. \quad (15)$$

Because according to Eq. (14) W_ψ is a product of a function of $\mathbf{r}_1$ and of a function of $\mathbf{r}_2$, the field is necessarily spatially completely coherent at frequency ω throughout the exterior of the source domain D (Ref. 1, Sec. 4.5.3, and Ref. 8). On the other hand, Eq. (13) does not factorize in this way and, therefore, the source distribution is necessarily only partially coherent.[9] One can show that as the number N of the nonradiating source modes is increased, i.e., as larger and larger numbers of such modes contribute to the field within the source region and if the coefficients λ_n are of the same order of magnitude, the source will become spatially highly incoherent. Yet all such sources will produce a completely spatially coherent field outside the source domain. This fact is illustrated in Fig. 1. The (nonnormalized) nonradiating modes were taken to be given by the expression

$$\phi_{\ell m N}(\mathbf{r}) = \sum_{n=1}^{N-1} \alpha_{\ell n}\Gamma_{\ell m n}(\mathbf{r}) - \frac{1}{\alpha_{\ell N}}\left[\sum_{n=1}^{N-1}[\alpha_{\ell n}]^2\right]\Gamma_{\ell m N}(\mathbf{r}),$$
$$(16)$$

where

$$\Gamma_{\ell m n}(\mathbf{r}) = \left[\frac{2}{a^3}\right]^{1/2}\frac{1}{|j_{\ell+1}(k_{\ell n}a)|}j_\ell(k_{\ell n}r)Y_\ell^m(\theta, \phi), \quad (17)$$

$$\alpha_{\ell n} = -\frac{k_{\ell n}}{k^2 - k_{\ell n}^2}\frac{j_{\ell+1}(k_{\ell n}a)}{|j_{\ell+1}(k_{\ell n}a)|}. \quad (18)$$

In Eq. (18) $k_{\ell n}$ is the nth zero of the spherical Bessel function of order ℓ, i.e., $j_\ell(k_{\ell n}a) = 0$, a is the radius of the spherical source, and k is defined by Eq. (6). Using the well-known identity[10]

$$\exp(i\mathbf{k} \cdot \mathbf{r}) = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}i^\ell j_\ell(kr)Y_\ell^m(\theta, \phi)$$
$$\times Y_\ell^m(\theta_k, \phi_k), \quad (19)$$

where $Y_{\ell m}(\theta, \phi)$ are the spherical harmonics, the functions $\phi_{\ell m n}(\mathbf{r})$ defined by Eq. (16) can be shown to be mutually orthogonal and to satisfy Eq. (10) for nonradiating sources.
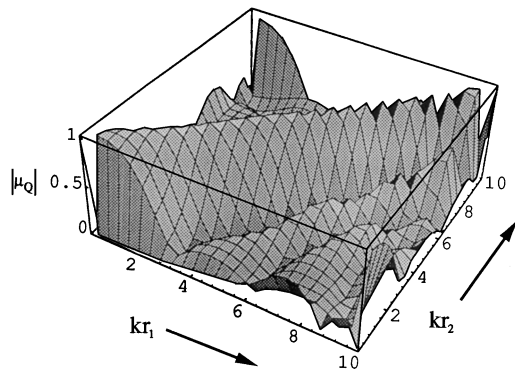
The result that we just derived has an important implication for spectroscopy of partially coherent sources. To see this we set $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}$ on the left-hand sides of Eqs. (13) and (14). The expressions then represent the source spectrum, $S_Q(\mathbf{r}, \omega)$, and the field spectrum, $S_\psi(\mathbf{r}, \omega)$:
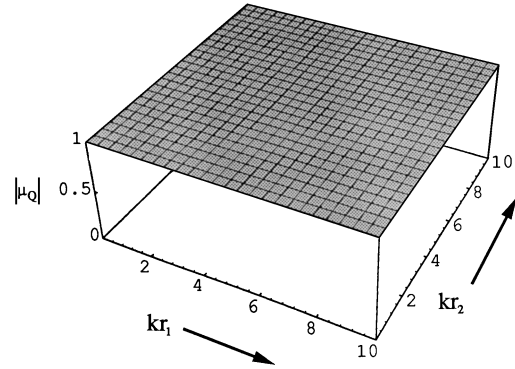
(a)   1 radiating mode
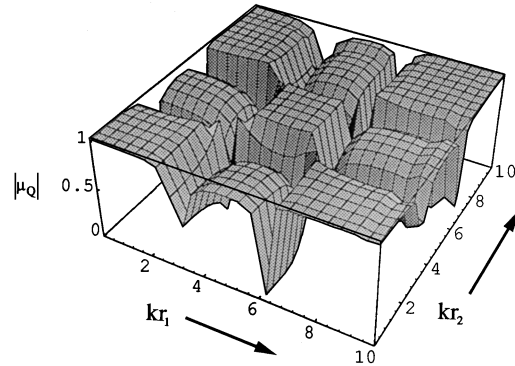
(b)   1 radiating mode
+ 1 nonradiating mode

(c)   1 radiating mode
+ 363 nonradiating modes

Fig. 1.   Absolute value of the spectral degree of coherence $\mu_Q(\mathbf{r}_1, \mathbf{r}_2, \omega)$ (see Ref. 1, p. 171) of spherically symmetric sources, consisting of suitable linear combinations of a radiating mode and of nonradiating modes. The nonradiating modes are given by Eq. (16), with appropriate normalization and with the choice of $ka = 10$. It is to be noted that in (c) the source is highly spatially incoherent. Yet all these sources produce a completely spatially coherent field outside the domain containing the source.

$$S_Q(\mathbf{r}, \omega) = \lambda_0(\omega)|\phi_0{}^R(\mathbf{r}, \omega)|^2 + \sum_{n=1}^{N} \lambda_n(\omega)|\phi_n^{\mathrm{NR}}(\mathbf{r}, \omega)|^2,$$

$$(20)$$

$$S_\psi(\mathbf{r}, \omega) = \lambda_0(\omega)|\psi_0^R(\mathbf{r}, \omega)|^2. \qquad (21)$$

Evidently the spectra $S_\psi(\mathbf{r}, \omega)$ and $S_Q(\mathbf{r}, \omega)$ differ from each other, even in the special cases when the source spectrum is independent of position $\mathbf{r}$. This result shows that, in general, one cannot determine the spectrum of a partially coherent source from measurements of the spectrum of the radiated field. This conclusion is perhaps not entirely surprising in view of the many new results obtained in recent years regarding the influence of source correlation on the spectra of radiated fields.[11] Clearly the development of a method for determining the spectra of partially coherent sources from measurements involving the radiated field outside the source is a basic unsolved problem of spectroscopy.

References

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9. That this is so is evident from the following simple argument. If Eq. (13) represented a completely spatially coherent source it would necessarily have to be expressible in the form $W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) = \Phi^*(\mathbf{r}_1, \omega)\Phi(\mathbf{r}_2, \omega)$. On substituting this expression into Eq. (2) one finds at once that, in this case, the integral equation would have only one eigenfunction (proportional to Φ). Hence such a source would consist of a single mode, which contradicts the assumption that it contained more than one mode as implied by Eq. (13).
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