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The quasi-homogeneous approximation for a class of three-dimensional primary sources

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Abstract

We investigate the validity of the quasi-homogeneous approximation for three-dimensional primary Gaussian Schell-model sources. It is shown that the quasi-homogeneous approximation is not valid for such a source unless two conditions are satisfied, one of which depends only upon the spatial characteristics of the source, and the other depends upon the wavelength of the radiation. The second of these conditions appears not to have been appreciated in previous work relating to the quasi-homogeneous approximation, and its significance to the foundations of radiometry is discussed. © 1999 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

The concept of a quasi-homogeneous source, originally introduced to describe planar secondary sources, [1] has been in use for some time now. Traditionally, the approximation has been introduced as follows. Consider a statistically stationary random radiation source Q, for which the cross-spectral density (Chap. 4 of Ref. [3]) can be written

$$W_Q(r_1,r_2,\omega) = \sqrt{S_Q(r_1,\omega)}\,\sqrt{S_Q(r_2,\omega)}\,g_Q(r_2-r_1,\omega). \tag{1}$$

In this formula $S_Q(r,\omega)$ represents the spectral density or intensity of the source at a point r at frequency ω, and $g_Q(r',\omega)$ represents the spectral degree of coherence of the source which depends only upon the vectorial distance between the source points. A source which can be modeled in this way is known as a *Schell-model source*. If the spectral degree of coherence g_Q is a very narrow function of position, and if S_Q is a slowly varying function of position in comparison to g_Q (see Fig. 1), it appears that the cross-spectral density is well-approximated by the factorized form

$$W_Q(r_1,r_2,\omega) \approx W_Q^{\mathrm{qh}}(r_1,r_2,\omega) \equiv S_Q\left(\frac{r_1+r_2}{2},\omega\right) g_Q(r_2-r_1,\omega). \tag{2}$$

This factorized form, with the assumptions on S_Q and g_Q just stated, is referred to as the quasi-homogeneous approximation, and it has been used extensively since its introduction. For example, it has been used to model various radiation sources, both two-dimensional [1,2] and three-dimensional [4], and has also been used to model spatially random scatterers [5]. The inverse problem for quasi-homogeneous sources has been investigated [6]. The quasi-homogeneous approximation has also been shown to be of

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[1] The quasi-homogeneous approximation was first stated in the form used here in Ref. [1]. Prior to that publication, very similar concepts were described in Ref. [2].

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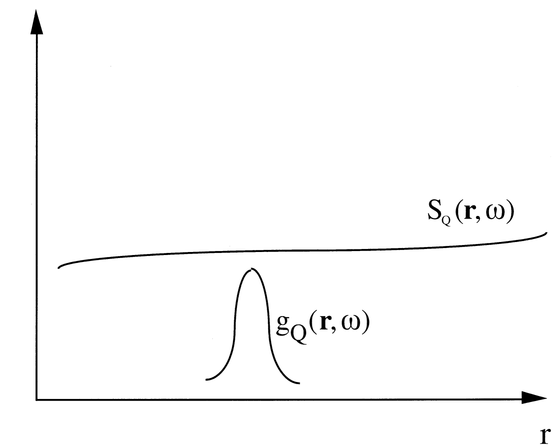

Fig. 1. Illustrating the conventional requirement for the validity of the quasi-homogeneous approximation. At a given frequency ω, the spectral density $S_Q(r,\omega)$ must be a 'slowly varying' function of position relative to the spectral degree of coherence $g_Q(r,\omega)$.

fundamental importance in clarifying the foundations of radiometry [7–9].

However, the qualitative arguments used in deriving this approximation leave significant questions unanswered. For example, what makes a function 'narrow' or 'slowly varying'? Also, are there any other requirements for the validity of the approximation? What is the nature and magnitude of any correction terms?

2. Discussion

We will investigate these questions by examining a simple type of quasi-homogeneous source. As we are interested in understanding the conditions under which the field generated by a quasi-homogeneous source gives an accurate approximation of the true radiation field, we will examine the cross-spectral density of the field far away from the source. In the far zone of a three-dimensional, statistically stationary source, the field cross-spectral density is given by the expression (Eq. (5.2-5) from Ref. [3])

$$W_V^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2,\omega) = (2\pi)^6 \frac{e^{ik(r_2-r_1)}}{r_1 r_2} \tilde{W}_Q(-k\mathbf{s}_1,k\mathbf{s}_2,\omega), \tag{3}$$

where $\mathbf{s}_1$ and $\mathbf{s}_2$ are unit vectors, r_1 and r_2 are the distances from the source to the field points (see Fig. 2) and

$$\tilde{W}_Q(k_1,k_2,\omega) = \frac{1}{(2\pi)^6}\int\int W_Q(r_1,r_2,\omega) \times e^{-i(k_1 r_1 + k_2 r_2)} d^3 r_1 d^3 r_2 \tag{4}$$

is the six-dimensional spatial Fourier transform of the cross-spectral density of the source. In Eq. (3) $k=\omega/c$ is the free-space wave number.

For a quasi-homogeneous source, the Fourier transform (4) of the source function (2) has the simple form

$$\tilde{W}_Q^{\mathrm{qh}}(-k\mathbf{s}_1,k\mathbf{s}_2,\omega) = \tilde{S}_Q(k(\mathbf{s}_2-\mathbf{s}_1))\,\tilde{g}_Q\left(k\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right), \tag{5}$$

where $\tilde{S}_Q$ and $\tilde{g}_Q$ are the three-dimensional Fourier transforms of S_Q, g_Q respectively, defined by the expressions

$$\tilde{S}_Q(K) = \frac{1}{(2\pi)^3}\int S_Q(r')e^{-iKr'}d^3r' \tag{6a}$$

$$\tilde{g}_Q(K') = \frac{1}{(2\pi)^3}\int g_Q(r')e^{-iK'r'}d^3r'. \tag{6b}$$

We will compare expression (5) with the Fourier transform of the unapproximated Schell-model source. On substituting the expression (1) for a Schell-model source into Eq. (4), and introducing the variables

$$R \equiv (r_1+r_2)/2,\ r \equiv r_2 - r_1, \tag{7}$$

we may rewrite the expression (4) for the Fourier transform as

$$\tilde{W}_Q(-k\mathbf{s}_1,k\mathbf{s}_2,\omega) = \frac{1}{(2\pi)^6}\int\int\sqrt{S_Q(R+r/2)}\sqrt{S_Q(R-r/2)}\,g_Q(r) \times e^{-ik(\mathbf{s}_2-\mathbf{s}_1)R}e^{-ik\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)r}d^3R d^3r. \tag{8}$$

In general, the R-integration in Eq. (8) is extremely difficult and cannot be evaluated analytically. One source

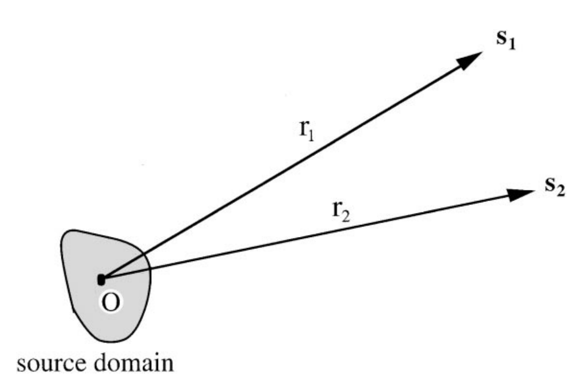

Fig. 2. Illustrating the notation used in connection with Eqs. (3) and (4).

commonly encountered for which the integration may be performed is a source with a Gaussian intensity profile, [2] i.e.,

$$S_Q(r,\omega) = S_0 e^{-r^2/\sigma_S^2}. \tag{9}$$

For this case, the R-integral in Eq. (8) can be evaluated, and that formula reduces to

$$\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2, \omega) = \frac{S_0(2\pi\sigma_S^2)^{3/2}}{(2\pi)^6} e^{-\sigma_S^2 k^2(\mathbf{s}_2-\mathbf{s}_1)^2/2} \times \int e^{-ik((\mathbf{s}_1+\mathbf{s}_2)/2)r} e^{-r^2/8\sigma_S^2} g_Q(r) d^3r. \tag{10}$$

Noting that the Fourier transform of the intensity (9) is

$$\tilde{S}_Q(K) = \frac{(2\pi)^{3/2}}{(2\pi)^3} S_0 \sigma_S^3 e^{-\sigma_S^2 K^2/2}, \tag{11}$$

we may write Eq. (10) in the suggestive form

$$\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2, \omega) = \tilde{S}_Q(k(\mathbf{s}_2-\mathbf{s}_1)) \frac{1}{(2\pi)^3} \times \int e^{-ik\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)r} e^{-r^2/8\sigma_S^2} g_Q(r) d^3r. \tag{12}$$

This expression has the form of the quasi-homogeneous approximation (5), save for the Gaussian term $\exp(-r^2/8\sigma_S^2)$ within the integral. This term represents the influence of the overlapped intensity integral from Eq. (8). Already it appears that the function g_Q must be significantly narrower than the width σ_S for Eq. (12) to have the form of the quasi-homogeneous approximation.

Let us simplify Eq. (12) by also choosing the spectral degree of coherence to be Gaussian, i.e.,

$$g_Q(r) = e^{-r^2/2\sigma_g^2}. \tag{13}$$

With this spectral degree of coherence, the Fourier transform of the cross-spectral density is readily found to be given by the expression

$$\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2, \omega) = \tilde{S}_Q(k(\mathbf{s}_2-\mathbf{s}_1)) \frac{\sigma_T^3}{(2\pi)^{3/2}} e^{-k^2\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)2\sigma 2_T/2}, \tag{14}$$

where

$$\sigma_T^2 = \frac{\sigma_S^2\sigma_g^2}{\sigma_g^2/4 + \sigma_S^2}. \tag{15}$$

By using Eqs. (9) and (13) in the quasi-homogeneous approximation, Eq. (5), we find that its Fourier transform is

$$\tilde{W}_Q^{qh}(-k\mathbf{s}_1, k\mathbf{s}_2, \omega) = \tilde{S}_Q(k(\mathbf{s}_2-\mathbf{s}_1)) \frac{\sigma_g^3}{(2\pi)^{3/2}} e^{-k^2\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)2\sigma 2_g/2}. \tag{16}$$

The difference between the unapproximated field (14) and the field given by the approximation (16) is given by

$$\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2, \omega) - \tilde{W}_Q^{qh}(-k\mathbf{s}_1, k\mathbf{s}_2, \omega) = \frac{\tilde{S}_Q(k(\mathbf{s}_2-\mathbf{s}_1))}{(2\pi)^{3/2}} \left[\sigma_T^3 e^{-k^2\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)\sigma 2_T/2} - \sigma_g^3 e^{-k^2\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)\sigma 2_g/2} \right]. \tag{17}$$

This difference represents the complete correction, in closed form, to the quasi-homogeneous approximation. We are interested in the magnitude of this term relative to the magnitude of the true Fourier transform of the source; when this term is small, the quasi-homogeneous approximation should be a good approximation. For a given pair of directions, this relative size may be characterized by the ratio

$$R(\mathbf{s}_1, \mathbf{s}_2) \equiv \left| \frac{\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2, \omega) - \tilde{W}_Q^{qh}(-k\mathbf{s}_1, k\mathbf{s}_2, \omega)}{\tilde{W}_Q^{qh}(-k\mathbf{s}_1, k\mathbf{s}_2, \omega)} \right|. \tag{18}$$

For our particular example, the ratio R is given by

$$R(\mathbf{s}_1, \mathbf{s}_2) = \left| 1 - \frac{\sigma_S^3}{\left[\sigma_S^2 + \sigma_g^2/4\right]^{3/2}} \times \exp\left[-\frac{K^2}{2} \left[\frac{\sigma_S^2\sigma_g^2}{\sigma_S^2 + \sigma_g^2/4} - \sigma_g^2 \right] \right] \right|, \tag{19}$$

where, $K = |K|$, and $K \equiv k(\mathbf{s}_1 + \mathbf{s}_2)/2$. Under what conditions, then, is this ratio negligible for all directions $\mathbf{s}_1$ and $\mathbf{s}_2$? Since $\mathbf{s}_1$ and $\mathbf{s}_2$ are unit vectors, $|K|$ may take on all values within the range

$$0 \le |K| \le k. \tag{20}$$

For the ratio to be roughly constant over all these values, the exponential must be slowly varying for all possible K values, implying that

$$\left| \frac{k^2}{2} \left[\frac{\sigma_S^2\sigma_g^2}{\sigma_S^2 + \sigma_g^2/4} - \sigma_g^2 \right] \right| \ll 1. \tag{21}$$

The expression on the left of Eq. (21) may be simplified and one finds

$$\frac{k^2\sigma_g^2}{1/4 + \sigma_S^2/\sigma_g^2} \ll 1. \tag{22}$$

[2] One might ask how to define the far zone of a source of infinite extent such as the one described in Eq. (9). One simple way to avoid difficulty is to consider Eq. (9) to be an approximation of a source truncated at a radius $L \ll \sigma_S$. This approximation, for sufficiently large L, will introduce negligible error as compared with the quasi-homogeneous approximation considered here.

The condition (22) is a *necessary* condition for the validity of the quasi-homogeneous approximation for our particular model. When this condition is satisfied, the exponential in Eq. (19) is approximately equal to unity for all possible K values. The ratio R will only be small, though, when

$$\frac{\sigma_S^3}{\left[\sigma_S^2+\sigma_g^2/4\right]^{3/2}}=\frac{1}{\left[1+\sigma_g^2/\left(4\sigma_S^2\right)\right]^{3/2}}\approx 1. \tag{23}$$

This approximate condition will only be satisfied when

$$\sigma_g/\sigma_S \ll 1. \tag{24}$$

Here we see the usual statement of the quasi-homogeneous approximation, that the width of the correlation function must be much smaller than the variation of the intensity profile. Our analysis has also suggested an additional constraint (22), which depends upon the wave number k. Noting that the denominator of Eq. (22) must be a very large quantity because of Eq. (24), Eq. (22) may be replaced by the simpler constraint

$$k^2\sigma_g^2=(2\pi)^2\frac{\sigma_g^2}{\lambda^2}\ll\sigma_S^2/\sigma_g^2. \tag{25}$$

This result appears to have been previously unappreciated in applications of the quasi-homogeneous approximation. It implies that if the wavelength of the radiation is significantly smaller than the range of correlations within the source, then the quasi-homogeneous approximation will not give an accurate representation of the radiation of the source. The meaning of this inequality can be understood as follows.

The Fourier transform at wavelength λ of a given function is sensitive to variations of that function over intervals on the order of a few wavelengths. In the quasi-homogeneous approximation, the width of the transformed function is σ_g, while the width of the modified correlation function is σ_{T}, given by Eq. (15). The difference between the two widths may be characterized by the expression

$$\Delta\sigma\equiv\sqrt{\left|\sigma_{\mathrm{T}}^2-\sigma_g^2\right|}\approx\sigma_g^2/2\sigma_S, \tag{26}$$

if Eq. (24) is satisfied. For the Fourier transform over the r variable to be insensitive to this difference, must be small relative to the wavelength, i.e.,

$$\Delta\sigma/\lambda\ll 1. \tag{27}$$

Using the relation $k=2\pi/\lambda$, this inequality is seen to be equivalent to Eq. (25).

The validity of the quasi-homogeneous approximation, therefore, depends not only upon the variations of the source intensity and correlation functions, but also upon the wavelength of the radiation. Eqs. (24) and (25) together comprise necessary and sufficient conditions for the validity of the quasi-homogeneous approximation for our particular model. Although these results were derived in this case for a simple class of sources, it seems clear that the quasi-homogeneous approximation will be influenced by the wavelength even for more complicated sources. Although that dependence may, in general, be quite complicated, the arguments leading to Eq. (27) indicate that the wavelength must be large compared to a distance $\Delta\sigma$ which will depend upon the width of the correlation function and the specific intensity profile.

3. Conclusion

This result is of particular interest for understanding the foundations of radiometry, in which quasi-homogeneous sources have played an important role [7–9]. It has been shown that one may construct a generalized radiance function from the cross-spectral density of a partially coherent source which in the limit $\lambda\to 0$ will satisfy all the requirements of radiometry *when the source is assumed to be quasi-homogeneous*. We have shown that the quasi-homogeneous approximation depends upon wavelength, and the satisfaction of our inequality (25) seems to be in conflict with the limit $\lambda\to 0$. Apparently, one must be more careful in using calculations with this radiometric limit. Eq. (25) can only be satisfied for $\lambda\to 0$ if one also takes $\sigma_g\to 0$, i.e., if one considers a completely incoherent source. Realistic sources, with small but nonzero correlation length and wavelength, will apparently approximately satisfy the postulates of radiometry provided they also satisfy the new condition (25). The true importance of this condition still remains to be fully clarified.

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