

Nonpropagating string excitations – finite length and damped strings

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Abstract

The existence of localized nonpropagating displacements of an infinite undamped string generated by a monochromatic driving force was recently demonstrated theoretically. In this paper, the theory of these nonpropagating excitations is generalized to the cases of (1) strings of finite length, (2) damped strings of infinite length, and (3) strings of infinite length driven by a force whose oscillations have a finite bandwidth. ©1999 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

In a previous publication [1], it was demonstrated that there exist localized monochromatic force densities which, when applied to an infinite undamped string, generate no displacement of the string outside the region of the applied force. These nonpropagating excitations are analogous to localized fields generated by non-radiating sources in radiation theory, usually discussed in three dimensions [2]¹. However, the circumstances under which excitations of this kind were shown to exist are not in practice realizable. A real string will be finite, and its vibrations will be damped. Moreover, the oscillations of the driving force will never be strictly monochromatic.

In this paper, we extend the theory of nonpropagating excitations to the cases of (1) strings of finite length, (2) damped strings of infinite length, and (3) strings of infinite length driven by a force whose oscillations have a finite bandwidth.

2. Nonpropagating excitations on a string of finite length

We first consider excitations upon a string of finite length, fixed at the endpoints $x = -L$, $x = L$. The displacement satisfies the wave equation [3]

$$\mu \frac{\partial^2 y(x, t)}{\partial t^2} - T \frac{\partial^2 y(x, t)}{\partial x^2} = f(x, t), \quad (1)$$

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¹ Further references on nonradiating sources can be found in [1].

where μ is the mass per unit length of the string, T is the tension in the string, and $f(x, t)$ is the applied force density (force per unit length). Waves on the string are subject to the boundary conditions

$$y(-L, t) = y(L, t) = 0. \quad (2)$$

To begin with, we restrict ourselves to simple harmonic driving forces, of the form

$$f(x, t) \equiv \operatorname{Re}\{f(x) e^{-i\omega t}\}, \quad (3)$$

where Re denotes the real part. The steady state solution of Eq. (1) will have the same form, viz.,

$$y(x, t) \equiv \operatorname{Re}\{y(x) e^{-i\omega t}\}. \quad (4)$$

Eq. (1) then reduces to the inhomogeneous one-dimensional Helmholtz equation,

$$\left[\frac{d^2}{dx^2} + k^2 \right] y(x) = q(x), \quad (5)$$

subject to the boundary conditions

$$y(-L) = y(L) = 0. \quad (6)$$

In Eq. (5),

$$k = \frac{\omega}{v}, \quad v = \sqrt{\frac{T}{\mu}}, \quad (7)$$

and

$$q(x) = -\frac{f(x)}{T}. \quad (8)$$

Here k represents the wave number and v represents the speed of the propagation of waves on the string.

The steady-state solution of Eq. (5), for a given force density $q(x)$ localized in the region $a \leq x \leq b$, is given by [3]

$$y(x) = \int_a^b G(x, x') q(x') dx', \quad (9)$$

with the Green's function

$$G(x, x') = \frac{1}{k \sin 2kL} \times \begin{cases} \sin k(x+L) \sin k(x'-L), & -L < x < x' < L, \\ \sin k(x-L) \sin k(x'+L), & -L < x' < x < L. \end{cases} \quad (10)$$

To begin with, we assume the string is driven off-resonance, i.e., $2kL \neq n\pi$, n being any positive integer. In such a case, the Green's function is well-behaved, and the displacement to the right (R) and left (L) of the applied force is given by the expressions

$$y(x)|_R = \frac{1}{k \sin 2kL} \sin k(x-L) \left[\int_a^b \sin k(x'+L) q(x') dx' \right], \quad (11)$$

$$y(x)|_L = \frac{1}{k \sin 2kL} \sin k(x+L) \left[\int_a^b \sin k(x'-L) q(x') dx' \right]. \quad (12)$$

Using an elementary trigonometric identity in Eqs. (11) and (12), we find that the solutions will be nonpropagating, i.e. $y(x) = 0$ for $x < a$ and $x > b$, when

$$\int_a^b \sin kx' q(x') dx' = \int_a^b \cos kx' q(x') dx' = 0 \quad (13)$$

or, equivalently, when

$$\int_a^b e^{-ikx'} q(x') dx' = \int_a^b e^{ikx'} q(x') dx' = 0. \quad (14)$$

On comparing these relations with Eqs. (9) and (10) of [1], we see that the conditions for nonpropagation on a finite string with end points fixed are the same as the conditions for an infinite string. This result is perhaps not surprising, because the nonpropagating solutions vanish identically outside a finite domain and it does not matter where the ends of the string are, or what boundary conditions are placed upon them. The constraint that the end points of the string are fixed does not influence the existence of nonpropagating excitations. We might expect, therefore, that the nonpropagating solutions should be independent of the length $2L$ of the string, and be well-behaved even for values of kL associated with resonance, i.e. when

$$2kL = n\pi. \quad (15)$$

Using Eq. (9) to determine the displacement in the interior of the region of the applied force (labeled by the subscript IN), we find that

$$\begin{aligned} y(x)|_{\text{IN}} = \frac{1}{k \sin 2kL} & \left\{ \sin kx \cos^2 kL \int_a^b q(x') \sin kx' dx' - \cos kx \sin^2 kL \int_a^b q(x') \cos kx' dx' \right. \\ & + \frac{1}{2} \cos kx \sin 2kL \int_a^b \text{sgn}(x' - x) q(x') \sin kx' dx' \\ & \left. - \frac{1}{2} \sin kx \sin 2kL \int_a^b \text{sgn}(x' - x) q(x') \cos kx' dx' \right\}, \end{aligned} \quad (16)$$

where

$$\text{sgn}(x' - x) = \begin{cases} 1 & x' > x \\ -1 & x' < x. \end{cases} \quad (17)$$

Since a nonpropagating excitation satisfies Eq. (13), the first two terms of Eq. (16) vanish and the displacement within the region of the applied force becomes

$$\begin{aligned} y(x)|_{\text{IN}} &= \frac{1}{2k} \left\{ \cos kx \int_a^b \text{sgn}(x' - x) q(x') \sin kx' dx' - \sin kx \int_a^b \text{sgn}(x' - x) q(x') \cos kx' dx' \right\} \\ &= \frac{1}{2k} \int_a^b \sin(k|x - x'|) q(x') dx', \end{aligned} \quad (18)$$

which, as expected, is independent of the length of the string.

3. Damped strings of infinite length

Let us now return to an infinitely long string, and consider the effect of a damping force per unit length $R\partial y/\partial t$ (where R is a constant) upon the existence and behavior of nonpropagating excitations. In this case we have, instead

of Eq. (1), the more general wave equation [3]

$$\mu \frac{\partial^2 y(x, t)}{\partial t^2} = T \frac{\partial^2 y(x, t)}{\partial x^2} - R \frac{\partial y(x, t)}{\partial t} + f(x, t). \quad (19)$$

Restricting ourselves to simple harmonic driving forces and the corresponding steady-state solutions (3) and (4), Eq. (19) reduces to a one-dimensional inhomogeneous Helmholtz equation with a complex wave number,

$$\frac{d^2 y(x)}{dx^2} + \left[\frac{\omega^2}{T/\mu} + i \frac{\omega R}{T} \right] y(x) = -\frac{f(x)}{T} = q(x). \quad (20)$$

The solution to Eq. (20) can be shown to be²

$$y(x) = \frac{1}{2i\beta} \int_a^b e^{i\beta|x-x'|} q(x') dx', \quad (21)$$

with

$$\beta \equiv \sqrt{\frac{\omega^2}{T/\mu} + i \frac{\omega R}{T}} = k \sqrt{\frac{1}{2} \left(\left[1 + \left(\frac{R}{\mu\omega} \right)^2 \right]^{1/2} + 1 \right)} + ik \sqrt{\frac{1}{2} \left(\left[1 + \left(\frac{R}{\mu\omega} \right)^2 \right]^{1/2} - 1 \right)}. \quad (22)$$

Here k , as before, is given by Eq. (7). This solution Eq. (21) represents exponentially damped waves propagating away from the region of the applied force. In the limit of weak damping, i.e. when

$$R \ll \mu\omega, \quad (23)$$

β may be approximated as

$$\beta \approx \frac{\omega}{v} + \frac{1}{2} \frac{iR}{\mu v} = k + i\alpha, \quad \alpha \equiv \frac{R}{2\mu v}. \quad (24)$$

This approximation is likely to hold for many situations of practical interest.

To the left (L) and right (R) of the applied force, the excitation is given by the expressions

$$y(x)|_L = \frac{1}{2i\beta} e^{-i\beta x} \int_a^b e^{i\beta x'} q(x') dx'. \quad (25)$$

$$y(x)|_R = \frac{1}{2i\beta} e^{i\beta x} \int_a^b e^{-i\beta x'} q(x') dx'. \quad (26)$$

These excitations will vanish outside the region of the applied force if and only if

$$\int_a^b e^{ikx' - \alpha x'} q(x') dx' = 0, \quad (27)$$

$$\int_a^b e^{-ikx' + \alpha x'} q(x') dx' = 0, \quad (28)$$

where the expression (24) for β was used. The appearance of the term α in the exponential term in Eqs. (27) and (28) results in a non-trivial departure from the theory of the undamped string.

² Eq. (19) is an obvious modification of the solution for the undamped infinite string, discussed in [3].

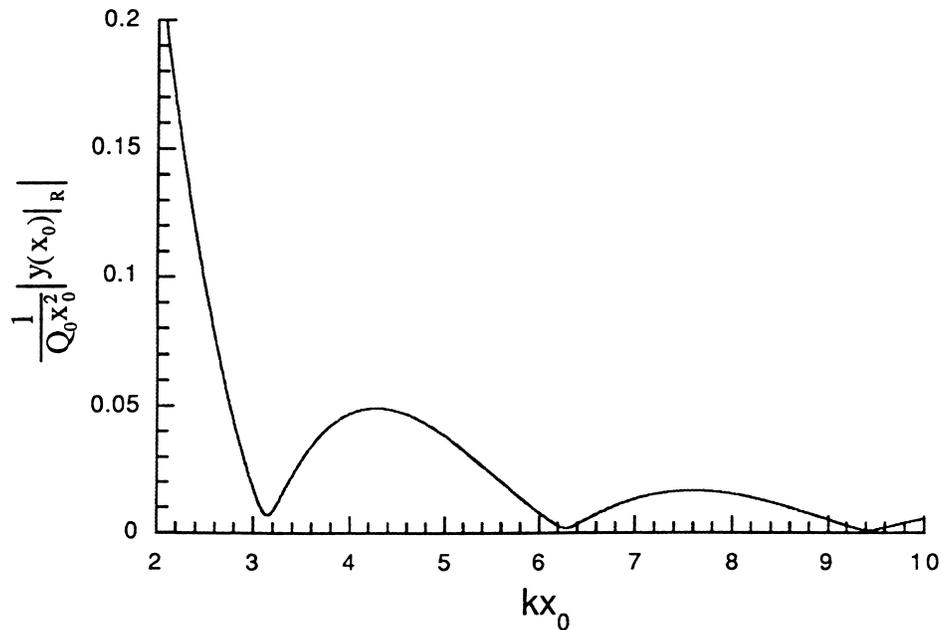


Fig. 1. The (normalized) amplitude $|y(x_0)|_R/Q_0x_0^2$ of waves propagating to the right of the region of the applied force, evaluated at the boundary x_0 . Though never strictly zero, the wave amplitude has a minimum, approximately when the frequency satisfies Eq. (30). In this example, $\alpha x_0 = 0.07$.

To see the difference, consider the step function force

$$q(x') = \begin{cases} Q_0 & |x'| \leq x_0 \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

In [1], the force density represented here by Eq. (29) was shown to give nonpropagating solutions only for values of kx_0 such that

$$kx_0 = n\pi, \quad (n = 1, 2, 3, \dots). \tag{30}$$

However, we are now interested in learning about nonpropagating solutions which may exist for this force distribution on a damped string. Using this force density in Eqs. (27) and (28), we find that the step function force will generate nonpropagating excitations if and only if

$$\tan kx_0 + i \tanh \alpha x_0 = 0. \tag{31}$$

This equation can be satisfied only for $\alpha = 0$, i.e. for the undamped case. On a string with damped oscillations, the step function force never generates nonpropagating excitations. The special values of kx_0 given by Eq. (30) are now only approximate minima in the displacement amplitude. In order to see this, let us consider the amplitude of the field to the right of the region of applied force in the limit as $x \rightarrow x_0$,

$$|y(x_0)|_R = \frac{1}{\sqrt{2}} \frac{Q_0}{[k^2 + \alpha^2]} e^{-\alpha x_0} [\cosh 2\alpha x_0 - \cos 2kx_0]^{1/2}. \tag{32}$$

If we vary the wave number k , we see that for $2\alpha x_0 \ll 1$, the minima of intensity occur roughly at the values of kx_0 given by Eq. (30) (See Fig. 1.)

This example raises the question of whether or not nonpropagating excitations can exist on a damped string. It is clear, though, that the displacement $y(x)$ and the force distribution that generates it are still related by a Helmholtz

equation. Hence, by the same arguments as those given in the Appendix of [1], the field and its derivative must be continuous at all points. Theorem 1 from that paper can then be modified for the case of a damped string as follows.

Theorem: *A nonpropagating excitation on an infinitely long damped string and the piecewise continuous force distribution confined to the region $a \leq x \leq b$ that generates it are related by the one-dimensional inhomogeneous Helmholtz equation, Eq. (20), with complex coefficients, subject to the conditions*

$$y(a) = y(b) = 0, \quad \left. \frac{dy}{dx} \right|_{x=a} = \left. \frac{dy}{dx} \right|_{x=b} = 0. \quad (33)$$

In general, the relation between the displacement of a localized excitation and the force distribution that generates it, is more complicated for solutions on the damped string, due to the complex coefficient in the Helmholtz equation (20). One can, however, create nonpropagating excitations with relatively simple force distributions, as the following example shows. Let

$$q(x) = \begin{cases} Q_0 & |x| \leq x_0 \\ Q_1 & x_0 \leq |x| \leq x_1 \\ 0 & x_1 \leq |x|. \end{cases} \quad (34)$$

Since this force distribution is symmetric about the point $x = 0$, we need only to satisfy one of the Eqs. (27) and (28). We will attempt to choose the coefficients in Eq. (34) to satisfy Eq. (27)

$$\int_{-x_1}^{x_1} e^{ikx' - \alpha x'} q(x') dx' = Q_0 \int_{-x_0}^{x_0} e^{ikx' - \alpha x'} dx' + Q_1 \left\{ \int_{-x_1}^{-x_0} e^{ikx' - \alpha x'} dx' + \int_{x_0}^{x_1} e^{ikx' - \alpha x'} dx' \right\}. \quad (35)$$

This relation leads, after some calculation, to the formula

$$\int_{-x_1}^{x_1} e^{ikx' - \alpha x'} q(x') dx' = \frac{2i}{ik - \alpha} \{ Q_1 [\sin kx_1 \cosh \alpha x_1 + i \sinh \alpha x_1 \cos kx_1] \\ + [Q_0 - Q_1] [\sin kx_0 \cosh \alpha x_0 + i \sinh \alpha x_0 \cos kx_0] \}. \quad (36)$$

To satisfy Eq. (27), we therefore require that the real and imaginary parts of Eq. (36) vanish, i.e. that

$$Q_1 \sin kx_1 \cosh \alpha x_1 + [Q_0 - Q_1] \sin kx_0 \cosh \alpha x_0 = 0, \quad (37)$$

$$Q_1 \sinh \alpha x_1 \cos kx_1 + [Q_0 - Q_1] \sinh \alpha x_0 \cos kx_0 = 0. \quad (38)$$

This set of equations has many solutions; we will only consider one of them. First, we note that if we choose

$$x_1 = \frac{n\pi}{k}, \quad x_0 = m\pi k, \quad n > m > 0, \quad (39)$$

Eq. (37) is automatically satisfied, and Eq. (38) reduces to

$$(-1)^{n-m} \frac{Q_1 - Q_0}{Q_1} = \frac{\sinh \alpha x_1}{\sinh \alpha x_0}. \quad (40)$$

Let us choose both m and n to be even. The right-hand side of Eq. (40) exceeds unity and it is therefore clear that there exist choices $Q_0 < 0$, $Q_1 > 0$ such that Eq. (40), and consequently Eqs. (27) and (28), are satisfied. An example of such a force distribution is shown in Fig. 2; unlike the step function force given by Eq. (29), it generates true nonpropagating excitations on a damped string.

We may also examine nonpropagating excitations on a finite damped string with fixed end points. It is clear, though, from the arguments given in this section and in Section 2, that the results are unchanged when the string is of finite length. The nonpropagating excitations, damped or undamped, are not influenced by the length of the string.

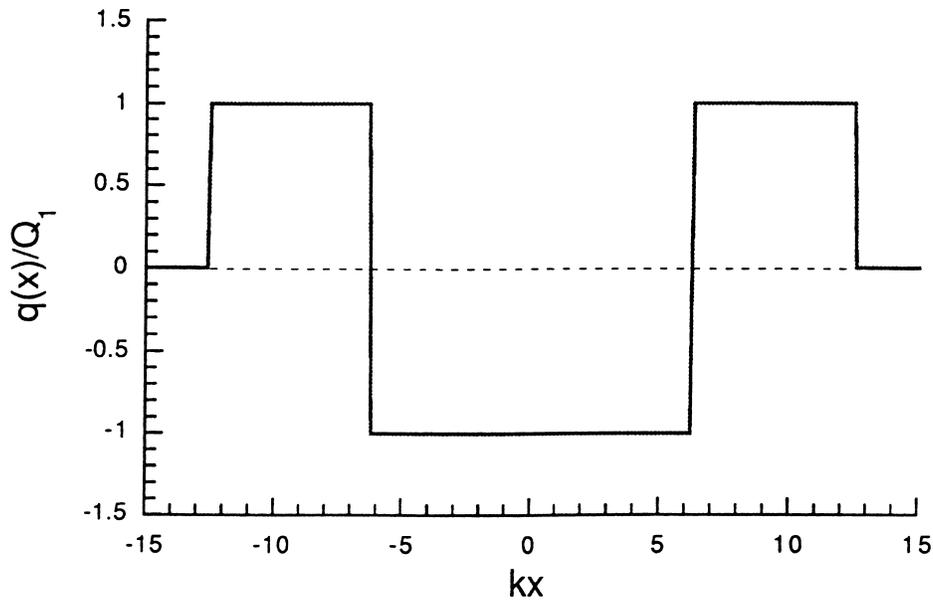


Fig. 2. An example of a force distribution which generates nonpropagating excitations on an infinite damped string, as given by Eq. (34). Here $kx_0 = 2\pi$, $kx_1 = 4\pi$, and $\alpha/k = 0.01$.

4. Quasi-monochromatic driving forces

In [1], it was demonstrated that a monochromatic step function driving force generates nonpropagating displacements only for certain frequencies, given by Eq. (30). We will now consider a similar force distribution on an infinite undamped string which has oscillations over a range of frequencies, in order to learn how nonpropagating solutions affect the behavior of waves on the string. More specifically, we consider a simple quasi-monochromatic force distribution with random phase fluctuations $\alpha(t)$ and center frequency ω_0 , given by the expression

$$f(x, t) = 2F(x) \cos [\alpha(t) - \omega_0 t], \tag{41}$$

where

$$F(x) = \begin{cases} F_0 & |x| \leq x_0 \\ 0 & |x| > x_0, \end{cases} \tag{42}$$

and $\alpha(t)$ is a stationary random function. The second order correlation properties of the force density are characterized by the function

$$\Gamma_f(x_1, x_2, \tau) = \langle f_A^*(x_1, t) f_A(x_2, t + \tau) \rangle, \tag{43}$$

where $f_A(x, t)$ is the complex analytic signal representation of $f(x, t)$ [4] and the sharp brackets denote an ensemble average. It can be shown that, for a quasi-monochromatic signal of the form of Eq. (41), one has to a good approximation [5]

$$f_A(x, t) = F(x) e^{i[\alpha(t) - \omega_0 t]}. \tag{44}$$

On substituting this expression into Eq. (43), one finds that

$$\Gamma_f(x_1, x_2, \tau) = F(x_1) F(x_2) \langle e^{-i\alpha(t)} e^{i\alpha(t+\tau)} \rangle e^{-i\omega_0 \tau}. \tag{45}$$

Let us assume that the ensemble average in Eq. (45) has the form of a Gaussian distribution, i.e. that

$$\langle e^{-i\alpha(t)} e^{i\alpha(t+\tau)} \rangle = e^{-\tau^2 \sigma^2 / 2}. \quad (46)$$

The cross-spectral density of the source is then given by the expression

$$W_f(x_1, x_2, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma_f(x_1, x_2, \tau) e^{i\omega\tau} d\tau = F(x_1)F(x_2) \frac{1}{\sqrt{2\pi}\sigma} e^{-(\omega-\omega_0)^2/2\sigma^2}. \quad (47)$$

We show in the Appendix that the cross-spectral density of the displacement at points to the right of the source is given by the formula

$$W_y(x_1, x_2, \omega) = \frac{(2\pi)^2}{4k^2 T^2} \tilde{W}_f(-k, k, \omega) e^{-ik(x_1-x_2)}, \quad (48)$$

where \tilde{W}_f is the two-dimensional Fourier transform of the force density, i.e.

$$\tilde{W}_f(-k_1, k_2, \omega) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_f(x_1, x_2, \omega) e^{ik_1 x_1 - ik_2 x_2} dx_1 dx_2. \quad (49)$$

The total ‘wave intensity’, here labeled $I(x)$, is given by the formula

$$I(x) = \langle y_A^*(x, t) y_A(x, t) \rangle \equiv \Gamma_y(x, x, 0) = \int_0^{\infty} W_y(x, x, \omega) d\omega, \quad (50)$$

where $y_A(x, t)$ is the complex analytic signal representation of $y(x, t)$. Substituting from Eqs. (43), (48) and (49) into Eq. (50), the wave intensity is found to be given by the expression

$$I(x) = \frac{F_0^2}{T^2} \int_0^{\infty} \frac{[\sin kx_0]^2}{k^4} \frac{1}{\sqrt{2\pi}\sigma} e^{-(\omega-\omega_0)^2/2\sigma^2} d\omega. \quad (51)$$

This expression is, of course, only approximate because of the approximation leading to Eq. (44). Any realistic force spectrum would not include a constant ($\omega = 0$) driving component. In the calculations which follow, we will neglect all components of the force spectrum beyond a distance 3σ from the center frequency.

It is clear from Eq. (51) that for this force density, nonpropagating solutions do not exist. Nevertheless, the nonpropagating phenomenon still affects the behavior of outgoing waves. To see this, consider a situation as described above for which the center frequency ω_0 of the driving force may be adjusted. If the bandwidth of the driving force is sufficiently narrow, one would expect the intensity $I(x)$ of the waves propagating away from the region of applied force to approach local minima as the center frequency approaches values for which $\sin(\omega_0 x_0/v) = 0$, i.e. values for which $\omega_0/v \rightarrow n\pi/x_0$. This effect can be seen in Fig. 3. These special frequencies correspond to the frequencies for which a constant monochromatic force distribution produces nonpropagating solutions.

The example presented here illustrates the phenomenon of correlation-induced spectral changes which has attracted a good deal of attention in recent years [7]. In the present case, the force spectrum is independent of position throughout the region of the applied force and is given by the expression

$$S_f(x, \omega) \equiv W_f(x, x, \omega) = |F_0|^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-(\omega-\omega_0)^2/2\sigma^2}, \quad |x| \leq x_0. \quad (52)$$

The spectrum of waves propagating away from the region of applied force, however, is given by the expression

$$S_y(x, \omega) \equiv W_y(x, x, \omega) = \frac{F_0^2}{T^2} \frac{[\sin kx_0]^2}{k^4} \frac{1}{\sqrt{2\pi}\sigma} e^{-(\omega-\omega_0)^2/2\sigma^2}, \quad (53)$$

which is clearly different from the spectrum S_f of the applied force.

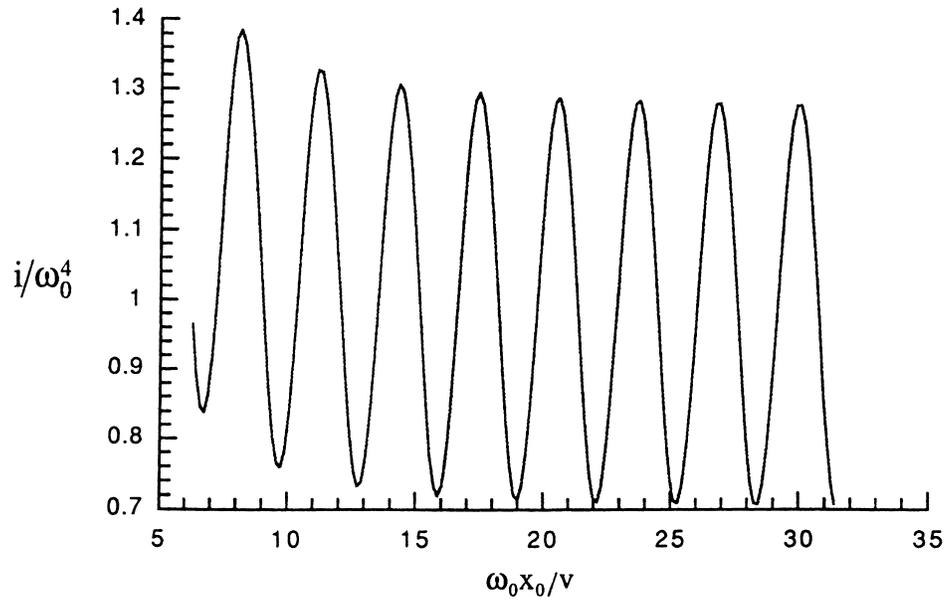


Fig. 3. The normalized intensity i of waves propagating away from the region of applied force, $i = \sqrt{2\pi}\sigma T^2 I(x)/F_0^2$, scaled by the fourth power of the center frequency ω_0 , with the choice $\sigma x_0/v = \pi/4$. $I(x)$ is given by Eq. (51), and i is independent of x .

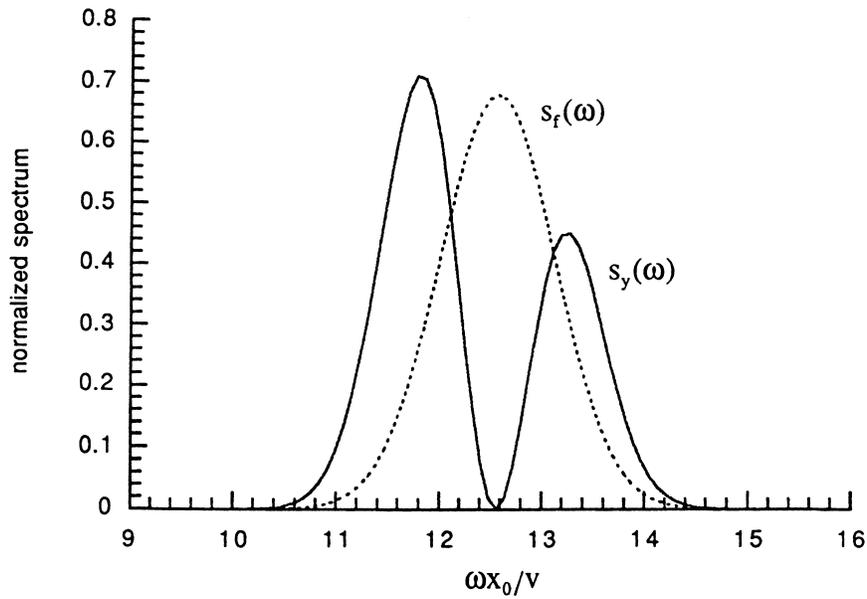


Fig. 4. The normalized spectrum of the applied force density and the string displacement, with the choice $\omega_0 x_0/v = 4\pi$, $\sigma x_0/v = \pi/4$. The normalized spectrum of the force density is defined by the expression $s_f(\omega) \equiv (S_f(x, \omega))/(\int_0^\infty S_f(x, \omega') d\omega')$ with a similar expression for $s_y(\omega)$, the normalized string displacement spectrum.

The normalized spectra $s_f(\omega)$ and $s_y(\omega)$ are plotted in Fig. 4 for the case when $\omega_0 x_0/v = 4\pi$, $\sigma x_0/v = \pi/4$. We see that whilst the force spectrum has a single peak, the wave spectrum has two peaks because the frequency $\omega = 4\pi v/x_0$ is nonpropagating.

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Appendix A. Derivation of Eq. 48

In order to derive the formula (48) we will use the space-frequency domain formulation of coherence theory [6]. According to this theory, the cross-spectral density of a fluctuating wavefield such as $y(x, t)$ may be represented as an average taken over a suitable chosen ensemble of strictly monochromatic wavefields $\{\tilde{y}(x, \omega)e^{-i\omega t}\}$, all of the same angular frequency ω , in the form

$$W_y(x_1, x_2, \omega) = \langle \tilde{y}^*(x_1, \omega) \tilde{y}(x_2, \omega) \rangle. \quad (\text{A.1})$$

A similar representation exists for the cross-spectral density function of the source

$$W_f(x_1, x_2, \omega) = \langle \tilde{f}^*(x_1, \omega) \tilde{f}(x_2, \omega) \rangle. \quad (\text{A.2})$$

It follows from Eqs. (6) and (8) of [1] that to the right of the source, $x > b$,

$$\tilde{y}(x, \omega) = -\frac{e^{ikx}}{2ikT} \int_a^b \tilde{f}(x', \omega) e^{-ikx'} dx'. \quad (\text{A.3})$$

Upon substituting this result into the left-hand side of Eq. (A.1) and using Eq. (A.2), we find that

$$W_y(x_1, x_2, \omega) = \frac{e^{ik(x_2-x_1)}}{(2kT)^2} \int_a^b \int_a^b W_f(x'_1, x'_2, \omega) e^{ik(x'_1-x'_2)} dx'_1 dx'_2 = \frac{(2\pi)^2}{(2kT)^2} \tilde{W}_f(-k, k, \omega) e^{ik(x_2-x_1)}, \quad (\text{A.4})$$

where \tilde{W}_f is defined by Eq. (49).

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