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Unpolarized sources that generate highly polarized fields outside the source

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Abstract. It is demonstrated that an unpolarized primary electromagnetic source may, under special conditions, produce a field outside the source domain that is almost completely polarized in nearly all directions. This result demonstrates that the polarization statistics of a random electromagnetic field may differ significantly from the polarization statistics of the source distribution that generates it, and may in fact be quite different in different directions of observation. An example of such a source is given.

In recent years it has been demonstrated that the spectrum of a field generated by a partially coherent source may differ from the spectrum of the source, and may in fact be different for different directions of observation. These correlation-induced spectral changes have been studied quite extensively since their discovery, both theoretically and experimentally [1].

More recently, it was shown that the spectral degree of coherence of a scalar field may differ significantly from the corresponding degree of coherence of the radiating source [2]. In particular, it was demonstrated that fluctuating scalar sources with quite different degrees of spatial coherence, even sources which are highly incoherent, can generate fields which are spatially completely coherent. This curious effect arises from the existence of non-radiating stochastic sources [3].

In this paper we demonstrate that the polarization properties of an electromagnetic field may be different than those of the source, and in fact radically different in different directions of observation. Such correlation-induced changes in the degree of polarization can be quite large, even for sources that are spatially highly incoherent. In particular, we will show that certain unpolarized electromagnetic sources can produce fields outside the source domain which are almost completely polarized in almost all directions.

Consider a fluctuating source polarization $\mathbf{P}(\mathbf{r}, t)$ that occupies a finite domain D, at a point specified by a position vector $\mathbf{r}$ in space at time t (see figure 1). We assume that the source fluctuations are statistically stationary, at least in the wide sense (see [4], p. 47).

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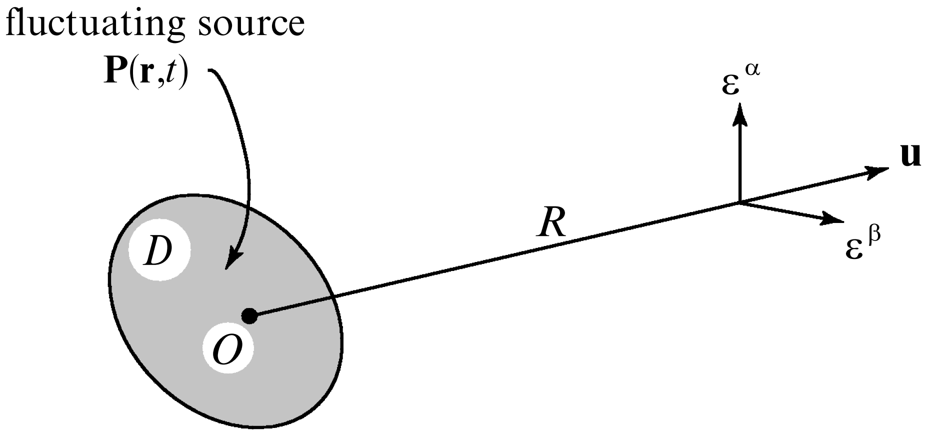

Figure 1. Illustrating notation relating to equations (1) through (4).

Let $W_{ij}^{(\mathrm{P})}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ be the cross-spectral density tensor (see [4], p. 371) of the source polarization at frequency ω. Here i, j represent Cartesian components of the source polarization. We will consider a polarization source for which the source fluctuations are quasi-homogeneous, i.e. such that $W_{ij}^{(\mathrm{P})}$ may be well-approximated in the form (see [4], p. 234)

$$W_{ij}^{(\mathrm{P})}(\mathbf{r}_1,\mathbf{r}_2,\omega) \approx S^{(\mathrm{P})}\Big(\frac{\mathbf{r}_1+\mathbf{r}_2}{2},\omega\Big)\mu_{ij}^{(\mathrm{P})}(\mathbf{r}_2-\mathbf{r}_1,\omega), \tag{1}$$

where $S^{(\mathrm{P})}(\mathbf{r},\omega)$ is the spectral density of the source and $\mu_{ij}^{(\mathrm{P})}(\mathbf{r}',\omega)$ is its spectral degree of coherence tensor. Moreover, $S^{(\mathrm{P})}(\mathbf{r},\omega)$ varies slowly with position and $\mu_{ij}^{(\mathrm{P})}(\mathbf{r}',\omega)$ is a very narrow function of position relative to $S^{(\mathrm{P})}(\mathbf{r},\omega)$ for all Cartesian components, labelled by the subscripts i, j.

Sufficiently far from the source domain D, the cross-spectral density tensor of the electric field may be expressed in the form [5]

$$W_{ij}^{(\mathrm{E})}(R\mathbf{u},R\mathbf{u},\omega) = \frac{k^4(2\pi)^6}{R^2}(\delta_{im}-u_iu_m)\big(\delta_{jn}-u_ju_n\big)\tilde{W}_{mn}^{(\mathrm{P})}(-k\mathbf{u},k\mathbf{u},\omega), \tag{2}$$

where R is the distance and $\mathbf{u}$ the direction from the source to the field point, u_i is the Cartesian component of $\mathbf{u}$, δ_{ij} is the Kronecker delta symbol and

$$\tilde{W}_{ij}^{(\mathrm{P})}(-k\mathbf{u},k\mathbf{u},\omega) = \frac{1}{(2\pi)^6}\int_D\int_D W_{ij}^{(\mathrm{P})}(\mathbf{r}_1,\mathbf{r}_2,\omega)\ \exp\left[-\mathrm{i}k\mathbf{u}\cdot(\mathbf{r}_2-\mathbf{r}_1)\right]\mathrm{d}^3r_1\mathrm{d}^3r_2 \tag{3}$$

is the six-dimensional spatial Fourier transform of the polarization tensor. Summation over repeated indices is implied. In the far zone, the electric field will only have components transverse to $\mathbf{u}$. The coherence matrix $M_{\alpha\beta}$ [6] (see also [4], p. 342) of the far field is defined as the projection of the far zone field tensor onto these transverse components, i.e.

$$M_{\alpha\beta}(R\mathbf{u},\omega) \equiv \epsilon_i^{\alpha}\epsilon_j^{\beta}W_{ij}^{(\mathrm{E})}(R\mathbf{u},R\mathbf{u},\omega), \tag{4}$$

where $\epsilon_i^{\alpha},\epsilon_j^{\beta}$ are the components of unit vectors $\epsilon^{\alpha},\epsilon^{\beta}$ perpendicular to $\mathbf{u}$. Substituting equation (2) into (4), and using the property that $\epsilon^{\alpha}\cdot\mathbf{u}=0$, the coherence matrix takes on the form

$$M_{\alpha\beta}(R\mathbf{u},\omega) = \frac{k^4(2\pi)^6}{R^2}\epsilon_i^\alpha \epsilon_j^\beta \tilde{W}_{ij}^{(\mathrm{P})}(-k\mathbf{u},k\mathbf{u},\omega). \tag{5}$$

The matrix $M_{\alpha\beta}$ describes the correlations which exist between components of the transverse electric field in the far zone. The *degree of polarization* of the field (see [4], p. 354) is then defined as

$$P(R\mathbf{u},\omega) = \left[1 - \frac{4\,\mathrm{Det}\left\{M_{\alpha\beta}(R\mathbf{u},\omega)\right\}}{\left(\mathrm{Tr}\left\{M_{\alpha\beta}(R\mathbf{u},\omega)\right\}\right)^2}\right]^{1/2}, \tag{6}$$

where Det and Tr denote the determinant and trace of the coherence matrix, respectively.

We will now demonstrate that it is possible for a source that is completely unpolarized to generate a field that is completely polarized in the far zone, i.e. a field such that $P(R\mathbf{u},\omega) = 1$ for nearly all directions of observation $\mathbf{u}$. A necessary and sufficient condition for the field to be completely polarized is that the coherence matrix has the form

$$M_{\alpha\beta} = \chi q_\alpha q_\beta, \tag{7}$$

where χ is a constant and $\mathbf{q}$ is a real 2-dimensional vector. A source is unpolarized if the cross-spectral density tensor at each point $\mathbf{r}$ obeys the following relationship:

$$W_{ij}^{(\mathrm{P})}(\mathbf{r},\mathbf{r},\omega) = \delta_{ij} S^{(\mathrm{P})}(\mathbf{r},\omega). \tag{8}$$

Now consider a quasi-homogeneous source polarization whose cross-spectral density tensor has the form

$$W_{ij}^{(\mathrm{P})}(\mathbf{r}_1,\mathbf{r}_2,\omega) \approx S^{(\mathrm{P})}\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2},\omega\right)\left\{\delta_{ij}A(\mathbf{r}_2-\mathbf{r}_1,\omega) + a_i a_j B(\mathbf{r}_2-\mathbf{r}_1,\omega)\right\}. \tag{9}$$

We will choose the functions A and B in such a way as to simultaneously satisfy equations (7) and (8). Substituting equation (9) into equation (5), the coherence matrix of the far-zone field radiated by such a source is given by the formula

$$M_{\alpha\beta}(R\mathbf{u},\omega) = \frac{k^4(2\pi)^6}{R^2}\tilde{S}^{(\mathrm{P})}(0,\omega)\left\{\epsilon_i^\alpha \epsilon_i^\beta \tilde{A}(k\mathbf{u},\omega) + (\epsilon_i^\alpha a_i)(\epsilon_j^\beta a_j)\tilde{B}(k\mathbf{u},\omega)\right\}. \tag{10}$$

On comparing equations (7) and (8) with equations (10) and (9) respectively, it is clear that the source is unpolarized if

$$B(0,\omega) = 0, \tag{11}$$

but the far-zone field of the source will be completely polarized if

$$\tilde{A}(k\mathbf{u},\omega) = 0. \tag{12}$$

The origin of this phenomenon is closely connected to the theory of non-radiating sources. Equation (12) indicates that the unpolarized part of the source does not manifest itself in any way in the far field. Some research on the theory of quasi-homogeneous sources, however, suggests that equation (12) cannot be satisfied exactly [7], i.e. that quasi-homogeneous sources are never exactly non-radiating. Nevertheless, a source of the form of equation (9) that satisfies equation (11) will produce an almost completely polarized field, provided that

$$\left|\tilde{A}(k\mathbf{u},\omega)\right| \ll \left|\tilde{B}(k\mathbf{u},\omega)\right|, \quad \text{for all } \mathbf{u}. \tag{13}$$

One might ask if a source which satisfies equations (9), (11) and (13) can possess all the properties of a valid correlation function. In the Appendix we demonstrate that this is so.

For a source which satisfies the inequality (13), the field will not be completely polarized in all directions. To see this, let us substitute equation (10) into equation (6). The degree of polarization is then found to be given by the expression

$$P(R\mathbf{u},\omega) = \frac{\left[(\epsilon^1\cdot\mathbf{a})^2+(\epsilon^2\cdot\mathbf{a})^2\right]\tilde{B}}{2\tilde{A}+\left[(\epsilon^1\cdot\mathbf{a})^2+(\epsilon^2\cdot\mathbf{a})^2\right]\tilde{B}}, \tag{14}$$

where the variables of $\tilde{A}$ and $\tilde{B}$ have been suppressed for brevity. In equation (14), ϵ^1, ϵ^2 are vectors perpendicular to $\mathbf{u}$ and therefore depend upon the direction $\mathbf{u}$. If θ denotes the angle between $\mathbf{a}$ and $\mathbf{u}$, equation (14) may be written in the simpler form

$$P(\theta,\omega) = \frac{\tilde{B}\sin^2\theta}{2\tilde{A}+\tilde{B}\sin^2\theta}. \tag{15}$$

If the Fourier transform of A vanishes for all directions $\mathbf{u}$, the degree of polarization will be unity for all directions. However, if $\tilde{A}(k\mathbf{u},\omega)$ is small compared to the Fourier transform of B but non-zero, the degree of polarization will vanish in directions parallel and antiparallel to $\mathbf{a}$.

As an example, consider the case when

$$A(\mathbf{r},\omega) = \frac{\sin qr}{qr}\exp[-r^2/2\sigma^2], \tag{16 a}$$

$$B(\mathbf{r},\omega) = \exp[-r^2/2\sigma^2] - \frac{\sin qr}{qr}\exp[-r^2/2\sigma^2]. \tag{16 b}$$

The radial dependence of these functions is displayed in figure 2. $A(\mathbf{r},\omega)$ has been chosen as the product of two non-negative definite, Hermitian functions, which

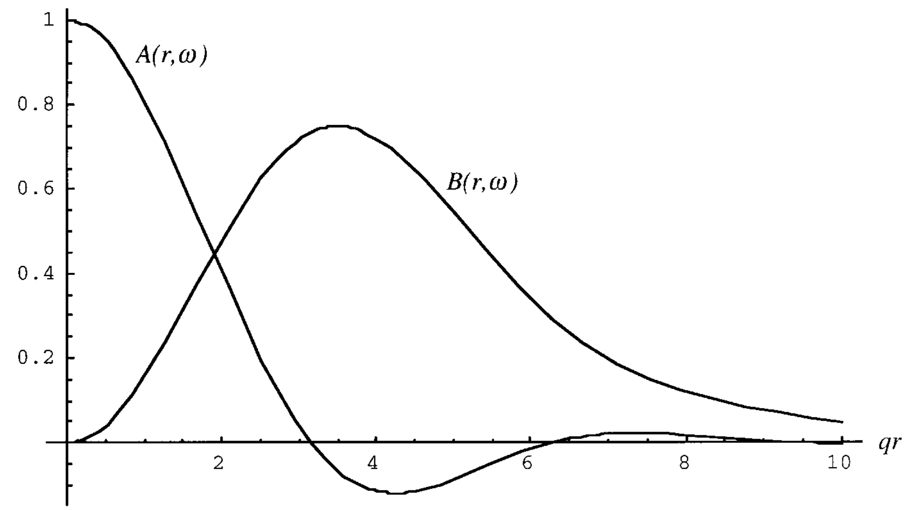

Figure 2. The radial dependence of the functions $A(\mathbf{r},\omega)$ and $B(\mathbf{r},\omega)$, given by equations (16 a) and (16 b), with $q\sigma = 4$.

will itself be non-negative definite and Hermitian [8]. The Fourier transforms of these functions are

$$\tilde{A}(k\mathbf{u},\omega) = \frac{\sigma^3}{(2\pi)^{3/2}} \exp\left(\frac{-k^2\sigma^2}{2}\right) \exp\left(\frac{-q^2\sigma^2}{2}\right) \frac{\sinh[kq\sigma^2]}{kq\sigma^2}, \tag{17a}$$

$$\tilde{B}(k\mathbf{u},\omega) = \frac{\sigma^3}{(2\pi)^{3/2}} \exp\left(\frac{-k^2\sigma^2}{2}\right) \left[1 - \exp\left(\frac{-q^2\sigma^2}{2}\right) \frac{\sinh[kq\sigma^2]}{kq\sigma^2}\right]. \tag{17b}$$

Both these functions are independent of the direction $\mathbf{u}$. Substituting these expressions into the inequality (13), it is clear that the source polarization will produce an almost perfectly polarized field if

$$\exp\left(\frac{-q^2\sigma^2}{2}\right) \frac{\sinh[kq\sigma^2]}{kq\sigma^2} \ll 1. \tag{18}$$

There are two undetermined parameters in this inequality, namely $q\sigma$ and $k\sigma$. Because of the rapid rate of decay of the Gaussian as compared to the growth of the sinh function, it is possible to choose these two parameters to satisfy the inequality (18). Such a field will, as discussed above, produce an almost perfectly polarized field for almost all directions of observation. Figure 3 shows the dependence of $P(\theta,\omega)$ upon θ for several values of $q\sigma$ and for a fixed value of $k\sigma$.

This example demonstrates that the polarization properties of a source distribution do not necessarily reflect themselves in the field generated by the source. This effect, like the correlation-induced spectral changes mentioned earlier [1], is a

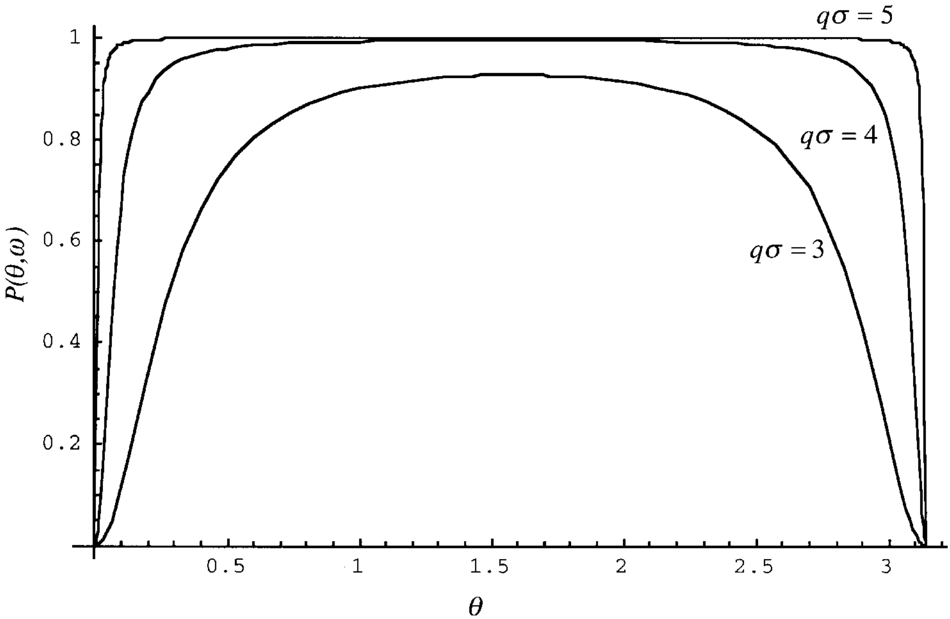

Figure 3. The degree of polarization $P(\theta,\omega)$ of the electric field in the far zone of the source, for various values of the parameter $q\sigma$. Here k$\sigma = 1$. The angle θ is the angle between the direction of observation $\mathbf{u}$ and the fixed unit vector $\mathbf{a}$. For larger values of the parameter $q\sigma$, the degree of polarization can be made arbitrarily close to unity for nearly all directions θ.

consequence of the spatial coherence of the source, described by the functions $A(\mathbf{r},\omega)$, $B(\mathbf{r},\omega)$, and some degree of anisotropy of the polarization of the source, as described by the vector a_i. Although, in general, any polarization changes will not be as extreme as those described here, any research involving sources with appreciable spatial coherence should take into account the possibility of such changes.

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Appendix: Non-negative definiteness and Hermicity of the polarization cross-spectral tensor

For a tensor $W_{ij}^{(\mathrm{P})}$ to be a valid cross-spectral density tensor of a source polarization, it must be Hermitian,

$$\left[W_{ij}^{(\mathrm{P})}(\mathbf{r}_1,\mathbf{r}_2,\omega)\right]^* = W_{ji}^{(\mathrm{P})}(\mathbf{r}_2,\mathbf{r}_1,\omega) \tag{A 1}$$

and it must be non-negative definite, i.e.

$$\int_D \int_D f_i^*(\mathbf{r}_1) f_j(\mathbf{r}_2) W_{ij}^{(\mathrm{P})}(\mathbf{r}_1,\mathbf{r}_2,\omega)\,\mathrm{d}^3 r_1\,\mathrm{d}^3 r_2 \geq 0 \tag{A 2}$$

for all well-behaved vector functions $f_i(\mathbf{r})$(see [4], section 6.6.1).

Let us consider the source distribution described by equation (9), namely,

$$W_{ij}^{(\mathrm{P})}(\mathbf{r}_1,\mathbf{r}_2,\omega) \approx S^{(\mathrm{P})}\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2},\omega\right)\left\{\delta_{ij}A(\mathbf{r}_2-\mathbf{r}_1,\omega) + a_i a_j B(\mathbf{r}_2-\mathbf{r}_1,\omega)\right\}. \tag{A 3}$$

This tensor will be Hermitian if the functions A, B are chosen to be real and dependent only upon the magnitude of the difference vector $\mathbf{r}_2 - \mathbf{r}_1$, i.e. one must have

$$A(\mathbf{r}_2-\mathbf{r}_1,\omega) = A(|\mathbf{r}_2-\mathbf{r}_1|,\omega), \quad B(\mathbf{r}_2-\mathbf{r}_1,\omega) = B(|\mathbf{r}_2-\mathbf{r}_1|,\omega). \tag{A 4}$$

As regards non-negative definiteness, it is to be noted that the Kronecker delta δ_{ij} may be written as the direct product of three orthogonal unit vectors,

$$\delta_{ij} = a_i a_j + a_i^{(2)} a_j^{(2)} + a_i^{(3)} a_j^{(3)}, \tag{A 5}$$

where a_i is the same unit vector as in equation (A 3). We may use this expression to rewrite equation (A 3) in the form

$$W_{ij}^{(\mathrm{P})}(\mathbf{r}_1,\mathbf{r}_2,\omega) = S^{(\mathrm{P})}\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2},\omega\right) \times \left\{[A(\mathbf{r},\omega)+B(\mathbf{r},\omega)]a_i a_j + A(\mathbf{r},\omega)[a_i^{(2)} a_j^{(2)} + a_i^{(3)} a_j^{(3)}]\right\}, \tag{A 6}$$

where $\mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1$. Substituting this expression into the non-negative definiteness condition, equation (A 2), we find that

$$\int_D \int_D S^{(\mathrm{P})}\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2},\omega\right)\{[A(\mathbf{r},\omega)+B(\mathbf{r},\omega)]f_a^*(\mathbf{r}_1)f_a(\mathbf{r}_2) + A(\mathbf{r},\omega)\left[f_{a^{(2)}}^*(\mathbf{r}_1)f_{a^{(2)}}(\mathbf{r}_2)+f_{a^{(3)}}^*(\mathbf{r}_1)f_{a^{(3)}}(\mathbf{r}_2)\right]\} \geq 0, \quad \text{(A 7)}$$

where $f_{a^{(i)}}(\mathbf{r})$ is the component of the function f along the given vector.

Because the components of f are arbitrary, the total tensor given by equation (A 3) will be non-negative definite if each of the two functions $A(\mathbf{r},\omega)$ and $[A(\mathbf{r},\omega)+B(\mathbf{r},\omega)]$ are chosen to be non-negative definite scalar functions.

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