15 January 2001

Optics Communications 187 (2001) 301–309

OPTICS COMMUNICATIONS

ELSEVIER

www.elsevier.com/locate/optcom

Uniqueness of the solution to the inverse source problem for quasi-homogeneous sources

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Received 7 August 2000; received in revised form 10 October 2000; accepted 14 November 2000

Abstract

The quasi-homogeneous approximation, often used but never rigorously justified, is carefully derived for primary, three-dimensional, scalar radiation sources. The derivation indicates that nonradiating quasi-homogeneous sources do not exist. The relevance of this result and its derivation for the inverse source problem is discussed. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Inverse source problem; Nonradiating sources; Quasi-homogeneous sources

1. Introduction

Consider the following question: Can an observer, given measurements of the wave field (acoustic, electromagnetic, or otherwise) generated by a three-dimensional primary radiation source (deterministic or random), determine worthwhile information about the structure of that source? Such an inverse problem is of potential importance in acoustics, optics, astronomy, and the earth sciences.

The near unanimous answer given to the above question by a wide variety of authors over the past 30 years is "no" [1]. The hypothetical existence of the ironically-named nonradiating sources [2], i.e. sources which produce no radiation outside their domain of support, implies that the inverse source problem is nonunique.

Nonradiating sources have been described for both scalar [2] and electromagnetic [3] radiation problems, and are a general feature of many systems with wavelike behavior. [1] They have been shown to exist in deterministic systems, in randomly fluctuating systems [5], and even in one-dimensional wave systems [6,7]. This widespread and very robust nonuniqueness in the inverse source problem has left it more a curiosity than a field of research.

However, it is known that for incoherent sources, whose spatial correlation properties may be represented by a delta function, the inverse problem is unique [8]. For such sources, a band limited version of the source intensity can be reconstructed from measurements of the cross-spectral density of the field. This uniqueness is not surprising in light of the fact that nonradiating

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[1] See Ref. [4] for a list of wave systems which exhibit nonradiating behavior.

PII: S0030-4018(00)01115-9

sources arise from a complicated interference phenomenon [9].

Furthermore, several papers have suggested that the inverse source problem for quasi-homogeneous sources is unique, allowing reconstruction of the source intensity or spectral degree of coherence from field cross-spectral density measurements, if one has sufficient prior knowledge of the source [10,11]. A quasi-homogeneous source is one whose cross-spectral density is well approximated by the form

$$W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) \approx I_Q\left(\frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \omega\right) \mu_Q(\mathbf{r}_2 - \mathbf{r}_1, \omega), \quad (1)$$

where I_Q, the source intensity, is a slowly varying function of position compared to the width of the spectral degree of coherence μ_Q (see Fig. 1). Such sources, which have an extremely small coherence volume, are said to be globally incoherent. Sources with delta correlations, as mentioned above, are a subclass of the set of quasi-homogeneous sources.

The quasi-homogeneous approximation has been used quite often since its introduction, both in modeling scatterers [12–14] as well as modeling sources [15]. It has also been used to elucidate the foundations of radiometry [16–18].

As prevalent as it has been in statistical optics, however, the quasi-homogeneous approximation has not as yet been put on firm mathematical ground. It seems that only one other paper to date has examined its foundations, and only for the class of Gaussian Schell-model sources [19]. Probably because of this, the question of uniqueness in the quasi-homogeneous inverse source problem is still open, [2] and few attempts have been made to investigate possible methods of inversion.

In this paper we will present an analysis of the quasi-homogeneous approximation for three-dimensional statistically stationary radiation sources in the space-frequency domain. From this analysis we find that nonradiating quasi-homogeneous sources do not exist. This result suggests that the inverse source problem is unique for the class of quasi-homogeneous sources. Furthermore, our analysis of the quasi-homogeneous approximation leads to simpler methods, requiring less prior knowledge, of solving the inverse quasi-homogeneous source problem than those presented in earlier work. These results therefore broaden the class of sources for which the inverse source problem is known to be uniquely solvable to include the class of quasi-homogeneous sources.

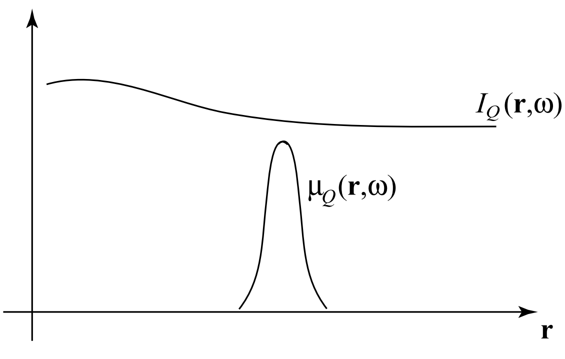

Fig. 1. Illustrating the conventional requirement for the validity of the quasi-homogeneous approximation. At a given frequency ω, the spectral density $I_Q(\mathbf{r}, \omega)$ must be a 'slowly varying' function of position relative to the spectral degree of coherence $\mu_Q(\mathbf{r}, \omega)$.

2. Derivation of the quasi-homogeneous approximation

Consider a three-dimensional, primary, random, scalar radiation source $q(\mathbf{r}, t)$, confined to a domain D (see Fig. 2). We assume that its fluctuations are stationary, at least in the wide sense [21, Section 2.2]. The mutual coherence function of the source distribution is defined by the formula

$$\Gamma_Q(\mathbf{r}_1, \mathbf{r}_2, \tau) = \langle q^*(\mathbf{r}_1, t) q(\mathbf{r}_2, t + \tau) \rangle, \quad (2)$$

where the angular brackets denote ensemble averaging. This function describes the correlation of the source fluctuations at pairs of points within the source. More useful for our purposes, however, is the cross-spectral density, defined as the Fourier transform of the mutual coherence function viz.,

$$W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) = \frac{1}{2\pi} \int \Gamma_Q(\mathbf{r}_1, \mathbf{r}_2, \tau) e^{i\omega\tau} \, d\tau. \quad (3)$$

The cross-spectral density may be expressed in the form

[2] Though one earlier paper [20] hinted that a certain class of quasi-homogeneous sources must radiate.

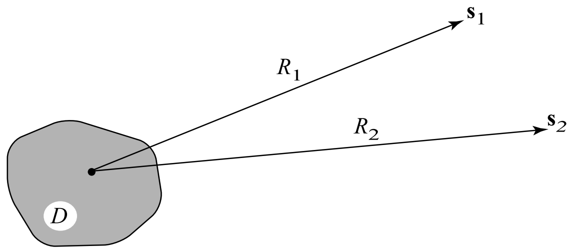

Fig. 2. Illustrating the notation used in describing radiation from a primary source $q(\mathbf{r}, t)$.

$$W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) = \sqrt{I_Q(\mathbf{r}_1, \omega)}\sqrt{I_Q(\mathbf{r}_2, \omega)}\mu_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) = h_Q(\mathbf{r}_1, \omega)h_Q(\mathbf{r}_2, \omega)\mu_Q(\mathbf{r}_1, \mathbf{r}_2, \omega), \quad (4)$$

where $h_Q(\mathbf{r}, \omega) \equiv \sqrt{I_Q(\mathbf{r}, \omega)}$. Because of the assumed stationarity, different frequency components of the cross-spectral density are uncorrelated; we will therefore confine our analysis to a single frequency component and will not display the dependence of the various quantities on ω. In Eq. (4), $I_Q(\mathbf{r})$ is the source intensity and $\mu_Q(\mathbf{r}_1, \mathbf{r}_2)$ is the spectral degree of coherence, which is defined as

$$\mu_Q(\mathbf{r}_1, \mathbf{r}_2) \equiv \frac{W_Q(\mathbf{r}_1, \mathbf{r}_2)}{\sqrt{W_Q(\mathbf{r}_1, \mathbf{r}_1)}\sqrt{W_Q(\mathbf{r}_2, \mathbf{r}_2)}}. \quad (5)$$

It is to be noted that $\mu_Q(\mathbf{r}_1, \mathbf{r}_2)$ is undefined for $\mathbf{r}_1, \mathbf{r}_2 \notin D$. The absolute value of the spectral degree of coherence can be shown to be restricted to the range

$$0 \leqslant |\mu_Q(\mathbf{r}_1, \mathbf{r}_2)| \leqslant 1. \quad (6)$$

The extreme value zero represents spatial incoherence and the value unity represents complete spatial coherence at frequency ω.

For a scalar source of this kind, the cross-spectral density of the radiated field far from the source can be shown to be given by the formula [21, Section 5.2]

$$W_U(R_1\mathbf{s}_1, R_2\mathbf{s}_2) = \frac{(2\pi)^6}{R_1R_2}\exp[\mathrm{i}k(R_2 - R_1)] \times \widetilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2), \quad (7)$$

where

$$\widetilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \frac{1}{(2\pi)^6}\int\int W_Q(\mathbf{r}_1, \mathbf{r}_2) \times \exp[-\mathrm{i}k(\mathbf{s}_2\cdot\mathbf{r}_2 - \mathbf{s}_1\cdot\mathbf{r}_1)] \times \mathrm{d}^3r_1\,\mathrm{d}^3r_2 \quad (8)$$

is the six-dimensional spatial Fourier transform of the source distribution, and $k = \omega/c$ is the wave number of the radiation.

It should be clear from Eq. (7) that all information about the source structure that is obtainable from the cross-spectral density of the field is contained within the function $\widetilde{W}_Q$, and we will therefore focus our investigation upon that function.

Substituting from Eq. (4) into Eq. (8), we may express Eq. (8) in the form

$$\widetilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \frac{1}{(2\pi)^6}\int\int h_Q(\mathbf{r}_1)h_Q(\mathbf{r}_2)\mu_Q(\mathbf{r}_1, \mathbf{r}_2) \times \exp[-\mathrm{i}k(\mathbf{s}_2\cdot\mathbf{r}_2 - \mathbf{s}_1\cdot\mathbf{r}_1)] \times \mathrm{d}^3r_1\,\mathrm{d}^3r_2. \quad (9)$$

Let us assume that the source correlations are homogeneous, i.e. that

$$\mu_Q(\mathbf{r}_1, \mathbf{r}_2) = \mu_Q(\mathbf{r}_2 - \mathbf{r}_1) \quad (10)$$

for all points within the source domain. A source for which Eq. (10) is satisfied is known as a Schell-model source. Changing the variables of integration to

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1, \qquad \mathbf{R} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \quad (11)$$

we may express Eq. (9) in the form

$$\widetilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \frac{1}{(2\pi)^6}\int M_Q[k(\mathbf{s}_2 - \mathbf{s}_1), \mathbf{r}]\mu_Q(\mathbf{r}) \times \exp\left[-\mathrm{i}k\left(\frac{\mathbf{s}_1 + \mathbf{s}_2}{2}\right)\cdot\mathbf{r}\right]\mathrm{d}^3r, \quad (12)$$

where

$$M_Q[\mathbf{K}, \mathbf{r}] \equiv \int h_Q\left(\mathbf{R} + \frac{\mathbf{r}}{2}\right)h_Q\left(\mathbf{R} - \frac{\mathbf{r}}{2}\right)\mathrm{e}^{-\mathrm{i}\mathbf{K}\cdot\mathbf{R}}\,\mathrm{d}^3R. \quad (13)$$

It is to be noted that M_Q is of finite extent with respect to the $\mathbf{r}$ variable, because the function h_Q is of finite extent. Also, because the function h_Q is nonnegative (h_Q describing, as before, the square root of the source intensity), M_Q satisfies the inequality

$$|M_Q[\mathbf{K},\mathbf{r}]| \leqslant M_Q[0,\mathbf{r}] \quad \text{for all } \mathbf{r}. \tag{14}$$

From these two properties it is clear that if we define a function $B(\mathbf{r})$ by the formula

$$\begin{aligned} B(\mathbf{r}) &= 0 \ \{\mathbf{r}: \ M_Q(0,\mathbf{r}) = 0\} \\ &= 1 \ \{\mathbf{r}: \ M_Q(0,\mathbf{r}) \neq 0\}, \end{aligned} \tag{15}$$

we may incorporate this function into the integrand of Eq. (12) without changing the value of that integral. This is allowable because the domain of support of $M_Q(\mathbf{K},\mathbf{r})$ is always smaller than and contained within the domain of support of $B(\mathbf{r})$.

On substituting $B(\mathbf{r})$ into Eq. (12), we have

$$\widetilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \frac{1}{(2\pi)^6}\int M_Q[k(\mathbf{s}_2-\mathbf{s}_1),\mathbf{r}]\mu_Q^B(\mathbf{r}) \times \exp\left[-\mathrm{i}k\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)\cdot\mathbf{r}\right]\mathrm{d}^3r, \tag{16}$$

where we have defined

$$\mu_Q^B(\mathbf{r}) \equiv B(\mathbf{r})\mu_Q(\mathbf{r}). \tag{17}$$

From the usual description of the quasi-homogeneous approximation, we expect that Eq. (16) will reduce to a quasi-homogeneous form if h_Q is a "slowly varying" function of position with respect to the "width" of μ_Q^B. This is a global requirement, however, in that it must hold for all locations within the source domain. It would seem more appropriate, then, to convert Eq. (16) into an integral involving the Fourier transforms of h_Q and μ_Q^B. We introduced the function $B(\mathbf{r})$ for this purpose; the function μ_Q is by itself undefined for values of $\mathbf{r}_1$, $\mathbf{r}_2$ not contained within the domain of support of $B(\mathbf{r})$ (see Eq. (5)).

As both M_Q and μ_Q^B are functions of finite support, they each have a Fourier representation, i.e.

$$M_Q[\mathbf{K}_0,\mathbf{r}] = \int \widetilde{M}_Q[\mathbf{K}_0,\mathbf{K}]\mathrm{e}^{\mathrm{i}\mathbf{K}\cdot\mathbf{r}}\,\mathrm{d}^3K, \tag{18}$$

and

$$\mu_Q^B(\mathbf{r}) = \int \tilde{\mu}_Q^B[\mathbf{K}]\mathrm{e}^{\mathrm{i}\mathbf{K}\cdot\mathbf{r}}\,\mathrm{d}^3K. \tag{19}$$

From Eq. (16), we see that $\widetilde{W}_Q$ is the Fourier transform of a product of two functions. By the convolution theorem, $\widetilde{W}_Q$ may therefore be written as the three-dimensional convolution of the Fourier transforms of these functions, so that

$$\widetilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \frac{1}{(2\pi)^3}\int \widetilde{M}_Q[k(\mathbf{s}_2-\mathbf{s}_1),\mathbf{K}] \times \tilde{\mu}_Q^B\left[k\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)-\mathbf{K}\right]\mathrm{d}^3K. \tag{20}$$

Substituting from Eq. (13) into Eq. (18), one can show that $\widetilde{M}_Q$ may be expressed in the form

$$\widetilde{M}_Q[\mathbf{K}_0,\mathbf{K}_1] = (2\pi)^3\tilde{h}_Q^*\left[-\mathbf{K}_1-\tfrac{1}{2}\mathbf{K}_0\right]\tilde{h}_Q\left[-\mathbf{K}_1+\tfrac{1}{2}\mathbf{K}_0\right]. \tag{21}$$

Substituting from this expression into Eq. (20), and changing the variable of integration from $\mathbf{K}$ to $-\mathbf{K}$, we arrive at the result that

$$\widetilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \int \tilde{h}_Q^*\left[\mathbf{K}-\frac{1}{2}k(\mathbf{s}_2-\mathbf{s}_1)\right] \times \tilde{h}_Q\left[\mathbf{K}+\frac{1}{2}k(\mathbf{s}_2-\mathbf{s}_1)\right] \times \tilde{\mu}_Q^B\left[\mathbf{K}+k\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)\right]\mathrm{d}^3K. \tag{22}$$

Eq. (22) is an *exact* expression for $\widetilde{W}_Q$, equivalent to our defining formula, Eq. (9). We have as yet made no approximations. Because each of the functions $\tilde{h}_Q(\mathbf{K})$ and $\tilde{\mu}_Q^B(\mathbf{K})$ is the Fourier transform of a function of finite support, each is the boundary value of an entire analytic function in three complex variables [22, p. 353]. A consequence of this analyticity is that if $h_Q(\mathbf{r})$ and $\mu_Q^B(\mathbf{r})$ are both nonnull functions (which is true if $W_Q(\mathbf{r}_1,\mathbf{r}_2)\neq 0$), then both $\tilde{h}_Q(\mathbf{K})$ and $\tilde{\mu}_Q^B(\mathbf{K})$ are functions of infinite support; neither may vanish on a domain in $\mathbf{K}$-space larger than a two-dimensional manifold. We will use this property shortly.

Although $\tilde{h}_Q(\mathbf{K})$ is of infinite support, it must be negligible for large values of $|\mathbf{K}|$, because its Fourier transform exists (which suggests that it decays sufficiently rapidly for large $\mathbf{K}$). Let us assume that $\tilde{h}_Q(\mathbf{K})$ is narrow enough that the integrand in

Eq. (22) is negligible for values of $|\mathbf{K}|$ larger than some parameter α, i.e. that

$$\tilde{h}_Q(\mathbf{K}) \approx 0 \quad \text{for all } |\mathbf{K}| \geqslant \alpha. \tag{23}$$

This requirement, which states that $h_Q(\mathbf{r})$ contains few high-frequency spatial Fourier components, suggests that it is slowly varying over spatial distances on the order of $1/\alpha$. It is not then difficult to show that the integrand in Eq. (22) will be appreciable only for $\mathbf{K}$ values such that $|\mathbf{K}| \leqslant \alpha$. We have already noted that $\tilde{\mu}_Q^B[k((\mathbf{s}_1+\mathbf{s}_2)/2)+\mathbf{K}]$ is the boundary value of an entire analytic function of three complex variables; it follows that it is differentiable to all orders and can be expanded in a Taylor series around the point $\mathbf{K}=0$, i.e. that

$$\tilde{\mu}_Q^B\left[k\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)+\mathbf{K}\right] = \sum_{n=0}^{\infty} \frac{1}{n!}(\mathbf{K}\cdot\nabla_{K'})^n \times \tilde{\mu}_Q^B(\mathbf{K}')|_{\mathbf{K}'=k(\mathbf{s}_1+\mathbf{s}_2)/2}. \tag{24}$$

If α is sufficiently small, the first term of this series will dominate the integral in Eq. (22). This contribution, which we denote by $\widetilde{W}_Q^0$, may be written as

$$\widetilde{W}_Q^0(-k\mathbf{s}_1, k\mathbf{s}_2) = \int \tilde{h}_Q^*\left[\mathbf{K}-\frac{1}{2}k(\mathbf{s}_2-\mathbf{s}_1)\right] \times \tilde{h}_Q\left[\mathbf{K}+\frac{1}{2}k(\mathbf{s}_2-\mathbf{s}_1)\right] \times \tilde{\mu}_Q^B\left[k\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)\right] \mathrm{d}^3K. \tag{25}$$

The term involving $\tilde{\mu}_Q^B$ is now independent of $\mathbf{K}$, and may be removed from the integrand. The integral may then be evaluated using the definition of the Fourier transform of h_Q, and $\widetilde{W}_Q^0$ may be written as

$$\widetilde{W}_Q^0(-k\mathbf{s}_1, k\mathbf{s}_2) = \tilde{I}_Q[k(\mathbf{s}_2-\mathbf{s}_1)]\tilde{\mu}_Q^B\left[k\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)\right], \tag{26}$$

where

$$\tilde{I}_Q(\mathbf{K}) = \frac{1}{(2\pi)^3}\int I_Q(\mathbf{r})\mathrm{e}^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}}\,\mathrm{d}^3r \tag{27}$$

is the Fourier transform of the source intensity $I_Q(\mathbf{r})$.

Eq. (26) is equivalent to the Fourier transform of the cross-spectral density of a quasi-homogeneous source, as can be seen by substituting Eq. (1) into Eq. (8). Note that this result differs from the usual statement of the quasi-homogeneous approximation by the appearance of the function $\tilde{\mu}_Q^B$, rather than the ill-defined $\tilde{\mu}_Q$. The quasi-homogeneous approximation therefore consists of using only the first term in the Taylor series expansion of $\tilde{\mu}_Q^B$ in the Fourier domain. As can be seen by considering Eq. (22), this approximation is only valid when $\tilde{\mu}_Q^B(\mathbf{K})$ is constant within a sphere of radius α centered on $\mathbf{K}=k(\mathbf{s}_1+\mathbf{s}_2)/2$. If the approximation is to be valid for all directions $\mathbf{s}_1$ and $\mathbf{s}_2$, then $\tilde{\mu}_Q^B(\mathbf{K})$ must be nearly constant within every sphere of radius α for all $\mathbf{K}$ values such that $|\mathbf{K}| \leqslant k$. Thus the value of $\tilde{\mu}_Q^B(\mathbf{K})$ cannot change significantly over any distance α, although it may, for small α, change considerably over a distance k. This assumption will form the basis of our analysis of the inverse quasi-homogeneous source problem, discussed in Section 4.

It is to be noted that these statements are in agreement with the usual justification of the quasi-homogeneous approximation, because of the reciprocal nature of a function and its transform. If $\tilde{\mu}_Q^B(\mathbf{K})$ is very slowly varying in comparison to $\tilde{h}_Q(\mathbf{K})$, then $h_Q(\mathbf{r})$ must be very slowly varying in comparison to $\mu_Q^B(\mathbf{r})$.

It would be remiss to talk about the quasi-homogeneous approximation and properties of quasi-homogeneous functions without some discussion of conditions under which a given source cross-spectral density is quasi-homogeneous. We will discuss this problem in Appendix A.

3. The nonexistence of quasi-homogeneous nonradiating sources

Two conclusions may be immediately drawn from our careful analysis of the quasi-homogeneous approximation. First, it is to be noted that a given source cross-spectral density will factorize in the form of Eq. (26) only if $\tilde{\mu}_Q^B(\mathbf{K})$ is constant for all $\mathbf{K}$ values. Formally, the inverse Fourier transform of a constant function is proportional to a delta function, and therefore a cross-spectral density will only factorize if it is delta correlated. For any other source with a sufficiently narrow correlation function $\mu_Q^B(\mathbf{r})$, this factorization is only approximate.

Second, it is to be noted that since the functions $\tilde{\mu}_Q^B$ and $\tilde{I}_Q$ are each the boundary value of an entire analytic function in three complex variables, neither function may vanish throughout a region of **K**-space with dimensionality greater than that of a surface, and likewise their product may only vanish on surfaces in **K**-space. It is therefore not possible for $\widetilde{W}_Q^0$ to vanish for all pairs of directions $\mathbf{s}_1$ and $\mathbf{s}_2$, unless W_Q^0 vanishes identically. Therefore nonradiating quasi-homogeneous sources do not exist.

This result has important consequences for the inverse source problem. The nonexistence of nonradiating quasi-homogeneous sources suggests that, if a source is quasi-homogeneous, some unique information about the source structure can be determined from measurements of the radiated field outside the source. We will discuss precisely what structural information can be recovered in the next section.

4. The inverse problem for quasi-homogeneous sources

In Section 2, we derived the quasi-homogeneous approximation through a careful analysis of radiation from globally incoherent sources. From this derivation we demonstrated the nonexistence of nonradiating quasi-homogeneous sources. This result suggests that the radiation of every quasi-homogeneous source possesses a unique "signature" that distinguishes it from every other, and that by measurements of the radiation emitted by such a source we may determine some of its structural features. We now consider briefly what sort of structural information may be obtained.

We have seen that when a source is quasi-homogeneous, the function $\tilde{\mu}_Q^B(\mathbf{K}+\mathbf{K}_0)$ must be effectively constant for all $|\mathbf{K}| \leqslant \alpha$, for every $|\mathbf{K}_0| \leqslant k$. Then the cross-spectral density of the field far from the source is proportional to $\widetilde{W}_Q^0(-k\mathbf{s}_1, k\mathbf{s}_2)$, given by Eq. (26).

Let us assume that measurements of the cross-spectral density of the field of a quasi-homogeneous source have been made for all directions $\mathbf{s}_1$ and $\mathbf{s}_2$. If we consider only field data for directions of observation such that

$$\left|k\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right|^2 \leqslant \alpha^2, \tag{28}$$

the Fourier transform of the spectral degree of coherence will be effectively constant over this range and may be replaced by its value at the origin, $\tilde{\mu}_Q^B(0)$. The function $\widetilde{W}_Q^0(-k\mathbf{s}_1, k\mathbf{s}_2)$ may then be written as

$$\widetilde{W}_Q^0(-k\mathbf{s}_1, k\mathbf{s}_2) = \tilde{I}_Q[k(\mathbf{s}_2-\mathbf{s}_1)]\tilde{\mu}_Q^B(0). \tag{29}$$

The inequality (28) is equivalent to considering only directions of observation in the range

$$4k^2 - 4\alpha^2 \leqslant k^2|\mathbf{s}_2-\mathbf{s}_1|^2 \leqslant 4k^2, \tag{30}$$

where the upper limit is determined by the maximum value of $k|\mathbf{s}_2-\mathbf{s}_1|$.

Using the values of $k(\mathbf{s}_2-\mathbf{s}_1)$ given by Eq. (30), we may determine, up to an arbitrary multiplicative constant $\tilde{\mu}_Q^B(0)$, the Fourier components of $\tilde{I}_Q[\mathbf{K}]$ which lie within the spherical shell defined by Eq. (30). From this information we may use Fourier inversion to reconstruct a "high pass" filtered version of the intensity function, $I_Q(\mathbf{r})$.

This reconstruction procedure has only two undetermined parameters which are unobtainable from field measurements: the value of $\tilde{\mu}_Q^B(0)$, as mentioned above, and α, which determines the allowed Fourier components, as in Eq. (30). Earlier quasi-homogeneous inversion methods described in the literature require knowledge of the value of $\tilde{\mu}_Q(\mathbf{K})$ over a continuous domain, either throughout the volume $|\mathbf{K}| \leqslant k$ [10] or along a radial line within that volume [11].

It is to be noted, however, that in deriving the quasi-homogeneous approximation, we have assumed that $\tilde{h}_Q(\mathbf{K})$ is negligible for all $|\mathbf{K}| > \alpha$; this suggests that $\tilde{I}_Q(\mathbf{K})$ is negligible for all $|\mathbf{K}| > 2\alpha$ (this can be shown by using the convolution theorem on $I_Q(\mathbf{r}) = |h_Q(\mathbf{r})|^2$). In order, then, that our reconstruction contains nonnegligible Fourier components of the source intensity $I_Q(\mathbf{r})$, we require that $4k^2 - 4\alpha^2 \leqslant 4\alpha^2$, i.e. that

$$\alpha^2 \geqslant \tfrac{1}{2}k^2. \tag{31}$$

This requirement indicates that the usefulness of reconstruction methods for quasi-homogeneous sources depends upon the relation between the width of I_Q and the wavelength $\lambda = 2\pi/k$. Other reconstruction schemes may be used which take better advantage of the relative values of these parameters, as well as the width of μ_Q^B.

We have so far only considered reconstruction of the intensity of the source; we now briefly examine the possibility of reconstructing the spectral degree of coherence. Let us assume that we know the source intensity I_Q. For a quasi-homogeneous source, the Fourier transform of the source intensity is negligible for all $|\mathbf{K}| > 2\alpha$; therefore the only field data available for reconstructing the degree of coherence is that for which

$$k|\mathbf{s}_2 - \mathbf{s}_1| \leqslant \min[2\alpha, 2k] \equiv \beta. \tag{32}$$

Eq. (32) may be rewritten to show that the only nonnegligible field data is that for which

$$\sqrt{k^2 - \frac{\beta^2}{4}} \leqslant k\left|\frac{\mathbf{s}_1 + \mathbf{s}_2}{2}\right| \leqslant k. \tag{33}$$

This formula defines a spherical shell within which are all the Fourier components which may be used to reconstruct the spectral degree of coherence. The radial width of this shell, however, is always comparable to α. For the quasi-homogeneous approximation to be valid, however, $\tilde{\mu}_Q^B$ must be constant across any radial distance α. The Fourier information available for reconstruction of the spectral degree of coherence, then, contains little or no information about that function's radial structure, and will not give an accurate reconstruction. From this argument it seems evident that the spectral degree of coherence cannot be reliably reconstructed for quasi-homogeneous sources.

Acknowledgements

This research was supported by the Engineering Research Program of the Office of Basic Energy Science at the Department of Energy under grant DE-FG02-90ER 14119 and by the Air Force Office of Scientific Research under grant F 49620-96-1-400.

Appendix A. Conditions for quasi-homogeneity

We have seen that the requirement for the validity of the quasi-homogeneous approximation is that the function $\tilde{\mu}_Q^B[\mathbf{K} + \mathbf{K}_0]$ be effectively constant for every value of $\mathbf{K}$ such that $|\mathbf{K}| \leqslant \alpha$, for any $|\mathbf{K}_0| \leqslant k$. In this appendix we will express this condition in a form which may be used in a straightforward manner to determine if a given correlation function may be considered quasi-homogeneous.

Instead of using the complete Taylor expansion for the spectral degree of coherence given by Eq. (24), let us consider the finite Taylor expansion given by

$$\tilde{\mu}_Q^B[\mathbf{K}_0 + \mathbf{K}] = \tilde{\mu}_Q^B[\mathbf{K}_0] + \int_0^1 \frac{\partial}{\partial l} \tilde{\mu}_Q^B[l\mathbf{K} + \mathbf{K}_0]\, \mathrm{d}l. \tag{A.1}$$

This expansion of $\tilde{\mu}_Q^B$ can be verified directly by carrying out the integration on the right-hand side of Eq. (A.1). The first term of this expansion results in the quasi-homogeneous approximation, and the second term is the correction to this approximation. It is therefore clear that a requirement for the validity of the quasi-homogeneous approximation is that

$$\frac{\left|\int_0^1 \frac{\partial}{\partial l} \tilde{\mu}_Q^B[l\mathbf{K} + \mathbf{K}_0]\, \mathrm{d}l\right|}{\left|\tilde{\mu}_Q^B(\mathbf{K}_0)\right|} \ll 1 \quad \text{for all } |\mathbf{K}| \leqslant \alpha, \quad |\mathbf{K}_0| \leqslant k. \tag{A.2}$$

This requirement guarantees that the quasi-homogeneous term will dominate the integral in Eq. (22) for all directions of observation $\mathbf{s}_1$ and $\mathbf{s}_2$. How small, in fact, the correction term must be to obtain a good solution to the inverse problem will evidently depend upon the desired accuracy of the reconstruction. In this sense the correction term may be considered "noise" in the field data.

We may use the triangle inequality, viz.

$$\left|\int_0^1 \frac{\partial}{\partial l} \tilde{\mu}_Q^B[l\mathbf{K} + \mathbf{K}_0]\, \mathrm{d}l\right| \leqslant \int_0^1 \left|\frac{\partial}{\partial l} \tilde{\mu}_Q^B[l\mathbf{K} + \mathbf{K}_0]\right| \mathrm{d}l \tag{A.3}$$

to simplify Eq. (A.2). It follows then that a weaker condition for the validity of the quasi-homogeneous approximation is that

$$\frac{\int_0^1 \left|\frac{\partial}{\partial l}\tilde{\mu}_Q^B[l\mathbf{K}+\mathbf{K}_0]\right| \mathrm{d}l}{\left|\tilde{\mu}_Q^B(\mathbf{K}_0)\right|} \ll 1 \quad \text{for all } |\mathbf{K}| \leqslant \alpha, \quad |\mathbf{K}_0| \leqslant k. \tag{A.4}$$

Because $\tilde{\mu}_Q^B(\mathbf{K})$ is the boundary value of an entire analytic function, it is everywhere continuous. It follows from this result that $|(\partial/\partial l)\tilde{\mu}_Q^B(l\mathbf{K}+\mathbf{K}_0)|$ is a continuous real function of the integration variable l. Therefore by the fundamental theorem of calculus, we may write

$$\int_0^1 \left|\frac{\partial}{\partial l}\tilde{\mu}_Q^B[l\mathbf{K}+\mathbf{K}_0]\right| \mathrm{d}l = \left|\frac{\partial}{\partial l}\tilde{\mu}_Q^B(l\mathbf{K}+\mathbf{K}_0)\right|_{l=l_1(\mathbf{K})}, \quad \text{where } 0 \leqslant l_1(\mathbf{K}) \leqslant 1. \tag{A.5}$$

This theorem states that the value of this dimensionless integral is equal to the value of the integrand evaluated at some point within the range of the integration. Using the chain rule for differentiation, we may rewrite this derivative in the form

$$\frac{\partial}{\partial l}\tilde{\mu}_Q^B(l\mathbf{K}+\mathbf{K}_0) = \mathbf{K}\cdot\nabla_{\mathbf{K}'}\tilde{\mu}_Q^B(\mathbf{K}'+\mathbf{K}_0)|_{\mathbf{K}'=l\mathbf{K}}; \tag{A.6}$$

our condition for the quasi-homogeneous approximation then becomes

$$\frac{\left|\mathbf{K}\cdot\nabla_{\mathbf{K}'}\tilde{\mu}_Q^B(\mathbf{K}'+\mathbf{K}_0)|_{\mathbf{K}'=l_1(\mathbf{K})\mathbf{K}}\right|}{\left|\tilde{\mu}_Q^B(\mathbf{K}_0)\right|} \ll 1 \quad \text{for all } |\mathbf{K}| \leqslant \alpha, \quad |\mathbf{K}_0| \leqslant k. \tag{A.7}$$

This inequality may be simplified further. We note that the maximum value of $|\mathbf{K}|$ is α, so if

$$\alpha\frac{\left|\hat{\mathbf{K}}\cdot\nabla_{\mathbf{K}'}\tilde{\mu}_Q^B(\mathbf{K}'+\mathbf{K}_0)|_{\mathbf{K}'=l_1(\mathbf{K})\mathbf{K}}\right|}{\left|\tilde{\mu}_Q^B(\mathbf{K}_0)\right|} \ll 1 \quad \text{for all } |\mathbf{K}| \leqslant \alpha, \quad |\mathbf{K}_0| \leqslant k \tag{A.8}$$

is satisfied, then Eq. (A.7) is satisfied. Here $\hat{\mathbf{K}}$ is a unit vector in the direction of $\mathbf{K}$. Also, if inequality (A.8) is satisfied for every l between 0 and 1, and not just l_1, then the inequality (A.8) will be satisfied. A final condition is therefore

$$\frac{\left|\hat{\mathbf{K}}\cdot\nabla_{\mathbf{K}'}\tilde{\mu}_Q^B(\mathbf{K}')|_{\mathbf{K}'=\mathbf{K}+\mathbf{K}_0}\right|}{\left|\tilde{\mu}_Q^B(\mathbf{K}_0)\right|} \ll \frac{1}{\alpha} \quad \text{for all } |\mathbf{K}| \leqslant \alpha, \quad |\mathbf{K}_0| \leqslant k. \tag{A.9}$$

This condition suggests that the absolute value of the gradient of $\tilde{\mu}_Q$ must be sufficiently small for all measurable values of $\mathbf{K}$. This is essentially the same statement, expressed mathematically, that $\tilde{\mu}_Q^B$ must be smoothly varying over all measurable values of $\mathbf{K}$; for this reason, we may consider Eq. (A.9) to be a general requirement for the validity of the quasi-homogeneous approximation. We may use this equation to determine whether a given model correlation function is or is not well described by the approximation.

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