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The Rayleigh range of Gaussian Schell-model beams

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Abstract. The concept of the Rayleigh range, well known in the theory of coherent beams, is generalized to a class of partially coherent beams. Curves are presented which show the dependence of the Rayleigh range on the spot size of the beam and on the spectral degree of coherence of the light in the plane of the waist.

1. Introduction

An important parameter used in the theory of laser beams is the so-called Rayleigh distance or Rayleigh range. The concept originated in the work of Lord Rayleigh towards the end of the 19th century, concerning images formed without reflection and refraction [1] and the theory of the pinhole camera [2]. Specifically, Rayleigh showed that images of a distant object can be formed without a lens at a distance d from the pinhole such that

$$d = \frac{2r^2}{\lambda}, \tag{1}$$

where r is the radius of the pinhole and λ is the wavelength of the light. This concept was later introduced in antenna theory to characterize the distance that a collimated beam will traverse before it begins to significantly diverge [3]. More recently the Rayleigh range has come to be used in connection with laser beams (see [4] or [5]). Specifically for a laser beam with a spot size w_0 at the beam waist, the Rayleigh range is defined as the distance z_R at which the diameter of the spot size increases by a factor $2^{1/2}$, i.e. at which the cross-sectional area is doubled from πw_0^2 to $2\pi w_0^2$ (see figure 1). It is given by the formula

$$z_R = \frac{\pi w_0^2}{\lambda}. \tag{2}$$

In recent years considerable attention has been paid to partially coherent beams. Such beams are finding useful applications, in part because they do not give rise to speckle effects as pronounced as with highly coherent beams (see [6], particularly section 4.4) and also because they are less affected by atmospheric turbulence [7]. They have also found use in lithography [8] and in laser fusion research [9, 10]. It is, therefore, of interest to try to extend the concept of the Rayleigh range to beams with an arbitrary state of coherence. In this note we do so

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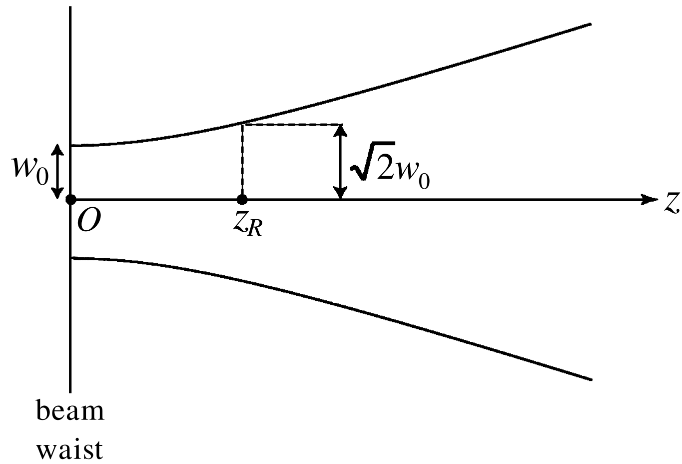

Figure 1. Depiction of the Rayleigh range for a Gaussian Schell-model beam with spot size w_0. For z values significantly less than the Rayleigh range, the beam has no appreciable spreading.

for a class of partially coherent beams, which have been extensively studied and used in recent years, the so-called Gaussian Schell-model beams.

2. The Rayleigh range of a Gaussian Schell-model beam†

Consider the field generated by a planar, secondary Gaussian Schell-model source, located in the plane $z = 0$ and radiating into the half-space $z > 0$. The intensity distribution $I^{(0)}(\boldsymbol{\rho}, \nu)$ across such a source and the spectral degree of coherence, $\mu^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \nu) \equiv g^{(0)}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1, \nu)$, (see [12], section 4.3.2), of the light at a pair of source points, at frequency ν, are Gaussian functions,

$$I^{(0)}(\boldsymbol{\rho}, \nu) = A^2 \exp(-\rho^2/2\sigma_{\mathrm{I}}^2), \tag{3}$$

and

$$g^{(0)}(\boldsymbol{\rho}', \nu) = \exp(-\rho'^2/2\sigma_{\mathrm{g}}^2), \tag{4}$$

where A, σ_{I} and σ_{g} are positive constants which, in general, depend on the frequency ν. With suitable choices of σ_{I} and σ_{g}, such a source will generate a beam, called a Gaussian Schell-model beam.

The spot size in any transverse plane is defined as the transverse radial distance at which the intensity falls to $1/e^2$ of its axial value. In particular, the spot size w_0 in the plane of the waist of the beam, located in the plane $z = 0$, is evidently

$$w_0 = 2\sigma_{\mathrm{I}}. \tag{5}$$

According to equation (5.6-98) of [12], with a trivial change of notation, the intensity generated by the source at a point specified by a two-dimensional position vector $\boldsymbol{\rho}$, in a plane $z =$ constant > 0, is given by the expression

† Since this paper was submitted for publication we discovered that related analysis was described in [11].

$$I(\boldsymbol{\rho}, z) = \frac{A^2}{[\Delta(z)]^2} \exp\left[-\frac{\rho^2}{2\sigma_{\mathrm{I}}^2[\Delta(z)]^2}\right], \tag{6}$$

where

$$\Delta(z) = \left[1 + (z/k\sigma_{\mathrm{I}}\delta)^2\right]^{1/2} \tag{7}$$

is known as the expansion coefficient of the beam,

$$\frac{1}{\delta^2} = \frac{1}{(2\sigma_{\mathrm{I}})^2} + \frac{1}{\sigma_{\mathrm{g}}^2} \tag{8}$$

and

$$k = \frac{2\pi\nu}{c} \tag{9}$$

is the free-space wave number, c being the speed of light in vacuum. In the above formula and elsewhere in this note the dependence of the various quantities on the frequency ν is suppressed.

According to equation (5.6-99) of [12], the beam radius $\overline{\rho}(z)$ in a cross-section $z = \text{constant} > 0$ is given by the expression

$$\overline{\rho}(z) = \sigma_{\mathrm{I}}\Delta(z)2^{1/2}. \tag{10}$$

In particular, according to equation (7), $\Delta(0) = 1$ and hence

$$\overline{\rho}(0) = \sigma_{\mathrm{I}}2^{1/2}. \tag{11}$$

From equations (10) and (11) it follows that

$$\frac{\overline{\rho}(z)}{\overline{\rho}(0)} = \Delta(z). \tag{12}$$

Introducing the Rayleigh range, z_{R}, for the partially coherent beam by analogy with the definition used for coherent beams, namely,

$$\frac{\overline{\rho}(z_{\mathrm{R}})}{\overline{\rho}(0)} = 2^{1/2}, \tag{13}$$

it follows from equations (13), (12) and (7) that

$$1 + \frac{z_{\mathrm{R}}}{k\sigma_{\mathrm{I}}\delta} = 2, \tag{14}$$

i.e. that

$$z_{\mathrm{R}} = k\sigma_{\mathrm{I}}\delta. \tag{15}$$

Recalling expression (8), which defines the parameter δ, and the fact, expressed by equation (5), that σ_{I} is half of the spot size w_0, expression (15) for the Rayleigh range becomes

$$z_{\mathrm{R}} = \frac{kw_0}{2}\left[\frac{1}{w_0^2} + \frac{1}{\sigma_{\mathrm{g}}^2}\right]^{-1/2}. \tag{16}$$

We note two limiting cases. For a completely spatially coherent source ($\sigma_g \to \infty$) generating a Gaussian Schell-model beam (which represents the lowest-order Hermite–Gaussian mode), formula (16) becomes

$$(z_R)_{coh} = \frac{kw_0^2}{2} \tag{17}$$

or, since $k = 2\pi/\lambda$, where λ is the wavelength,

$$(z_R)_{coh} = \frac{\pi w_0^2}{\lambda}, \tag{18}$$

in agreement with the usual expression (2). In the other extreme case of a field generated by a completely incoherent source, $\sigma_g \to 0$ and formula (16) gives

$$(z_R)_{incoh} = 0. \tag{19}$$

For a partially coherent source $0 < \sigma_g < \infty$, and equations (16) and (17) imply that

$$0 \leq z_R \leq (z_R)_{coh}, \tag{20}$$

demonstrating that the fully spatially coherent Gaussian Schell-model beam has the greatest Rayleigh range, given by the usual expression (18).

In figure 2 a three-dimensional plot of the scaled Rayleigh range z_R as a function of the scaled spot size w_0 and the scaled rms width σ_g of the spectral degree of coherence is shown, calculated from equation (16). The corresponding contours are shown in figure 3. By straightforward calculations expression (16) for the Rayleigh range of a Gaussian Schell-model beam can be rewritten in the form

$$z_R = (z_R)_{coh}\left[1 + \left(\frac{w_0}{\sigma_g}\right)^2\right]^{-1/2}. \tag{21}$$

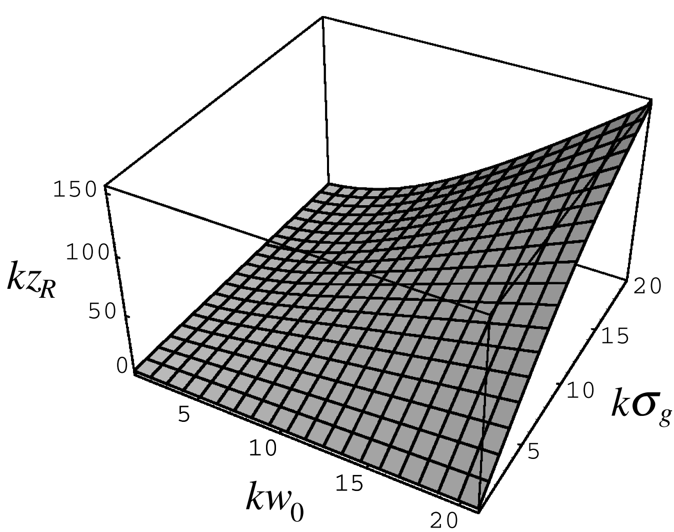

Figure 2. The three-dimensional plot of the (scaled) Rayleigh range kz_R, as a function of the scaled spot size kw_0 and the scaled rms width $k\sigma_g$ of the spectral degree of coherence.

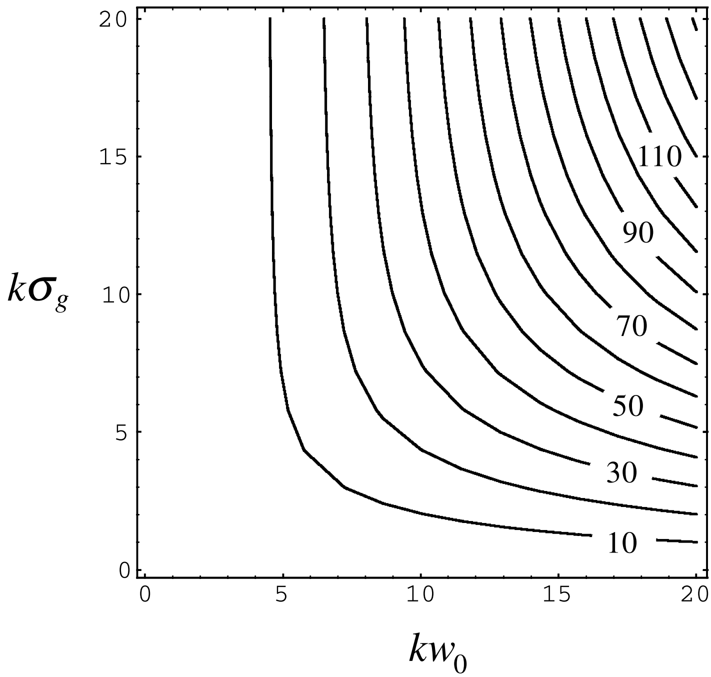

Figure 3. The contours of the (scaled) Rayleigh range kz_R as a function of kw_0 and $k\sigma_g$.

The ratio $z_R/(z_R)_{coh}$ is seen to be a function of a single parameter

$$\alpha = \frac{w_0}{\sigma_g}. \tag{22}$$

For beams with the same spot size w_0 but different values of σ_g (different spectral degrees of coherence) this parameter can take on any values in the range

$$0 \le \alpha < \infty, \tag{23}$$

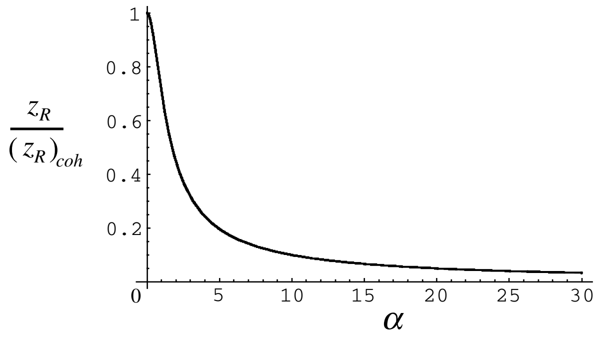

Figure 4. The ratio of the Rayleigh range z_R for a partially coherent beam to the Rayleigh range of a fully coherent beam, $(z_R)_{coh}$, as a function of the parameter $\alpha = w_0/\sigma_g$.

the extreme value zero representing the fully coherent case, the other extreme value, $\alpha \to \infty$, representing the completely incoherent case. In figure 4, the ratio $z_R/(z_R)_{coh}$ is displayed as a function of the parameter α.

3. Some theorems concerning the Rayleigh range

We will now derive two simple results involving the Rayleigh range of Gaussian Schell-model beams.

3.1. *A reciprocity relation*

The angular spread $\overline{\theta}$ of a Gaussian Schell-model beam is given by the formula (see [12], equation (5.6-102))

$$\overline{\theta}^2 = \frac{2}{k^2\delta^2}. \tag{24}$$

According to equation (15), the Rayleigh range is expressible in the form

$$z_R = \frac{1}{2} k w_0 \delta. \tag{25}$$

From these two equations we obtain at once the reciprocity relation

$$z_R = \frac{w_0}{2^{1/2}\overline{\theta}}, \tag{26}$$

which shows that the Rayleigh range is inversely proportional to the angular spread of the beam. Formula (26) also shows that the Rayleigh range is directly proportional to the spot size w_0.

3.2. *The Rayleigh range associated with sources that generate the same angular distribution of radiant intensity*

It has been demonstrated many years ago that, with suitable choices of the parameters, different Gaussian Schell-model sources may generate beams which have the same normalized far-zone intensity distribution (see [13]; see also [12], section 5.4.2). Such 'equivalent' sources have the same value of the parameter

$$E \equiv \frac{1}{w_0^2} + \frac{1}{\sigma_g^2}. \tag{27}$$

According to equations (16) and (27) the Rayleigh range of a Gaussian Schell-model beam is expressible in the form

$$z_R = \frac{\pi w_0}{\lambda E^{1/2}}. \tag{28}$$

This formula shows that all Gaussian Schell-model beams which have the same (normalized) angular distribution of radiation in the far zone (and consequently for which the parameter E has the same value) have Rayleigh ranges which are proportional to their spot size w_0.

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