Relation between computed tomography and diffraction tomography

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The relationship between computed tomography (CAT) and diffraction tomography (DT) is investigated. A simple condition with a clear physical meaning is derived for the applicability of CAT. Corrections due to scattering are incorporated into CAT, and it is shown that the effect of scattering may be characterized by a two-dimensional fractional Fourier transform. The implications of these results for the three-dimensional imaging of weakly scattering objects are also discussed. © 2001 Optical Society of America

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1. INTRODUCTION

During the past 30 years or so, tomographic methods have developed into techniques of great importance in the three-dimensional reconstruction (loosely referred to as three-dimensional imaging) of absorbing and scattering objects. Foremost among them is computed tomography, also known as computed axial tomography, abbreviated as CAT (see, for example, Sec. 4.11 of Ref. 1, or Refs. 2 and 3). It is a basic diagnostic technique in medicine, for which it was originally developed, but it has found numerous applications in other fields. Computed tomography is based on a geometrical model of the propagation of radiation. Such a model is appropriate for the interpretation of results of measurements with x rays that are passed through certain organs of the human body such as the brain or the kidney. It is, however, not appropriate when the organ is composed of soft tissues, such as women's breasts. To detect objects such as cancerous tumors in these regions it is more appropriate to use ultrasonic waves rather than x rays. Because the wavelengths of ultrasound used in such applications are of the same order of magnitude as the features to be detected, one can no longer use a geometrical model for propagation of the waves; i.e., one has to take into account diffraction and scattering. Instead of CAT one can then use so-called diffraction tomography (DT) to reconstruct the features of the object of interest (see Ref. 4 or Sec. 13.2 of Ref. 1). Just like CAT, diffraction tomography has found uses in many different fields, but unlike CAT, DT requires measurement not only of the intensity (squared amplitude) of the diffracted wave but also of its phase. However, phase measurements of optical waves present formidable practical difficulties.[5]

Since geometrical optics represents a short-wavelength limit of wave optics, and since the theory of CAT is based on a geometrical model of propagation (usually of x rays), one might suspect that DT reduces to CAT in the short-wavelength limit.[6] It would therefore seem that a clearer understanding of the range of validity of CAT might be obtained, as well as an indication of how to broaden its range of applicability, by studying its relationship to DT. It is the purpose of this paper to elucidate this question.

2. COMPUTED TOMOGRAPHY AND DIFFRACTION TOMOGRAPHY

Consider a monochromatic scalar plane wave $U_i(\mathbf{r}, t) = U_i(\mathbf{r})\exp(-i\omega t)$ of frequency ω and wave number $k = \omega/c$ (c being the speed of light in vacuum), with spatial dependence $U_i(\mathbf{r}) = \exp(ik\mathbf{s}_0 \cdot \mathbf{r})$ ($\mathbf{s}_0$ being the unit vector in the direction of propagation), incident upon a scattering object characterized by a potential $F(\mathbf{r})$, occupying a volume V (see Fig. 1). The time-independent total field $U(\mathbf{r})$ (incident plus scattered) satisfies the equation (Ref. 1, Sec. 13.1),

$$[\nabla^2 + k^2]U(\mathbf{r}) = -4\pi F(\mathbf{r})U(\mathbf{r}). \tag{1}$$

If the scattering potential is sufficiently weak, the total field may be represented in the form

$$U(\mathbf{r}) \approx U_i(\mathbf{r})\exp[\psi(\mathbf{r})], \tag{2}$$

where

$$\psi(\mathbf{r}) = \int_V F(\mathbf{r}')\frac{\exp(ik|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} \times \exp[-ik\mathbf{s}_0 \cdot (\mathbf{r} - \mathbf{r}')]\mathrm{d}^3r'. \tag{3}$$

This approximation to the total field is known as the first Rytov approximation (Ref. 1, Sec. 13.5).

We consider the arrangement shown in Fig. 1, in which the measurement plane $z = d$ is oriented perpendicular to the direction of incidence $\mathbf{s}_0$. This arrangement is often referred to as the *classical measurement configuration*. We may rewrite expression (3) in a form suitable for evaluation on the measurement plane by making use of an expansion of a spherical wave due to Weyl (Ref. 7, Sec. 3.2), which for $z > z'$ takes the form

$$\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} = \frac{ik}{2\pi}\int\int \frac{1}{m}\exp[ik[p\mathbf{s}_1 + q\mathbf{s}_2 + m\mathbf{s}_0]\cdot(\mathbf{r}-\mathbf{r}')]\mathrm{d}p\,\mathrm{d}q, \quad (4)$$

where $\mathbf{r} = x\mathbf{s}_1 + y\mathbf{s}_2 + z\mathbf{s}_0$, $\mathbf{r}' = x'\mathbf{s}_1 + y'\mathbf{s}_2 + z'\mathbf{s}_0$, $(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_0)$ is a triad of mutually orthogonal unit vectors,

$$m = \begin{cases} \sqrt{1-p^2-q^2} & \text{when } p^2+q^2 \leqslant 1 \\ i\sqrt{p^2+q^2-1} & \text{when } p^2+q^2 > 1 \end{cases}, \quad (5)$$

and the integration in Eq. (4) is taken over the complete p, q plane. By substituting expansion (4) into Eq. (3), one finds that the complex phase $\psi(\mathbf{r})$ on the plane $\mathbf{s}_0\cdot\mathbf{r} = d$ may be expressed in the form

$$\psi(x, y; d) = \frac{ik}{2\pi}\int_V \mathrm{d}^3r' \int\int \frac{1}{m}F(\mathbf{r}') \times \exp[ik(m-1)(d-z')] \times \exp\{ik[p(x-x') + q(y-y')]\}\mathrm{d}p\,\mathrm{d}q. \quad (6)$$

By carrying out the $\mathbf{r}'$ integration, one may express Eq. (6) in the simple form

$$\psi(x, y; d) = (2\pi)^2 ik \int\int \frac{1}{m}\tilde{F}(k[p\mathbf{s}_1 + q\mathbf{s}_2 + (m-1)\mathbf{s}_0]) \times \exp[ik(m-1)d]\exp[ik(px+qy)]\mathrm{d}p\,\mathrm{d}q, \quad (7)$$

where

$$\tilde{F}[\mathbf{K}] = \frac{1}{(2\pi)^3}\int_V F(\mathbf{r})\exp(-i\mathbf{K}\cdot\mathbf{r})\mathrm{d}^3r \quad (8)$$

is the three-dimensional Fourier transform of the scattering potential. Equation (7), which relates the complex phase of the scattered field on the plane $\mathbf{s}_0\cdot\mathbf{r} = d$ to the three-dimensional Fourier transform of the scattering potential, forms the basis of diffraction tomography in the first Rytov approximation.

We are interested in the short-wavelength limit of this formulation, for which intensity data alone may be used to determine the structure of the object. Let us define an intensity data function $D(x, y; d)$ by the formula

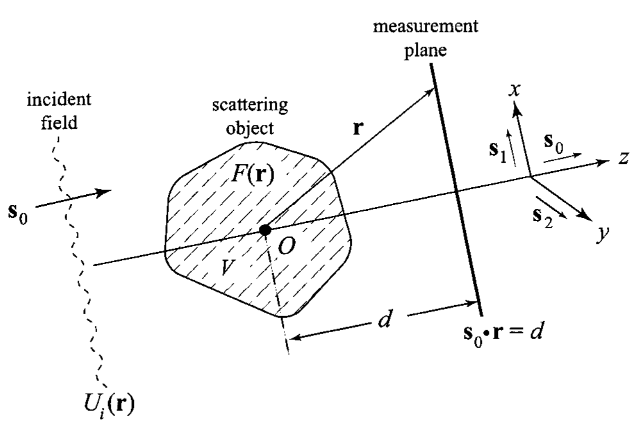

Fig. 1. Depiction of the arrangement and notation.

$$D(x, y; d) \equiv \ln\left\{\frac{|U(\mathbf{r})|^2}{|U_i(\mathbf{r})|^2}\right\}_{z=d} = 2\,\mathrm{Re}\{\psi(x, y; d)\} \quad (9)$$

(Re denoting the real part), where expression (2) was used. Suppose that the scattering potential $F(\mathbf{r})$ varies slowly over distances comparable to or less than some distance that which we will denote by σ. Under these circumstances it is reasonable to assume that the Fourier transform of the scattering potential is negligible for all $|\mathbf{K}| \gtrsim 2\pi/\sigma$, i.e., that

$$\tilde{F}[\mathbf{K}] \approx 0, \quad |\mathbf{K}| \gtrsim 2\pi/\sigma. \quad (10)$$

When the wavelength is sufficiently short, the wave number $k = 2\pi/\lambda$ will be large and consequently

$$k\sigma \equiv 2\pi\left(\frac{\sigma}{\lambda}\right) \gg 1. \quad (11)$$

When inequality (11) is valid, the scattering potential changes slowly over distances on the order of the wavelength.

Let us consider the approximate form of data function (9) when Eqs. (10) and (11) are satisfied. Using inequalities (10), one can show by straightforward calculation that the integrand of Eq. (7) and hence also the integrand of Eq. (6) is likely to be appreciable only for values of p and q such that

$$p^2 + q^2 \ll 1. \quad (12)$$

In this small p, q limit, the term $(1/m)\exp[ik(m-1) \times (d-z')]$ in the integrand of Eq. (6) may be expanded in a Taylor series about $p = q = 0$, and one then has

$$\frac{1}{m}\exp[ik(m-1)(d-z')] \approx 1 + [1 - ik(d-z')]\left(\frac{p^2+q^2}{2}\right) + O[(p^2+q^2)^2], \quad (13)$$

where O represents the "order" symbol common in asymptotic analysis. On substitution from Eq. (13) into Eq. (6), the data function may be expressed in a series form

$$D(x, y; d) = D_0(x,y;d) + D_1(x, y; d) + D_2(x, y; d) + \ldots. \quad (14)$$

The first term, D_0, which is expected to be the dominant term when inequality (11) is satisfied, is given by the expression

$$D_0(x, y; d) \approx 2\,\mathrm{Re}\left\{\frac{ik}{2\pi}\int_V \mathrm{d}^3r' \int\int F(\mathbf{r}') \times \exp\{ik[p(x-x') + q(y-y')]\}\mathrm{d}p\,\mathrm{d}q\right\}. \quad (15)$$

The p, q integrations may be evaluated by using the Fourier representation of the Dirac delta function, viz.,

$$\delta(x - x') = \frac{1}{2\pi}\int \exp[iu(x - x')]\mathrm{d}u. \tag{16}$$

Using Eq. (16), and recalling that $x' = \mathbf{s}_1 \cdot \mathbf{r}'$, $y' = \mathbf{s}_2 \cdot \mathbf{r}'$, one may then express the dominant term in the data function $D(x, y; d)$ in the simple form

$$D_0(x, y; d) \approx -\frac{4\pi}{k}\int \mathrm{Im}\{F(\mathbf{r}')\}\delta(x - \mathbf{s}_1 \cdot \mathbf{r}') \times \delta(y - \mathbf{s}_2 \cdot \mathbf{r}')\mathrm{d}^3r'. \tag{17}$$

Expression (17) has the form of the data function used in the theory of CAT studies,[8] if one makes the identification

$$\alpha(\mathbf{r}) = \frac{4\pi}{k}\mathrm{Im}\{F(\mathbf{r})\}, \tag{18}$$

where $\alpha(\mathbf{r})$ is the absorption coefficient of the object. One can obtain corrections to the CAT limit that are due to scattering by including higher-order terms in the Taylor series of expression (13). Using methods similar to those employed in deriving expression (17) and noting that

$$p^2 \exp[ikp(x - x')] = \frac{-1}{k^2}\frac{\partial^2}{\partial x'^2}\exp[ikp(x - x')], \tag{19}$$

with a similar expression involving q^2, one can express the first correction due to scattering beyond the CAT limit in the form

$$D_1(x, y; d) = \frac{2\pi}{k^3}\int [\mathrm{Im}\{\nabla_T^2 F(\mathbf{r}')\} - k\,\mathrm{Re}\{\nabla_T^2 F(\mathbf{r}')\} \times (d - \mathbf{s}_0 \cdot \mathbf{r}')]\delta(x - \mathbf{s}_1 \cdot \mathbf{r}') \times \delta(y - \mathbf{s}_2 \cdot \mathbf{r}')\mathrm{d}^3r', \tag{20}$$

where ∇_T^2 is the Laplacian with respect to the coordinates transverse to $\mathbf{s}_0$. It is interesting to note that Eq. (20) is comparable to a formula derived elsewhere[9] by a geometrical optics method for the intensity variations of a ray bundle propagating in a medium with real refractive index. In our calculation, if the refractive index is real valued, we expect that Eq. (20) will be the dominant term of the series in Eq. (14) in the short-wavelength limit.

Returning to Eq. (6), we may derive another, potentially more useful, expression for the data function valid in the short-wavelength limit. Instead of expanding the exponential in Eq. (6), let us expand the *exponent* in Eq. (7) in a Taylor series. As we have seen, when inequality (11) is satisfied, inequality (12) is also satisfied and, to a good approximation, we may write that

$$(m - 1) \approx [-\tfrac{1}{2}(p^2 + q^2)]. \tag{21}$$

Using Eqs. (7), (9), and (21), we may express the data function in the form

$$D(x, y; d) \approx \frac{(2\pi)^3 k}{\pi}\mathrm{Re}\left\{\int\int \frac{i}{m}\tilde{F}(k[p\mathbf{s}_1 + q\mathbf{s}_2 + (m - 1)\mathbf{s}_0])\exp\{ik[px + qy - d(p^2 + q^2)/2]\}\mathrm{d}p\,\mathrm{d}q\right\}. \tag{22}$$

Let us compare expression (22) with the definition of a two-dimensional fractional Fourier transform (FracFT) $\mathcal{F}_\theta^{(2)}$ (Refs. 10–14) of a function $f(u, v)$, viz.,

$$\mathcal{F}_\theta^{(2)} f(u, v) = \frac{i\exp(-i\theta)}{2\pi\sin\theta}\exp\left[-\frac{i}{2}\cot\theta(u^2 + v^2)\right] \times \int\int \exp\left[\frac{i}{\sin\theta}(uu' + vv')\right] \times \exp\left[-\frac{i}{2}\cot\theta(u'^2 + v'^2)\right] \times f(u', v')\mathrm{d}u'\mathrm{d}v', \tag{23}$$

where u, v and u', v' are dimensionless variables and θ is the *order* of the transform. It can be shown that

$$\mathcal{F}_0^{(2)} f(u, v) = f(u, v) \quad \text{(identity)}, \tag{24}$$

$$\mathcal{F}_{\theta_1}^{(2)}\mathcal{F}_{\theta_2}^{(2)} f(u, v) = \mathcal{F}_{\theta_1+\theta_2}^{(2)} f(u, v) \quad \text{(linearity)}, \tag{25}$$

$$\mathcal{F}_{\pi/2}^{(2)} f(u, v) = \hat{f}(u, v), \tag{26}$$

where $\hat{f}$ is the two-dimensional Fourier transform of f, defined by the formula

$$\hat{f}(u, v) \equiv \frac{1}{2\pi}\int\int f(u', v')\exp[i(uu' + vv')]\mathrm{d}u'\mathrm{d}v'. \tag{27}$$

To make the comparison, let us set

$$\cot\theta = kd, \tag{28}$$

$$\beta = k\sin\theta, \tag{29}$$

and

$$\Gamma(p, q) = \frac{1}{m}\tilde{F}(k[p\mathbf{s}_1 + q\mathbf{s}_2 + (m - 1)\mathbf{s}_0]). \tag{30}$$

By use of Eqs. (28), (29), (30), and (23), data function (22) may be expressed in the form

$$D(x, y; d) = 2(2\pi)^3\beta\,\mathrm{Re}\left\{\exp(i\theta)\exp\left(\frac{i}{2}\cot\theta[(\beta x)^2 + (\beta y)^2]\right)\mathcal{F}_\theta^{(2)}\Gamma(\beta x, \beta y)\right\}. \tag{31}$$

It is to be noted that the parameter θ depends only on quantities that are known, namely, the wavelength $\lambda = 2\pi/k$ and the position of the measurement plane, specified by d.

Equation (31) is the main result of this paper. It demonstrates that, under the conditions indicated by inequalities (10) and (11), the data function $D(x, y; d)$ is related to the three-dimensional Fourier transform of the scattering potential by a two-dimensional fractional Fourier transformation.

We may use Eq. (31) to investigate the relation between the CAT and DT methods. Let us first consider the form of the data function in the limit as $\lambda \rightarrow 0$, with σ and d fixed. From Eq. (28) it is clear that in this limit $\theta \rightarrow \pi/2$. Furthermore, as $\lambda \rightarrow 0$, inequality (11) is satisfied and consequently inequality (12) is also satisfied, and hence $m \approx 1$ for all values of p, q for which $\Gamma \neq 0$. In this limit.

$$\Gamma(p, q) \approx \tilde{F}[k(p\mathbf{s}_1 + q\mathbf{s}_2)] \equiv \Gamma_0(p, q). \tag{32}$$

It therefore follows from Eq. (31), since $\beta \rightarrow k$ as $\theta \rightarrow \pi/2$, that

$$\lim_{\lambda \rightarrow 0} D(x, y; d) = 2(2\pi)^3 k \,\mathrm{Re}\{i\mathcal{F}_{\pi/2}^{(2)}\Gamma_0(kx, ky)\}. \tag{33}$$

By use of Eqs. (26) and (27), Eq. (33) is seen to imply that

$$\lim_{\lambda \rightarrow 0} D(x, y; d) = -\frac{4\pi}{k}\int \mathrm{Im}\{F[\mathbf{r}']\}\delta(x - x') \times \delta(y - y')\mathrm{d}^3r', \tag{34}$$

which, as evident from Eq. (17), is the expression appropriate to CAT.

The wavelength λ of the incident light is, of course, never strictly zero. We would like to investigate under what conditions the data function is *well approximated* by inequality (17). For this purpose we note that in both DT and CAT the Fourier components of the scattering potential are recovered by taking a two-dimensional Fourier transform of the data function. If we set

$$\mathcal{D}[kx, ky; d] \equiv D[x, y; d] \tag{35}$$

for notational convenience, we may express the two-dimensional Fourier transform of $\mathcal{D}$ with respect to the dimensionless variables kx, ky as

$$\hat{\mathcal{D}}[u, v; d] = \frac{1}{2\pi}\int\int \exp[-ik(ux + vy)] \times \mathcal{D}[kx, ky; d]\mathrm{d}(kx)\mathrm{d}(ky). \tag{36}$$

On substituting from Eq. (31) into Eq. (36) and using definition (23) of the two-dimensional FracFT, we find that the Fourier transform of the data function may be expressed in the form

$$\hat{\mathcal{D}}[u, v; d] = (2\pi)^3 ik\left\{\exp\left[-\frac{i}{2}\cot\theta(u^2 + v^2)\right]\Gamma(u,v) - \exp\left[\frac{i}{2}\cot\theta(u^2 + v^2)\right][\Gamma(-u, -v)]^*\right\}, \tag{37}$$

the asterisk denoting the complex conjugate. If the CAT approximation is to be valid, this expression must be nearly equal to that of the Fourier transform of the CAT data function, which, on substituting from expression (17) into Eq. (36) and using Eq. (32), is found to be

$$\hat{\mathcal{D}}_0[u, v; d] = (2\pi)^3 ik\{\tilde{F}(ku\mathbf{s}_1 + kv\mathbf{s}_2) - [\tilde{F}(-ku\mathbf{s}_1 - kv\mathbf{s}_2)]^*\}. \tag{38}$$

We have already seen that, provided that inequality (11) is satisfied, we may approximate $(m - 1)$ as in expression (21). By use of expression (21), along with Eqs. (30) and (8), we may express the Fourier transform (37) of the data function in the form

$$\hat{\mathcal{D}}[u, v; d] = \frac{ik}{m}\left\{\int_V F(\mathbf{r}')\exp[-ik(ux' + vy')] \times \exp\left[\frac{i}{2}k(u^2 + v^2)(d - z')\right]\mathrm{d}^3r' - \int_V F^*(\mathbf{r}')\exp[-ik(ux' + vy')] \times \exp\left[-\frac{i}{2}k(u^2 + v^2)(d - z')\right]\mathrm{d}^3r'\right\}. \tag{39}$$

Equation (39) differs from Eq. (38) because of the presence of the complex z' exponentials in the integrands in Eq. (39). The CAT approximation will therefore be valid only if the exponents are sufficiently small (much less than $\pi/2$, at which value the exponentials become pure imaginary) for all values of u, v such that $\tilde{F}(ku\mathbf{s}_1 + kv\mathbf{s}_2) \neq 0$, as well as for all $z' \in V$, or that

$$\frac{1}{2}k(d - z')[u^2 + v^2] \ll \frac{\pi}{2} \quad \text{for all} \quad u^2 + v^2 \leqslant \frac{(2\pi)^2}{(k\sigma)^2} = \left(\frac{\lambda}{\sigma}\right)^2, \quad \text{and for all} \quad z' \in V. \tag{40}$$

If this inequality is satisfied for the maximum value of $u^2 + v^2$, it will be satisfied for all values of $u^2 + v^2$; similarly, if it is satisfied for the maximum value of $d - z'$, it will be satisfied for all values of $d - z'$. The maximum value of $u^2 + v^2$ is given in inequality (40); the maximum value of $d - z'$ is the distance along the direction $\mathbf{s}_0$ from the measurement plane to the far side of the scatterer, as shown in Fig. 2. Denoting this distance by L, inequality (40) may be expressed in the simple form

$$L \ll \frac{\sigma^2}{2\lambda}. \tag{41}$$

The physical significance of this inequality may readily be understood if we note that the quantity on the right-hand

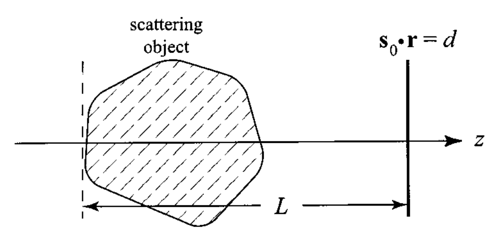

Fig. 2. Definition of the distance L relevant to inequality (40).

side of inequality (41) is of the order of the Rayleigh range,[15] well known in the theory of antenna design and in the theory of laser beam propagation.[17] In physical terms, inequality (41) implies that the detector plane must be well within this range. Under these circumstances, the Huygens secondary wavelets which proceed from the scatterer will essentially cancel each other out by destructive interference, *except in the forward direction*, as can be deduced by the use of the principle of stationary phase (Ref. 7, Sec. 3.3). The cancellation implies that within the region specified by inequality (41), the propagation is well described by geometrical optics. Conversely, when inequality (41) is not satisfied, the geometrical optics model does not apply and CAT will not provide reliable reconstruction of the object.

In tomographic experiments using x rays, which have photon energies of the order of 50 KeV, the wavelength of the incident radiation is of the order of 2.5×10^{-11} m. It is reasonable to assume that the variation σ of the index of refraction of objects typically imaged is of the order of 1 mm. For this choice of parameters, the Rayleigh range is found to be 20 km.

In contrast, in tomographic experiments using ultrasound, the ultrasonic waves typically have wavelengths of the order of 0.5 cm. Again, assuming that the variation σ is of the order of 1 mm, and assuming propagation of waves through water (for which the speed of propagation is approximately 1500 m/s), the Rayleigh range is found to be 0.1 mm.

3. EXAMPLE

We will now illustrate these results by an example. We consider scattering from an object with a scattering potential of Gaussian shape, i.e.,

$$F(\mathbf{r}) = A \exp(-r^2/2\sigma_0^2), \tag{42}$$

where A and σ_0 are real-valued positive constants. The three-dimensional spatial Fourier transform of this potential is given by the expression

$$\tilde{F}(\mathbf{K}) = \frac{A\sigma_0^3}{(2\pi)^{3/2}} \exp(-K^2\sigma_0^2/2), \tag{43}$$

implying that the scattering potential will be negligible for all values of $\mathbf{K}$ such that $|\mathbf{K}| \gtrsim 2\pi/\sigma_0$. In the limiting situation in which CAT is valid, we expect that the

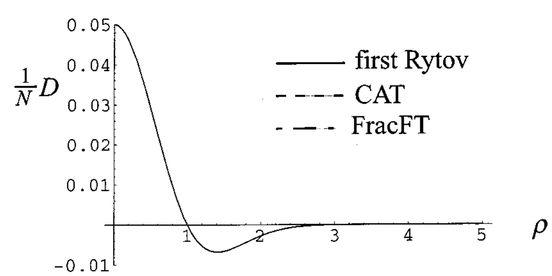

Fig. 3. First Rytov data function, CAT data function, and fractional Fourier data function for $k\sigma_0 = 40$, $kd = 40$. In the figure, the variable $\rho = (x^2 + y^2)^{1/2}/(\sqrt{2}\sigma_0)$ and the normalization constant $N = A(2\pi)^{3/2}\sigma_0/k$. For this particular choice of $k\sigma_0$ and kd, all three plots practically coincide.

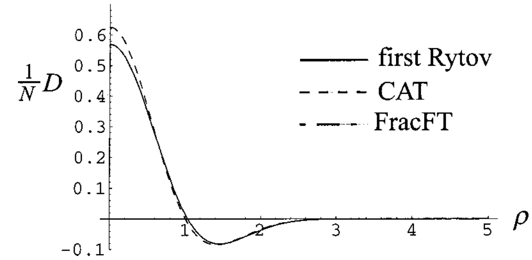

Fig. 4. First Rytov data function, CAT data function, and fractional Fourier data function for $k\sigma_0 = 40$, $kd = 500$.

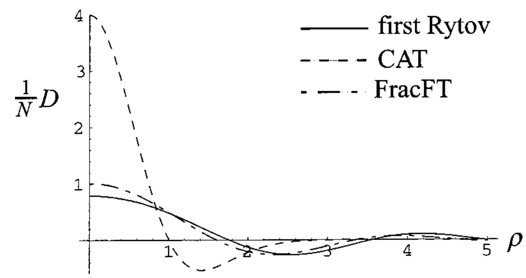

Fig. 5. First Rytov data function, CAT data function, and the fractional Fourier data function for $k\sigma_0 = 2$, $kd = 8$.

data function will have the form of the first nonzero term in series (14); on inspection of Eqs. (17) and (20) it can be seen that this term is given by Eq. (20). The effective width of the scatterer is $2\sigma_0$, as can be seen from Eq. (42); the distance L then satisfies the relation $L = \sigma_0 + d$.

We have seen that for the CAT approximation to be valid, inequalities (11) and (41) must be satisfied. In Fig. 3 we display the first Rytov data function,[16] [from Eqs. (7) and (9)], the FracFT approximation to that function [Eq. (31)], and the data function used in CAT studies [Eq. (20)] for the case when $k\sigma_0 = 40$, $kd = 40$. We see that with this choice of parameters, the three data functions practically coincide.

Figure 4 shows the three data functions for $k\sigma_0 = 40$, $kd = 500$. In this case, inequality (11) is satisfied but inequality (41) is not. One can therefore expect that the CAT data function will not provide a good approximation to the scattered field, and this, indeed, is evident in the figure. The fractional Fourier data function, which should be valid if inequality (11) is satisfied, is seen again to practically coincide with the first Rytov data function.

Figure 5 shows the three data functions when $k\sigma_0 = 2$, $kd = 8$; in this case neither inequality (11) nor inequality (41) is satisfied. It can be seen that there are significant differences between the first Rytov data function, the FracFT function, and CAT. It is to be observed, however, that even though inequality (11) is not satisfied, the FracFT data function illustrates fairly well the general behavior of the predictions of the first Rytov theory.

4. CONCLUSIONS

We have studied, within the accuracy of the first Rytov approximation, the relationship between computerized axial tomography (CAT) and diffraction tomography (DT). We found that when certain conditions are satisfied, CAT is a short-wavelength limit of DT. The conditions are the following: (1) the scattering potential varies slowly over distances of the order of a wavelength and (2) the measurement plane is at a distance small compared with a certain characteristic length, which turns out to be the so-called Rayleigh range, well known in antenna theory and, in a more restricted form, in the theory of laser beam propagation.[17]

Corrections to the CAT limit that account for the finite wavelength of the incident radiation have also been briefly discussed.

Our analysis brings into evidence an intimate relationship that exists between the theory of weak scattering [for which the first Rytov approximation and inequality (11) are satisfied] and fractional Fourier transforms. The expression of the data function based on the fractional Fourier transform [Eq. (31)] is distinct from both the CAT data function and the general DT data function and serves as an intermediate form between the two. The existence of a distinct intermediate form of the data function may eventually lead to the formulation of hybrid CAT–DT methods of reconstructing the scattering potential.

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15. For a thorough discussion of the Rayleigh range see, for instance, J. F. Ramsey, "Tubular beams from radiating apertures," in *Advances in Microwaves*, L. F. Young, ed. (Academic, New York, 1968), Vol. 3, pp. 127–221.
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