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The Rayleigh range of partially coherent beams

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Abstract

The concept of the Rayleigh range, well known in the theory of coherent beams, is extended to partially coherent beams. A simple formula is derived, which expresses it in terms of the rms widths of the source intensity and of the intensity of the field far from the source. In the special cases when the beam is completely coherent or is of the Gaussian Schell-model type, our formula reduces to known expressions. It is also shown that a partially coherent beam will always have a Rayleigh range that is shorter than that of a fully coherent beam with the same intensity distribution in the beam waist. © 2001 Published by Elsevier Science B.V.

Keywords: Rayleigh range; Beam quality; Partial coherence; Beam spreading

The Rayleigh range is used in antenna theory [1] and in the theory of lasers [2] to characterize the distance a beam propagates without spreading appreciably. When the source plane is the waist plane of a beam or an aperture illuminated by a plane wave, the Rayleigh range is a measure of the distance over which a beam may be considered effectively non-spreading. Although an important characteristic of any beam, this quantity has not been closely studied for partially coherent beams, except in two recent investigations concerning the so-called Gaussian Schell-model beams [3,4]. In this paper we investigate the Rayleigh range for partially coherent beams only.

We obtain a simple relation that expresses the Rayleigh range in terms of the second moments of the intensity in the source plane and of the far field. We show that, for the case of a coherent Gaussian beam, our formula reduces to the usual one. We investigate Schell-model beams in some detail.

We consider a field propagating into the half-space $z > 0$ (see Fig. 1). The plane $z = 0$ may represent the waist of an incident beam or an opaque screen with an aperture. The cross-spectral density $W_V(\mathbf{r}_1, \mathbf{r}_2, \omega)$ [5, Section 4.3.2] of the field at frequency ω at any pair of points $\mathbf{r}_1$, $\mathbf{r}_2$ in the half-space $z > 0$ may be expressed in terms of the cross-spectral density of the field $W_0(\boldsymbol{\rho}'_1, \boldsymbol{\rho}'_2, \omega)$ in the plane $z = 0$ as

$$W_V(\mathbf{r}_1, \mathbf{r}_2, \omega) = \frac{1}{(2\pi)^2} \int\int W_0(\boldsymbol{\rho}'_1, \boldsymbol{\rho}'_2, \omega) \frac{\partial}{\partial z'_1} \left\{ \frac{\exp(-\mathrm{i}k|\mathbf{r}_1 - \mathbf{r}'_1|)}{|\mathbf{r}_1 - \mathbf{r}'_1|} \right\}_{z'_1=0} \frac{\partial}{\partial z'_2} \left\{ \frac{\exp(\mathrm{i}k|\mathbf{r}_2 - \mathbf{r}'_2|)}{|\mathbf{r}_2 - \mathbf{r}'_2|} \right\}_{z'_2=0} \mathrm{d}^2\rho'_1 \, \mathrm{d}^2\rho'_2. \quad (1)$$

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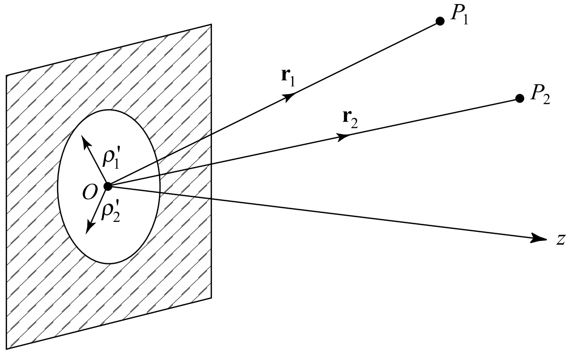

Fig. 1. Illustrating the notation relating to radiation from a partially coherent planar secondary source.

This formula readily follows from the representation of the cross-spectral density of the field as the ensemble average of a monochromatic field, $W_V(\mathbf{r}_1, \mathbf{r}_2, \omega) = \langle V^*(\mathbf{r}_1, \omega) V(\mathbf{r}_2, \omega) \rangle$ [5, Section 4.7.2], and from the Rayleigh–Sommerfeld formula of the first kind [6, Section 8.11.2]. We may make use of the Weyl representation for the free-space Green's function [5, Section 3.2],

$$\frac{\exp(\mathrm{i}k|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} = \frac{\mathrm{i}k}{2\pi} \int\int \frac{1}{s_z} \exp(\mathrm{i}k[s_x(x - x') + s_y(y - y') + s_z|z - z'|])\, \mathrm{d}^2 s, \tag{2}$$

where

$$s_z = \begin{cases} \sqrt{1 - s_x^2 - s_y^2}, & \text{when } |s_x^2 + s_y^2| \leqslant 1, \\ \mathrm{i}\sqrt{s_x^2 + s_y^2 - 1}, & \text{when } |s_x^2 + s_y^2| > 1, \end{cases} \tag{3}$$

and

$$k = \frac{\omega}{c}, \tag{4}$$

c being the speed of light in vacuum. The integration in Eq. (2) is taken over the entire s_x, s_y plane, with $\mathrm{d}^2 s$ representing the element $\mathrm{d}s_x\,\mathrm{d}s_y$. On substituting from Eq. (2) into Eq. (1), the latter equation takes on the form

$$\begin{aligned} W_V(\mathbf{r}_1, \mathbf{r}_2) = \frac{k^4}{(2\pi)^4} \int\int\int\int & W_0(\boldsymbol{\rho}_1', \boldsymbol{\rho}_2') \exp(-\mathrm{i}k[s_{1x}(x_1 - x_1') + s_{1y}(y_1 - y_1') + s_{1z}^* z_1]) \\ & \times \exp(\mathrm{i}k[s_{2x}(x_2 - x_2') + s_{2y}(y_2 - y_2') + s_{2z} z_2])\, \mathrm{d}^2\rho_1'\, \mathrm{d}^2\rho_2'\, \mathrm{d}^2 s_1\, \mathrm{d}^2 s_2, \end{aligned} \tag{5}$$

where $\mathrm{d}^2 s_i = \mathrm{d}s_{ix}\,\mathrm{d}s_{iy}$ $(i = 1, 2)$. From now on we do not display the frequency dependence of the cross-spectral density. We may integrate with respect to the primed variables and find that

$$W_V(\mathbf{r}_1, \mathbf{r}_2) = k^4 \int\int \widetilde{W}_0(-k\mathbf{s}_{1\perp}, k\mathbf{s}_{2\perp}) \exp(-\mathrm{i}k[s_{1x}x_1 + s_{1y}y_1 + s_{1z}^* z_1]) \exp(\mathrm{i}k[s_{2x}x_2 + s_{2y}y_2 + s_{2z}z_2])\, \mathrm{d}^2 s_1\, \mathrm{d}^2 s_2, \tag{6}$$

where

$$\widetilde{W}_0(\mathbf{K}_1, \mathbf{K}_2) = \frac{1}{(2\pi)^4} \int\int W_0(\boldsymbol{\rho}_1', \boldsymbol{\rho}_2') \exp(-\mathrm{i}[\mathbf{K}_1 \cdot \boldsymbol{\rho}_1' + \mathbf{K}_2 \cdot \boldsymbol{\rho}_2'])\, \mathrm{d}^2\rho_1'\, \mathrm{d}^2\rho_2' \tag{7}$$

is the four-dimensional spatial Fourier transform of the cross-spectral density of the field in the plane $z = 0$, and $\mathbf{s}_{i\perp} = s_{ix}\hat{x} + s_{iy}\hat{y}$ $(i = 1, 2)$, $\hat{x}$, $\hat{y}$ being unit vectors which, together with the z-direction, form a right-handed orthogonal coordinate system.

It is well known [5, Section 5.6] that a partially coherent field is beam-like only if the Fourier transform of the cross-spectral density of the field satisfies the approximate relation

$$\widetilde{W}_0(\mathbf{K}_1, \mathbf{K}_2) \approx 0 \quad \text{when } |\mathbf{K}_1|^2 > \alpha^2 \text{ or } |\mathbf{K}_2|^2 > \alpha^2, \tag{8}$$

for some $\alpha \ll k$. When relation (8) holds,

$$s_{iz} \approx 1 - \tfrac{1}{2}(s_{ix}^2 + s_{iy}^2). \tag{9}$$

Eq. (6) for the cross-spectral density of the field may then be expressed as

$$\begin{aligned} W_V(\mathbf{r}_1, \mathbf{r}_2) = k^4 \exp[\mathrm{i}k(z_2 - z_1)] \int\int \widetilde{W}_0(-k\mathbf{s}_{1\perp}, k\mathbf{s}_{2\perp}) \exp(-\mathrm{i}k[s_{1x}x_1 + s_{1y}y_1]) \exp(\mathrm{i}k[s_{2x}x_2 + s_{2y}y_2]) \\ \times \exp\left(\frac{\mathrm{i}k}{2}[s_{1x}^2 + s_{1y}^2]z_1\right) \exp\left(-\frac{\mathrm{i}k}{2}[s_{2x}^2 + s_{2y}^2]z_2\right) \mathrm{d}^2 s_1 \, \mathrm{d}^2 s_2. \end{aligned} \tag{10}$$

The intensity of the field at a point $(\boldsymbol{\rho}, z \geqslant 0)$ is given by the expression

$$I(\boldsymbol{\rho}, z) = W_V(\boldsymbol{\rho}, z; \boldsymbol{\rho}, z). \tag{11}$$

On substituting from Eq. (10) into this expression, it follows that

$$\begin{aligned} I(\boldsymbol{\rho}, z) = k^4 \int\int \widetilde{W}_0(-k\mathbf{s}_{1\perp}, k\mathbf{s}_{2\perp}) \exp(-\mathrm{i}k[s_{1x} - s_{2x}]x) \exp(-\mathrm{i}k[s_{1y} - s_{2y}]y) \\ \times \exp\left(\frac{\mathrm{i}k}{2}[s_{1x}^2 + s_{1y}^2 - s_{2x}^2 - s_{2y}^2]z\right) \mathrm{d}^2 s_1 \, \mathrm{d}^2 s_2. \end{aligned} \tag{12}$$

The Rayleigh range of a beam is defined as the distance from the plane $z = 0$ at which the width of the beam increases by a factor of $\sqrt{2}$. The width of the beam may be specified by the its normalized spatial variance,

$$\overline{\rho^2(z)} \equiv \frac{R_2}{R_0}, \tag{13}$$

where

$$R_0 = \int I(\boldsymbol{\rho}, z) \, \mathrm{d}^2\rho, \tag{14}$$

and

$$R_2 = \int \rho^2 I(\boldsymbol{\rho}, z) \, \mathrm{d}^2\rho \tag{15}$$

are the zeroth and second moments of the intensity in the plane $z = \text{const}$. We have assumed for simplicity that the the beam is centered at $\boldsymbol{\rho} = 0$ for all $z \geqslant 0$. The evaluation of R_0 is straightforward. On substituting from Eq. (12) into Eq. (14), it follows that

$$\begin{aligned} R_0 = k^4 \int\int\int \widetilde{W}_0(-k\mathbf{s}_{1\perp}, k\mathbf{s}_{2\perp}) \exp(-\mathrm{i}k[s_{1x} - s_{2x}]x) \exp(-\mathrm{i}k[s_{1y} - s_{2y}]y) \\ \times \exp\left(\frac{\mathrm{i}k}{2}[s_{1x}^2 + s_{1y}^2 - s_{2x}^2 - s_{2y}^2]z\right) \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}^2 s_1 \, \mathrm{d}^2 s_2. \end{aligned} \tag{16}$$

By using the Fourier representation of the Dirac delta function, i.e.

$$\delta(u_1-u_2)=\frac{1}{2\pi}\int \exp[\mathrm{i}(u_1-u_2)x]\,\mathrm{d}x, \tag{17}$$

expression (16) for R_0 becomes

$$R_0=(2\pi)^2k^4\int\int \widetilde{W}_0(-k\mathbf{s}_{1\perp},k\mathbf{s}_{2\perp})\delta(ks_{1x}-ks_{2x})\delta(ks_{1y}-ks_{2y})\exp\left(\frac{\mathrm{i}k}{2}[s_{1x}^2+s_{1y}^2-s_{2x}^2-s_{2y}^2]z\right)\mathrm{d}^2s_1\,\mathrm{d}^2s_2. \tag{18}$$

Upon integrating over one of the **s**-variables, it follows that [1]

$$R_0=(2\pi)^2k^2\int \widetilde{W}_0(-k\mathbf{s}_{\perp},k\mathbf{s}_{\perp})\,\mathrm{d}^2s. \tag{19}$$

It is to be noted that this expression is independent of z; the z-dependence of $\overline{\rho^2(z)}$ is entirely determined by R_2. Furthermore, it follows from Eqs. (14) and (19) that

$$\int I(\boldsymbol{\rho},0)\,\mathrm{d}^2\rho=(2\pi)^2k^2\int \widetilde{W}_0(-k\mathbf{s}_{\perp},k\mathbf{s}_{\perp})\,\mathrm{d}^2s. \tag{20}$$

The calculation for R_2 is more involved. Noting that $\rho^2=x^2+y^2$, R_2 may be expressed in the form

$$R_2=k^4\int\int\int \widetilde{W}_0(-k\mathbf{s}_{1\perp},k\mathbf{s}_{2\perp})\exp(-\mathrm{i}k[s_{1x}-s_{2x}]x)\exp(-\mathrm{i}k[s_{1y}-s_{2y}]y)$$
$$\times\exp\left(\frac{\mathrm{i}k}{2}[s_{1x}^2+s_{1y}^2-s_{2x}^2-s_{2y}^2]z\right)(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}^2s_1\,\mathrm{d}^2s_2. \tag{21}$$

Furthermore, we may make use of the identity

$$-\frac{1}{2k^2}(\nabla_{s1}^2+\nabla_{s2}^2)\exp(-\mathrm{i}k[s_{1x}-s_{2x}]x)\exp(-\mathrm{i}k[s_{1y}-s_{2y}]y)$$
$$=(x^2+y^2)\exp(-\mathrm{i}k[s_{1x}-s_{2x}]x)\exp(-\mathrm{i}k[s_{1y}-s_{2y}]y) \tag{22}$$

in the integrand of Eq. (21). Here ∇_{si} $(i=1,2)$, is the gradient with respect to $\mathbf{s}_i$. On substituting from Eq. (22) into Eq. (21), and applying twice Green's first identity, we find that

$$R_2=-\frac{k^2}{2}\int\int\int \exp(-\mathrm{i}k[s_{1x}-s_{2x}]x)\exp(-\mathrm{i}k[s_{1y}-s_{2y}]y)$$
$$\times(\nabla_{s1}^2+\nabla_{s2}^2)\left[\widetilde{W}_0(-k\mathbf{s}_{1\perp},k\mathbf{s}_{2\perp})\exp\left(\frac{\mathrm{i}k}{2}[s_{1x}^2+s_{1y}^2-s_{2x}^2-s_{2y}^2]z\right)\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}^2s_1\,\mathrm{d}^2s_2. \tag{23}$$

We may now use Eq. (17) again, and find that

$$R_2=-\frac{(2\pi)^2k^2}{2}\int\int \delta(ks_{1x}-ks_{2x})\delta(ks_{1y}-ks_{2y})$$
$$\times(\nabla_{s1}^2+\nabla_{s2}^2)\left[\widetilde{W}_0(-k\mathbf{s}_{1\perp},k\mathbf{s}_{2\perp})\exp\left(\frac{\mathrm{i}k}{2}[s_{1x}^2+s_{1y}^2-s_{2x}^2-s_{2y}^2]z\right)\right]\mathrm{d}^2s_1\,\mathrm{d}^2s_2. \tag{24}$$

[1] It can be shown from Eq. (19) that the quantity R_0 represents, within the accuracy of the paraxial approximation, the power radiated across a large hemisphere in the half-space $z\geqslant 0$, centered on the origin. See, for instance, the discussion preceding Eq. (40) below.

Upon integrating over $\mathbf{s}_1$, this expression may be further simplified to

$$R_2 = -\frac{(2\pi)^2}{2} \int \left\{ (\nabla_{s1}^2 + \nabla_{s2}^2) \left[\widetilde{W}_0(-k\mathbf{s}_{1\perp}, k\mathbf{s}_{2\perp}) \exp\left(\frac{\mathrm{i}k}{2} [s_{1x}^2 + s_{1y}^2 - s_{2x}^2 - s_{2y}^2] z \right) \right] \right\}_{\mathbf{s}_1 = \mathbf{s}_2 = \mathbf{s}} \mathrm{d}^2 s. \tag{25}$$

The derivatives with respect to $\mathbf{s}_1$ and $\mathbf{s}_2$ can be readily evaluated by the use of Eq. (7). One then finds that

$$R_2 = A_2 + B_2 z + C_2 z^2, \tag{26}$$

where

$$A_2 = \int W_0(\boldsymbol{\rho}, \boldsymbol{\rho}) \rho^2 \, \mathrm{d}^2 \rho, \tag{27}$$

$$C_2 = (2\pi)^2 k^2 \int \widetilde{W}_0(-k\mathbf{s}_\perp, k\mathbf{s}_\perp) s_\perp^2 \, \mathrm{d}^2 s, \tag{28}$$

and

$$B_2 = \frac{k^2}{(2\pi)^2} \int \int \int W_0(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) [\boldsymbol{\rho}_1 \cdot \mathbf{s} + \boldsymbol{\rho}_2 \cdot \mathbf{s}] \exp(-\mathrm{i}k[\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1] \cdot \mathbf{s}) \, \mathrm{d}^2\rho_1 \, \mathrm{d}^2\rho_2 \, \mathrm{d}^2 s. \tag{29}$$

The quantity B_2 may be simplified further. We first note that

$$\boldsymbol{\rho}_1 \cdot \nabla_1 \exp(-\mathrm{i}k[\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1] \cdot \mathbf{s}) = \mathrm{i}(k\boldsymbol{\rho}_1 \cdot \mathbf{s}) \exp(-\mathrm{i}k[\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1] \cdot \mathbf{s}), \tag{30}$$

where ∇_1 is the gradient taken with respect to $\boldsymbol{\rho}_1$, with a similar expression for $\boldsymbol{\rho}_2$. We may then express B_2 as

$$B_2 = \frac{1}{(2\pi)^2 \mathrm{i}k} k^2 \int \int \int W_0(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) [-\boldsymbol{\rho}_1 \cdot \nabla_1 + \boldsymbol{\rho}_2 \cdot \nabla_2] \exp(-\mathrm{i}k[\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1] \cdot \mathbf{s}) \, \mathrm{d}^2\rho_1 \, \mathrm{d}^2\rho_2 \, \mathrm{d}^2 s. \tag{31}$$

The right-hand side may be evaluated by use of the divergence theorem, and one finds that

$$B_2 = \frac{1}{\mathrm{i}k} \int \{[-\boldsymbol{\rho}_1 \cdot \nabla_1 + \boldsymbol{\rho}_2 \cdot \nabla_2] W_0(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)\}_{\boldsymbol{\rho}_1 = \boldsymbol{\rho}_2 = \boldsymbol{\rho}} \, \mathrm{d}^2 \rho. \tag{32}$$

On substituting from Eq. (26) into Eq. (13), we may express the beam width at distance z as

$$\overline{\rho^2(z)} = \frac{A_2 + B_2 z + C_2 z^2}{R_0}, \tag{33}$$

where R_0 is given by Eq. (19), and A_2, B_2, and C_2 are given by Eqs. (27), (32), and (28), respectively. The functional form of Eq. (33) is perhaps not surprising; it has been known for some time that fully coherent beams satisfy such a relation [7]. Eq. (33) was derived for partially coherent beams in a slightly different form in Ref. [8]. It is to be noted that A_2, B_2 and C_2 are all real. This fact is obvious for A_2 and C_2 from their definitions; for B_2 it follows from the Hermiticity of the cross-spectral density,

$$W_0(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = W_0^*(\boldsymbol{\rho}_2, \boldsymbol{\rho}_1), \tag{34}$$

that Eq. (32) may be expressed in the form

$$B_2 = -\frac{2}{k} \int \mathrm{Im}\{\boldsymbol{\rho}' \cdot \nabla' W_0(\boldsymbol{\rho}', \boldsymbol{\rho})\}_{\boldsymbol{\rho}' = \boldsymbol{\rho}} \, \mathrm{d}^2 \rho, \tag{35}$$

where Im denotes the imaginary part and ∇' is the gradient with respect to $\boldsymbol{\rho}'$. It is to be noted that $B_2 = 0$ if W_0 is real, which is the case if the field in the plane $z = 0$ contains the waist of the beam or is an aperture illuminated by a normally incident plane wave.

Recalling the fact that the Rayleigh range is the distance z_R at which the width of the beam increases to $\sqrt{2}$ the width of the beam in the plane $z = 0$, we have

$$\frac{\overline{\rho^2(z_R)}}{\overline{\rho^2(0)}} = 2. \tag{36}$$

On substituting from Eq. (33) into Eq. (36), it follows that

$$C_2 z_R^2 + B_2 z_R - A_2 = 0, \tag{37}$$

so that

$$z_R = \frac{-B_2^2 + \sqrt{B_2^2 + 4A_2C_2}}{2C_2}. \tag{38}$$

In the case when $B_2 = 0$, the Rayleigh range takes on the simple form

$$z_R = \sqrt{\frac{A_2}{C_2}} = \frac{1}{2\pi k}\sqrt{\frac{\int W_0(\boldsymbol{\rho},\boldsymbol{\rho})\rho^2\,\mathrm{d}^2\rho}{\int \widetilde{W}_0(-k\mathbf{s}_\perp, k\mathbf{s}_\perp)s_\perp^2\,\mathrm{d}^2s}}. \tag{39}$$

The denominator on the right of Eq. (39) has a simple meaning. To see this we recall that the radiant intensity $J(\mathbf{s})$ generated by the source in the direction specified by the unit vector $\mathbf{s}$, i.e. the average power at frequency ω radiated by the source into the unit solid angle around the $\mathbf{s}$-direction is given by the expression [5, (Eq. 5.3-8)]

$$J(\mathbf{s}) = (2\pi k)^2\widetilde{W}_0(-k\mathbf{s}_\perp, k\mathbf{s}_\perp)\cos^2\theta, \tag{40}$$

where $\mathbf{s}_\perp$ is the projection (considered as a two-dimensional vector) of the unit vector $\mathbf{s}$ onto the plane $z = 0$, and θ is the angle which the unit vector $\mathbf{s}$ makes with the z-axis. For a beam which propagates close to the z-axis, $\cos\theta \approx 1$, and it follows from Eqs. (39) and (40) that the Rayleigh range is given by the simple formula

$$z_R = \frac{\sigma_I}{\sigma_J}, \tag{41}$$

where

$$\sigma_I^2 = \frac{\int \rho^2 I_0(\boldsymbol{\rho})\,\mathrm{d}^2\rho}{\int I_0(\boldsymbol{\rho})\,\mathrm{d}^2\rho}, \tag{42}$$

and

$$\sigma_J^2 = \frac{\int s_\perp^2 J(\mathbf{s})\,\mathrm{d}^2s}{\int J(\mathbf{s})\,\mathrm{d}^2s} \tag{43}$$

are the normalized second moments of the source intensity $I_0(\boldsymbol{\rho}) \equiv I(\boldsymbol{\rho}, 0)$ and of the radiant intensity $J(\mathbf{s})$, respectively.

The simple formula (41) is the main result of this paper. We note two special cases of it.

(1) *Fully coherent beams:* When the field is spatially fully coherent, the cross-spectral density of the field factorizes in the form [5, Section 4.5.3]

$$W_0(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = U_0^*(\boldsymbol{\rho}_1)U_0(\boldsymbol{\rho}_2). \tag{44}$$

On substituting from Eq. (44) into Eq. (39), the Rayleigh range for a fully coherent beam may be written in terms of U_0 as

$$z_{\mathrm{R}} = \frac{1}{2\pi k}\sqrt{\frac{\int |U_0(\boldsymbol{\rho})|^2 \rho^2 \,\mathrm{d}^2\rho}{\int |\widetilde{U}_0(k\mathbf{s}_\perp)|^2 s_\perp^2 \,\mathrm{d}^2 s}}, \tag{45}$$

where

$$\widetilde{U}_0(\mathbf{K}) = \frac{1}{(2\pi)^2}\int U_0(\boldsymbol{\rho})\exp(-\mathrm{i}\mathbf{K}\cdot\boldsymbol{\rho})\,\mathrm{d}^2\rho \tag{46}$$

is the two-dimensional Fourier transform of the field in the plane $z = 0$.

As an example, consider a fully coherent beam with a Gaussian profile and no phase front curvature in the plane $z = 0$. Then

$$U_0(\boldsymbol{\rho}) = \exp(-\rho^2/w_0^2). \tag{47}$$

On substitution from Eq. (47) into Eq. (45) one finds that the Rayleigh range for such a beam is given by the expression

$$z_{\mathrm{R}} = \frac{\pi w_0^2}{\lambda}, \tag{48}$$

where $\lambda = 2\pi/k$ is the wavelength of the field. Eq. (48) is the well-known expression for the Rayleigh range of a coherent Gaussian beam [2, p. 668].

(2) *Schell-model beams:* If B_2 is zero, as we now assume, the Rayleigh range is given by formula (39). Let us also assume that the field in the plane $z = 0$ is of the Schell-model type [5, Section 5.3.2], i.e. that W_0 has the form

$$W_0(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \sqrt{I_0(\boldsymbol{\rho}_1)}\sqrt{I_0(\boldsymbol{\rho}_2)}\,\mu_0(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1), \tag{49}$$

where $I_0(\boldsymbol{\rho})$ is the intensity of the field in the plane $z = 0$, and $\mu_0(\boldsymbol{\rho}')$ is the spectral degree of coherence of the field at two points in that plane, assumed to depend only upon the difference between the position vectors $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ of the two points. It is to be noted that the numerator of Eq. (39) depends only upon the intensity of the field in the plane $z = 0$. For a given source intensity profile, the influence of the state of coherence of the beam upon the Rayleigh range is determined entirely by the denominator of Eq. (39). We therefore consider the quantity [2]

$$\overline{s^2} \equiv (2\pi)^4 \int \widetilde{W}_0(-k\mathbf{s}_\perp, k\mathbf{s}_\perp) s_\perp^2 \,\mathrm{d}^2 s, \tag{50}$$

where the factor $(2\pi)^4$ has been introduced for convenience. By use of Eq. (7), this formula may be re-written as

$$\overline{s^2} = \int\int\int W_0(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)\exp[-\mathrm{i}k\mathbf{s}\cdot(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1)] s_\perp^2 \,\mathrm{d}^2 s\,\mathrm{d}^2\rho_1\,\mathrm{d}^2\rho_2. \tag{51}$$

On substituting from Eq. (49) into Eq. (51), and introducing the variables

$$\boldsymbol{\rho} = \frac{\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2}{2}, \qquad \boldsymbol{\rho}' = \boldsymbol{\rho}_2 - \boldsymbol{\rho}_1, \tag{52}$$

Eq. (51) may be expressed in the form

$$\overline{s^2} = \int\int\int h_0(\boldsymbol{\rho} + \boldsymbol{\rho}'/2) h_0(\boldsymbol{\rho} - \boldsymbol{\rho}'/2)\mu_0(\boldsymbol{\rho}')\exp(-\mathrm{i}k\mathbf{s}\cdot\boldsymbol{\rho}') s_\perp^2 \,\mathrm{d}^2 s\,\mathrm{d}^2\rho\,\mathrm{d}^2\rho', \tag{53}$$

[2] This quantity has been evaluated for Schell-model beams with one-dimensional cross-sections in investigations of the M^2 factor [9].

where

$$h_0(\boldsymbol{\rho}) \equiv \sqrt{I_0(\boldsymbol{\rho})}. \tag{54}$$

By introducing the *source integrated intensity*,

$$C_0(\boldsymbol{\rho}') = \int h_0(\boldsymbol{\rho} + \boldsymbol{\rho}'/2) h_0(\boldsymbol{\rho} - \boldsymbol{\rho}'/2)\, \mathrm{d}^2\rho, \tag{55}$$

we may express $\overline{s^2}$ as

$$\overline{s^2} = \int\int C_0(\boldsymbol{\rho}')\mu_0(\boldsymbol{\rho}') \exp(-\mathrm{i}k\mathbf{s}\cdot\boldsymbol{\rho}') s_\perp^2\, \mathrm{d}^2 s\, \mathrm{d}^2\rho'. \tag{56}$$

Noting that

$$s_\perp^2 \exp(-\mathrm{i}k\mathbf{s}\cdot\boldsymbol{\rho}') = -\frac{1}{k^2}\nabla_{\rho'}^2 \exp(-\mathrm{i}k\mathbf{s}\cdot\boldsymbol{\rho}'), \tag{57}$$

where $\nabla_{\rho'}$ is the gradient with respect to $\boldsymbol{\rho}'$, we may express Eq. (56) in the form

$$\overline{s^2} = -\frac{1}{k^2}\int\int C_0(\boldsymbol{\rho}')\mu_0(\boldsymbol{\rho}')\nabla_{\rho'}^2 \exp(-\mathrm{i}k\mathbf{s}\cdot\boldsymbol{\rho}')\, \mathrm{d}^2 s\, \mathrm{d}^2\rho'. \tag{58}$$

Let us assume that the intensity I_0 and the spectral degree of coherence μ_0 are continuous functions of position. This assumption excludes the case of fields truncated by hard-edged apertures, for which it is known that the far-zone second moments of the fields do not exist [10]. Furthermore, let us assume that the (two-dimensional) gradient of the spectral degree of coherence is a continuous function of position. Under these circumstances, we may use Green's theorem with respect to the integration over $\boldsymbol{\rho}'$, and rewrite Eq. (58) as

$$\begin{aligned}\overline{s^2} = & -\frac{1}{k^2}\int\int \exp(-\mathrm{i}k\mathbf{s}\cdot\boldsymbol{\rho}')\nabla_{\rho'}^2\left[C_0(\boldsymbol{\rho}')\mu_0(\boldsymbol{\rho}')\right] \mathrm{d}^2 s\, \mathrm{d}^2\rho' - \frac{1}{k^2}\int\int_L \left[C_0(\boldsymbol{\rho}')\mu_0(\boldsymbol{\rho}')\nabla_{\rho'}\exp(-\mathrm{i}k\mathbf{s}\cdot\boldsymbol{\rho}')\right. \\ & \left. - \exp(-\mathrm{i}k\mathbf{s}\cdot\boldsymbol{\rho}')\nabla_{\rho'}(C_0(\boldsymbol{\rho}')\mu_0(\boldsymbol{\rho}'))\right]\cdot\mathbf{n}\, \mathrm{d}^2 s\, \mathrm{d}l.\end{aligned} \tag{59}$$

In the second term on the right of Eq. (59), the integral with respect to $\boldsymbol{\rho}'$ is over the perimeter L of the aperture, $\mathbf{n}$ is the outward normal to the perimeter, and $\mathrm{d}l$ is an infinitesimal line element along the perimeter. Because of our assumptions about the continuity of the intensity and of the spectral degree of coherence in the plane $z = 0$, this integral over the perimeter of the aperture vanishes and we are left with

$$\overline{s^2} = -\frac{1}{k^2}\int\int \exp(-\mathrm{i}k\mathbf{s}\cdot\boldsymbol{\rho}')\nabla_{\rho'}^2\left[C_0(\boldsymbol{\rho}')\mu_0(\boldsymbol{\rho}')\right] \mathrm{d}^2 s\, \mathrm{d}^2\rho'. \tag{60}$$

This expression may be further simplified by use of the Fourier representation of the two-dimensional Dirac delta function, i.e.

$$\delta^{(2)}(\boldsymbol{\rho}) = \frac{k^2}{(2\pi)^2}\int \exp(\mathrm{i}k\boldsymbol{\rho}\cdot\mathbf{s})\, \mathrm{d}^2 s. \tag{61}$$

On substituting from Eq. (61) into Eq. (60), and integrating over $\boldsymbol{\rho}'$, we find that

$$\overline{s^2} = -\frac{(2\pi)^2}{k^4}\nabla_{\rho'}^2\left[C_0(\boldsymbol{\rho}')\mu_0(\boldsymbol{\rho}')\right]_{\rho'=0}. \tag{62}$$

The right-hand side of this expression may be expanded in the form

$$\overline{s^2} = -\frac{(2\pi)^2}{k^4}\left(\mu_0(0)\nabla_{\rho'}^2 C_0(\boldsymbol{\rho}')|_{\rho'=0} + 2\nabla_{\rho'}\mu_0(\boldsymbol{\rho}')\cdot\nabla_{\rho'}C_0(\boldsymbol{\rho}')|_{\rho'=0} + C_0(0)\nabla_{\rho'}^2\mu_0(\boldsymbol{\rho}')|_{\rho'=0}\right). \tag{63}$$

Because μ_0 takes on the maximum value of unity at $\boldsymbol{\rho}' = 0$, and is stationary at that point, we may discard the second term in the above equation, and simplify the first term, resulting in the formula

$$\overline{s^2} = -\frac{(2\pi)^2}{k^4}(\nabla^2_{\rho'} C_0(\boldsymbol{\rho}')|_{\boldsymbol{\rho}'=0} + C_0(0)\nabla^2_{\rho'}\mu_0(\boldsymbol{\rho}')|_{\boldsymbol{\rho}'=0}). \tag{64}$$

The first term on the right is independent of the coherence properties of the source. It may be shown by Fourier decomposition of $C_0(\boldsymbol{\rho}')$ that

$$\nabla^2_{\rho'} C_0(\boldsymbol{\rho}')|_{\boldsymbol{\rho}'=0} = -(2\pi)^2 k^4 \int s_\perp^2 \tilde{h}_0^*(k\mathbf{s}_\perp)\tilde{h}_0(k\mathbf{s}_\perp)\,\mathrm{d}^2 s. \tag{65}$$

Noting that the radiant intensity of a fully coherent field with amplitude $h_0(\boldsymbol{\rho})$ and constant phase in the plane $z = 0$ is (assuming again that $\cos\theta \approx 1$)

$$[J(\mathbf{s})]_{\mathrm{coh}} = (2\pi k)^2 \tilde{h}_0^*(k\mathbf{s}_\perp)\tilde{h}_0(k\mathbf{s}_\perp), \tag{66}$$

Eq. (65) may be expressed as

$$\nabla^2_{\rho'} C_0(\boldsymbol{\rho}')|_{\boldsymbol{\rho}'=0} = -k^2 \int [J(\mathbf{s})]_{\mathrm{coh}} s_\perp^2\,\mathrm{d}^2 s. \tag{67}$$

Furthermore, it is clear from the definition (55) that

$$C_0(0) = \int I_0(\boldsymbol{\rho})\,\mathrm{d}^2\rho. \tag{68}$$

We then find, on substituting from Eqs. (67) and (68) into Eq. (64), and from Eq. (64) into Eq. (39), that

$$z_{\mathrm{R}} = \sqrt{\frac{\int I_0(\boldsymbol{\rho})\rho^2\,\mathrm{d}^2\rho}{\left[\int [J(\mathbf{s})]_{\mathrm{coh}} s_\perp^2\,\mathrm{d}^2 s - \nabla^2_\rho \mu_0(\boldsymbol{\rho})|_{\boldsymbol{\rho}=0}\int I_0(\boldsymbol{\rho})\,\mathrm{d}^2\rho/k^2\right]}}. \tag{69}$$

This formula may be simplified by the use of Eqs. (42) and (43). One then obtains for the Rayleigh range the expression

$$z_{\mathrm{R}} = \frac{\sigma_I}{\sqrt{[\sigma_J^2]_{\mathrm{coh}} - \frac{1}{k^2}\nabla^2_\rho \mu_0(\boldsymbol{\rho})|_{\boldsymbol{\rho}=0}}}. \tag{70}$$

Evidently the effect of the state of coherence on the Rayleigh range is entirely contained within the second term in the denominator of Eq. (70). Because $\mu_0(\boldsymbol{\rho}')$ has a maximum at $\boldsymbol{\rho}' = 0$, the Laplacian at that point will always be negative. A partially coherent beam will therefore always have a Rayleigh range shorter than a fully coherent one for beams with the same intensity across the waist or within the aperture.

As an example let us consider Gaussian Schell-model sources. Then

$$I_0(\boldsymbol{\rho}) = A^2 \exp(-2\rho^2/w_0^2) \tag{71}$$

and

$$\mu_0(\boldsymbol{\rho}') = \exp(-\rho'^2/2\sigma_\mu^2), \tag{72}$$

where w_0 and σ_μ are positive constants. On substituting from these formulas into Eq. (70), it is readily found that

$$z_{\mathrm{R}} = \frac{kw_0}{2}\sqrt{\frac{1}{1/w_0^2 + 1/\sigma_\mu^2}}, \tag{73}$$

which is in agreement with results derived previously [3].

In closing, it is worthwhile to compare the Rayleigh range to the M^2 factor which has been accepted as a standard measure of laser beam quality [7]. In our notation, M^2 may be written as [3]

$$M^2 = k\sigma_I\sigma_J, \tag{74}$$

where σ_I and σ_J are defined in Eqs. (42) and (43), respectively. It can be seen from Eqs. (41) and (74) that knowledge of M^2 and z_R is equivalent to knowledge of σ_I and $k\sigma_J$, because one pair of the parameters may be derived from the other. Though similar methods were used in obtaining z_R as have been used to calculate M^2 previously, the two quantities should not be considered to be equivalent. In fact, it was shown in an earlier paper that both z_R and M^2 should be specified to give a satisfactory description of a beam [11].

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[3] Our choice of normalization is based on the requirement that $M^2 = 1$ for a fully coherent Gaussian beam.