# Anomalous Behavior of Spectra near Phase Singularities of Focused Waves

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It is shown that remarkable spectral changes take place in the neighborhood of phase singularities near the focus of a converging, spatially fully coherent polychromatic wave diffracted at an aperture. In particular, when the spectrum of the wave in the aperture consists of a single line with a narrow Gaussian profile, the spectrum near a phase singularity (i.e., near points of zero intensity of some particular spectral component) changes drastically along a closed loop around the singularity. The spectrum is redshifted at some points, blueshifted at others, and is split into two lines elsewhere.

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During the past few years, a great deal of attention has been paid to the structure of wave fields in the neighborhood of points where the field amplitude has zero value. At such points the phase of the wave is singular. Studies of phenomena associated with phase singularities are gradually developing into a new branch of physical optics, sometimes called *singular optics* [1]. It is a rich subject [2], because many different kinds of behavior may exist near singular points, such as wave front dislocations [3] and optical vortices. The analysis of such phenomena can probably best be understood within the framework of topology and catastrophe theory [4].

The majority of publications concerned with singular optics deal with monochromatic waves [5]. In the present paper, we show that a new kind of anomalous behavior may take place in the neighborhood of a phase singularity when the field is polychromatic. More specifically, we show that, in the focal region of a spatially fully coherent converging polychromatic spherical wave diffracted at an aperture, the spectrum changes drastically along a closed loop around a phase singularity. In particular, when the spectrum of the incident light consists of a single line of Gaussian profile centered at a frequency $\omega_0$, the spectrum of the focused field along a closed loop around a phase singularity of the spectral component of frequency $\omega_0$ is redshifted at some points, blueshifted at others, and splits into two lines elsewhere. We illustrate these *diffraction-induced* spectral changes by numerical examples.

Consider first a monochromatic, spherical wave emerging from a circular aperture of radius $a$ and converging towards the geometrical focal point $O$ (see Fig. 1). The field at a point $Q$ specified by position vector $\mathbf{r}'$ on the wave front $\mathcal{W}$ which momentarily fills the aperture is given by the expression

$$V^{(0)}(\mathbf{r}', t) = U^{(0)}(\mathbf{r}', \omega) e^{-i\omega t}, \tag{1}$$

where

$$U^{(0)}(\mathbf{r}', \omega) = \frac{A(\omega)}{f} e^{-ikf}, \tag{2}$$

$f$ being the distance between the point $Q$ on the wave front in the aperture and the geometrical focus $O$, and

$$k = 2\pi/\lambda = \omega/c\,, \tag{3}$$

where $k$ is the wave number associated with frequency $\omega$, $c$ is the speed of light in vacuum, and $\lambda$ is the wavelength. For simplicity, we have assumed that the amplitude $A(\omega)/f$ of the wave in the aperture is the same at every point on the wave front $\mathcal{W}$. According to the Huygens-Fresnel principle in the paraxial domain, the field at any point $P(\mathbf{r})$ in the region of the geometrical focus is given by the expression [6]

$$U(\mathbf{r}, \omega) = -\frac{i}{\lambda f} A(\omega) e^{-ikf} \iint_{\mathcal{W}} \frac{e^{ikR}}{R} d^2 r', \tag{4}$$

where

$$R = |\mathbf{r} - \mathbf{r}'|\,. \tag{5}$$

It follows that the spectral intensity $S(\mathbf{r}, \omega) = |U(\mathbf{r}, \omega)|^2$ of the field at $P$ is given by the formula,

$$S(\mathbf{r}, \omega) = \frac{S^{(i)}(\omega)}{\lambda^2} \left| \iint_{\mathcal{W}} \frac{e^{ikR}}{R} d^2 r' \right|^2, \tag{6}$$

where

$$S^{(i)}(\omega) = \frac{|A(\omega)|^2}{f^2} \tag{7}$$

is the spectral intensity of the incident field on the wave front $\mathcal{W}$ in the aperture.

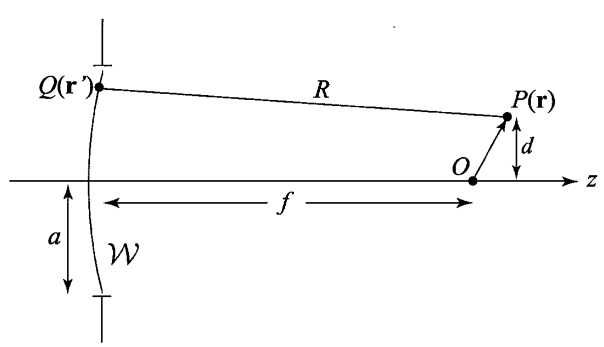


FIG. 1. Notation relating to the focusing configuration.

  

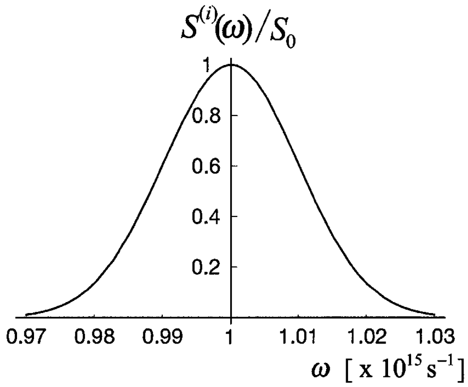


FIG. 2. The spectrum $S^{(i)}(\omega) = S_0 e^{-(\omega-\omega_0)^2/2\sigma^2}$ of the incident field in the aperture, with $\omega_0 = 10^{15}$ s$^{-1}$ and $\sigma = 10^{13}$ s$^{-1}$, normalized by $S_0$.

Suppose now that the incident field is not monochromatic, but is polychromatic and is spatially fully coherent. The spectral intensity in the focal region is then given by the expression

$$S(\mathbf{r}, \omega) = S^{(i)}(\omega) M(\mathbf{r}, \omega), \tag{8}$$

where the factor

$$M(\mathbf{r}, \omega) = \frac{1}{\lambda^2} \left| \iint_{\mathcal{W}} \frac{e^{ikR}}{R} d^2 r' \right|^2 \tag{9}$$

may be called the *spectral modifier*. It indicates how the spectrum $S^{(i)}(\omega)$ of the incident wave is modified by diffraction. Its dependence on $\mathbf{r}$ shows that, in general, the spectrum of the field in the focal region will differ from the spectrum of the field incident on the aperture and will be different at different points.

Suppose that the spectrum of the incident field on the wave front $\mathcal{W}$ in the aperture consists of a single line of Gaussian profile, centered at frequency $\omega_0$ and with rms width $\sigma$, i.e.,

$$S^{(i)}(\omega) = S_0 e^{-(\omega-\omega_0)^2/2\sigma^2}, \tag{10}$$

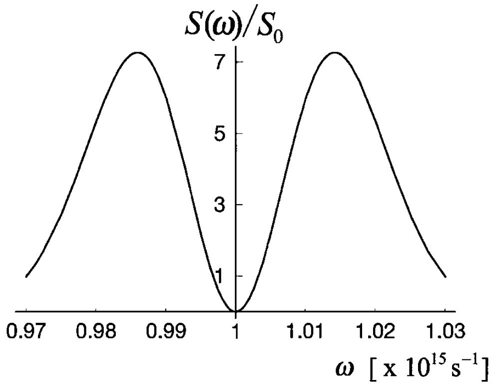


FIG. 3. The normalized spectrum $S[u_1(\omega_0), \omega]/S_0$ at the first axial zero of the mean frequency component $\omega_0$. The Fresnel number $N = a^2/\lambda f$ was taken to have the value 100 at $\omega = \omega_0$.

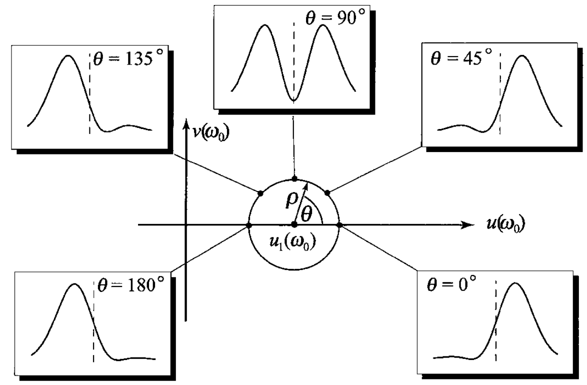


FIG. 4. Schematic illustration of the changes of the spectra along the loop $[u(\omega_0) - u_1(\omega_0)]^2 + v(\omega_0)^2 = \rho^2$ with $\rho = 0.15$, for selected values of the polar angle $\theta$ defined by Eq. (14).

where $S_0$ is a constant (see Fig. 2). Assuming that the Fresnel number $N = a^2/\lambda f$ of the focusing geometry is large compared to unity, the spectral intensity at frequency $\omega$ at any point in the focal region may be calculated from classic expressions due to Lommel [Ref. [6], p. 489, Eqs. (21) and (22)]. For monochromatic light of frequency $\omega$, a point $P$ in the focal region may be specified by the Lommel variables,

$$u(\omega) = \frac{\omega}{c}\left(\frac{a}{f}\right)^2 z, \qquad v(\omega) = \frac{\omega}{c}\left(\frac{a}{f}\right) d, \tag{11}$$

where $z$ and $d$ are the projections of the vector $OP$ along and perpendicular to the $z$ axis, respectively.

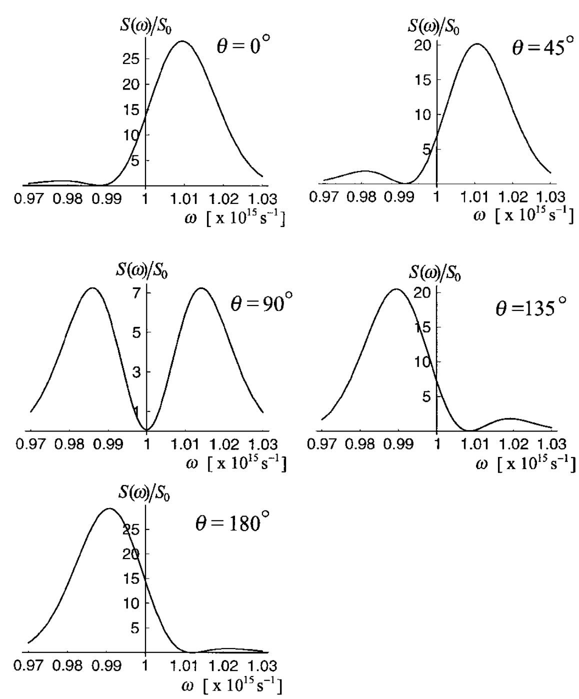


FIG. 5. The normalized spectra along the loop defined in the caption of Fig. 4, at points with polar angle $\theta$, defined by Eq. (14), for $\theta = 0°$, 45°, 90°, 135°, and 180°. The Fresnel number $N$ is again taken to have the value 100 at $\omega = \omega_0$.

It can be shown ([6], p. 491) that when the point $P$ lies on the $z$ axis the spectral intensity at frequency $\omega$ has zero value when

$$u(\omega) = u_n(\omega) \equiv 4\pi n \qquad n = \pm 1, \pm 2, \ldots. \tag{12}$$

We have made use of Eqs. (8) and (9) to determine the following.

1. The spectrum of the light at the first axial zero, $u_1(\omega_0)$, which is shown in Fig. 3. Because the spectral intensity at frequency $\omega_0$ is zero at this point, the spectrum is split into two lines.

2. The spectra at several points $P[u(\omega_0), v(\omega_0)]$ around a closed loop,

$$[u(\omega_0) - u_1(\omega_0)]^2 + v(\omega_0)^2 = \rho^2, \tag{13}$$

centered at the first axial zero of the center frequency $\omega_0$, and with $\rho$ a positive constant [7].

Figure 4 shows the spectra at various points along the loop (13). The points are labeled by the appropriate angle $\theta$, where

$$\cos\theta = u(\omega_0)/\rho, \qquad \sin\theta = v(\omega_0)/\rho. \tag{14}$$

Figure 5 shows in detail the spectra for selected angular directions. We see that when $\theta = 0°$ the spectrum is blueshifted with respect to the spectrum of the incident field in the aperture. When $\theta = 45°$ the spectrum is also blueshifted, whereas when $\theta = 90°$ the spectrum is split into two lines. When $\theta = 135°$ and $\theta = 180°$, the spectrum is redshifted with respect to the spectrum in the aperture.

Using Eq. (9), the spectrum may be evaluated throughout the focal region, not just on the $z$ axis. Figure 6 shows the mean frequency of the spectrum plotted as a function of $u(\omega_0)$, $v(\omega_0)$. The rather spectacular structure of the spectrum will be discussed in more detail in a future publication.

It is of interest to note that substantial spectral changes of the form which we have demonstrated take place even when the incident light has a very narrow bandwidth (e.g., $\sigma/\omega_0 = 0.01$). That such drastic changes can occur even for narrow band light is clearly due to the presence of zeros in the spectral modifier function, $M(\mathbf{r}, \omega)$.

Let us summarize our results. As is well known, the phase of a converging, monochromatic spherical wave emerging from an aperture undergoes a rapid change by half-a-period in the region of the geometrical focus ([6], Sec. 8.8.4). This phenomenon is often called the *phase anomaly near focus*, and, more recently, the *Gouy phase,*

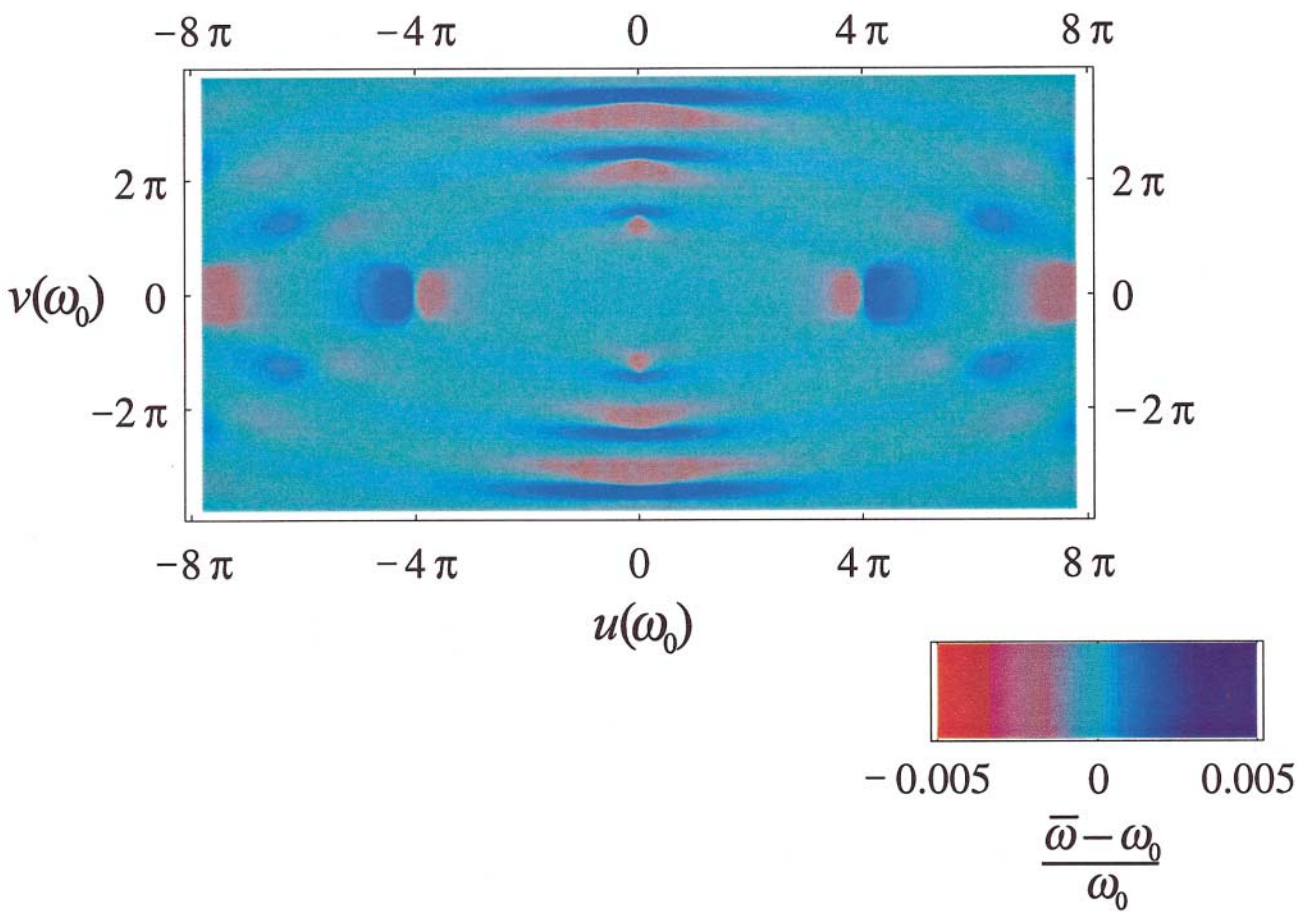


FIG. 6 (color). Color-coded plot of the mean frequency $\overline{\omega}$ of the spectrum in the focal region as a function of $u(\omega_0)$, $v(\omega_0)$. The color is more red or blue as the spectrum is more redshifted or blueshifted, respectively.

after Gouy who first observed this effect more than 100 years ago. Along the axis of symmetry, at points where the intensity is zero, the phase is discontinuous.

In the present paper, we have shown that, when the converging wave incident on the aperture is a spatially fully coherent polychromatic wave rather than a monochromatic wave, the spectrum in the focal region also exhibits an anomalous behavior. In particular, we have shown that, if the spectrum of the field in the aperture consists of a narrow spectral line centered at frequency $\omega_0$, the spectrum of the focused field along a small closed loop enclosing a phase singularity of the spectral component of frequency $\omega_0$ undergoes rapid changes as the point of observation moves along the loop. The spectrum along the loop is redshifted at some points, blueshifted at other points, and is split into two lines elsewhere. These spectral modifications are due to diffraction and must be distinguished from so-called correlation-induced spectral changes which may be exhibited by partially coherent light on propagation in free space, even in the absence of a limiting aperture [8].

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[1] A review of singular optics was recently given by M. S. Soskin and M. V. Vasnetsov, in *Progress in Optics,* edited by E. Wolf (Elsevier, Amsterdam, 2001), Vol. 42, p. 219.

[2] This fact is evident from numerous examples given by J. F. Nye, in *Natural Focusing and the Fine Structure of Light* (Institute of Physics, Bristol, 1999).

[3] A pioneering paper on this subject is due to J. F. Nye and M. V. Berry, Proc. R. Soc. London A **336**, 165 (1974).

[4] T. Poston and I. Stewart, *Catastrophe Theory and its Applications* (Dover, Mineola, 1996).

[5] A notable exception is the paper by M. V. Berry and S. Klein, Proc. Natl. Acad. Sci. U.S.A. **93**, 2614 (1996).

[6] M. Born and E. Wolf, *Principles of Optics* (Cambridge University Press, Cambridge, England, 1999), 7th ed., Chap. 8.8.

[7] It readily follows from Eq. (13) and expressions (11) for the Lommel variables $u$ and $v$ that, in coordinate space, the loop is the ellipse $\alpha^2[z - z_1(\omega_0)]^2 + d^2 = C_0^2\rho^2$, where $\alpha^2 = (a/f)^2$, and $C_0^2 = (\lambda_0 f/2\pi a)^2$.

[8] For a review of this subject, see E. Wolf and D. F. V. James, Rep. Prog. Phys. **59**, 771 (1996).