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Determination of the scattering amplitude and of the extinction cross-section from measurements at arbitrary distances from the scatterer

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Abstract

It is shown that the scattering amplitude for any direction of incidence and any direction of scattering and, consequently, also the extinction cross-section, of a scattering object may be determined from measurements of the scattered field over a plane at an arbitrary distance from it.

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A basic quantity in the theory of scattering from a localized object or from a finite-range potential is the scattering amplitude. It is defined in terms of the far-zone behavior of the scattered field. Specifically, let

$$\Psi^{(i)}(\mathbf{r}, t) = \psi^{(i)}(\mathbf{r})e^{-\mathrm{i}\omega t}, \tag{1}$$

with

$$\psi^{(i)}(\mathbf{r}) = e^{\mathrm{i}k\mathbf{s}_0\cdot\mathbf{r}}, \tag{2}$$

($k = \omega/c$, c being the vacuum speed of light) be a plane monochromatic wave of unit amplitude, incident on the object in free space, in a direction specified by a unit vector $\mathbf{s}_0$. The space-dependent part of the far field, in the direction specified by a unit vector $\mathbf{s}$, has the form (see, for example, Section 13.1, Eqs. (19) and (20) of Ref. [1])

$$\psi(r\mathbf{s}, \mathbf{s}_0) \sim e^{\mathrm{i}kr\mathbf{s}\cdot\mathbf{s}_0} + f(\mathbf{s}, \mathbf{s}_0)\frac{e^{\mathrm{i}kr}}{r} \quad (kr \to \infty,\ \mathbf{s}\ \text{fixed}), \tag{3}$$

where $f(\mathbf{s}, \mathbf{s}_0)$ is the scattering amplitude. In terms of the scattering amplitude one may determine the extinction cross-section, Q say, by the optical cross-section theorem viz. (Ref. [1], Section 13.3, Eq. (18))

$$Q = \frac{4\pi}{k}\,\mathrm{Im}\, f(\mathbf{s}_0, \mathbf{s}_0), \tag{4}$$

where Im denotes the imaginary part.

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In this Letter we show how the scattering amplitude for any direction of incidence and any direction of scattering and, consequently, also the extinction cross-section, may be determined from the knowledge of the scattered field on a plane at an arbitrary distance from the scatterer. The possibility of determining the scattering amplitude from measurements at arbitrary distances from the scatterer is perhaps not surprising in view of its analytic properties [2]. What is, however, not so obvious, is that an algorithm may be developed which does not include contributions from evanescent waves, i.e., waves whose amplitudes decay exponentially with increasing distance from the scatterer. Such waves are well-known to introduce instabilities in the solution of inverse reconstruction problems [4]. Because our algorithm does not involve evanescent waves, it is stable. Our result might be expected to be useful, for example, in connection with inverse scattering problems with acoustical waves, because in such cases the far zone may be far outside the laboratory. Our results may also be of interest in the rapidly expanding field of near-field optics [5], as well as in the new technique of power-extinction diffraction tomography [6].

Suppose that the scatterer is located in the strip $0 \leqslant z \leqslant Z$ (see Fig. 1). Let us represent the scattered field in the half-spaces $z < 0$ and $z > Z$, denoted by $\mathcal{R}^-$ and $\mathcal{R}^+$, respectively, by angular spectra of plane waves viz. (Ref. [7], Section 3.2)

$$\psi^{(s)}(\mathbf{r};\mathbf{s}_0) = \iint_{-\infty}^{\infty} a^{(\pm)}(s'_x, s'_y; s_{0x}, s_{0y}) \times e^{\mathrm{i}k[s'_x x + s'_y y \pm s'_z z]}\,\mathrm{d}s'_x\,\mathrm{d}s'_y, \tag{5}$$

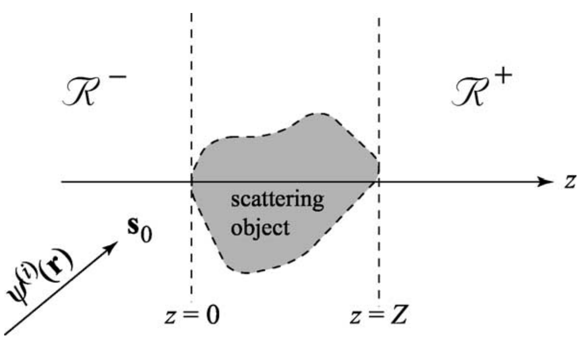

Fig. 1. Illustrating the notation.

where $\mathbf{r} \equiv (x, y, z)$, $\mathbf{s}_0 \equiv (s_{0x}, s_{0y}, s_{0z})$, $\mathbf{s}' \equiv (s'_x, s'_y, s'_z)$, and $s'_z = \sqrt{1 - s'^2_x - s'^2_y}$ when $s'^2_x + s'^2_y \leqslant 1$ and $s'_z = \mathrm{i}\sqrt{s'^2_x + s'^2_y - 1}$ when $s'^2_x + s'^2_y > 1$. The plane waves in the integrand of Eq. (5) for which s'_z is real are ordinary (homogeneous) waves; those for which s'_z is pure imaginary are evanescent waves, whose amplitude decays exponentially with increasing $|z|$. The upper or lower signs are taken in Eq. (5) according as the point $\mathbf{r}$ is in the half-space $\mathcal{R}^+$ or $\mathcal{R}^-$, respectively.

The far zone behavior of the angular spectrum representation (5) as $kr \to \infty$ in a fixed direction specified by a unit vector $\mathbf{s}$ is known to be (Ref. [7], Eq. (3.2-22))

$$\psi^{(s)}(r\mathbf{s};\mathbf{s}_0) \sim -\frac{2\pi\mathrm{i}}{k} s_z a^{(\pm)}(s_x, s_y; s_{0x}, s_{0y}) \frac{e^{\mathrm{i}kr}}{r}, \tag{6}$$

and hence, according to Eqs. (3) and (6), the scattering amplitude $f(\mathbf{s},\mathbf{s}_0)$ is related to the angular spectrum amplitude by the formula

$$f(\mathbf{s},\mathbf{s}_0) = -\frac{2\pi\mathrm{i}}{k} s_z a^{(\pm)}(s_x, s_y; s_{0x}, s_{0y}), \tag{7}$$

the upper or lower sign being taken on the right according as $s_z > 0$ or $s_z < 0$.

With the direction of incidence $\mathbf{s}_0$ being fixed, the spectral amplitude $a^{(\pm)}(s_x, s_y; s_{0x}, s_{0y})$ may be determined, for any direction of scattering, from the knowledge of the scattered field in any arbitrary plane $z = \zeta$ outside the scatterer (i.e., $\zeta < 0$ or $\zeta > Z$) by making use of the two-dimensional Fourier inverse of Eq. (5), viz.

$$a^{(\pm)}(s_x, s_y; s_{0x}, s_{0y}) = k^2 \tilde{\psi}^{(s)}(ks_x, ks_y; \mathbf{s}_0; \zeta) e^{\mp \mathrm{i}ks_z\zeta}, \tag{8}$$

where

$$\tilde{\psi}^{(s)}(u, v; \mathbf{s}_0; \zeta) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \psi^{(s)}(x, y; \mathbf{s}_0; \zeta) e^{-\mathrm{i}(ux+vy)}\,\mathrm{d}x\,\mathrm{d}y \tag{9}$$

is the two-dimensional spatial Fourier transform of the scattered field in the plane $z = \zeta$.

From Eqs. (7) and (8) it immediately follows that the scattering amplitude may be expressed in the form

$$f(\mathbf{s},\mathbf{s}_0) = -2\pi\mathrm{i}ks_z \tilde{\psi}^{(s)}(ks_x, ks_y; \mathbf{s}_0; \zeta) e^{\mp \mathrm{i}ks_z\zeta}. \tag{10}$$

On substituting from Eq. (10) into the formula (4) we obtain the following expression for the extinction cross-section

$$Q = -8\pi^2 \operatorname{Re} s_z \tilde{\psi}^{(s)}(ks_{0x}, ks_{0y}; \mathbf{s}_0; \zeta) e^{\mp i k s_{0z}\zeta}, \quad (11)$$

where Re denotes the real part.

The formulas (10) and (11) are the main results of this note. They show that the scattering amplitude for any direction $\mathbf{s}$ and also the scattering cross-section can be determined from the knowledge of a single two-dimensional spatial Fourier component of the scattered field in any plane $z = \zeta$ outside of the scatterer. In these formulas the upper or lower sign is taken in the exponents on the right-hand sides according as $\zeta > Z$ or $\zeta < 0$ respectively.

We stress that Eqs. (10) and (11) are rigorous consequences of the theory of potential scattering. It is to be noted that because in Eq. (7) s_z is necessarily real, the formulas (10) and (11) do not contain any contributions from evanescent waves. Consequently our method for determining the scattering amplitude and the extinction cross-section from measurements of the scattered field in any plane outside the scatterer is stable.

As an example, we consider scattering from a homogeneous sphere of radius a, with $ka = 20$ and refractive index $n = 1.4$. The scattered field was determined numerically by a partial wave expansion, and evaluated on planes perpendicular to $\mathbf{s}_0$ at distances $kd = \pm 25$, measured from the center of the sphere. To demonstrate the stability of the method, complex Gaussian noise was added to the scattered field with a variance equal to 10% of the average amplitude of the scattered field. A discrete version of Eq. (9) was implemented to determine the Fourier transform of the scattered field, and the scattering amplitude was determined using Eq. (10). In Fig. 2 the actual and reconstructed forms of the scattering amplitude are shown. In this figure the forward measurement plane was used, and $s_\perp = \sin\theta$, where θ is the angle between the unit vectors $\mathbf{s}_0$ and $\mathbf{s}$. It can be seen that there is good agreement between the actual and reconstructed forms of the amplitude. In Fig. 3 the actual and reconstructed forms of the scattering amplitude are shown when the rear measurement plane was used for reconstruction. Here $s_\perp = \sin\theta'$, where θ' is the angle between $-\mathbf{s}_0$ and $\mathbf{s}$. Again, we see that there is good agreement.

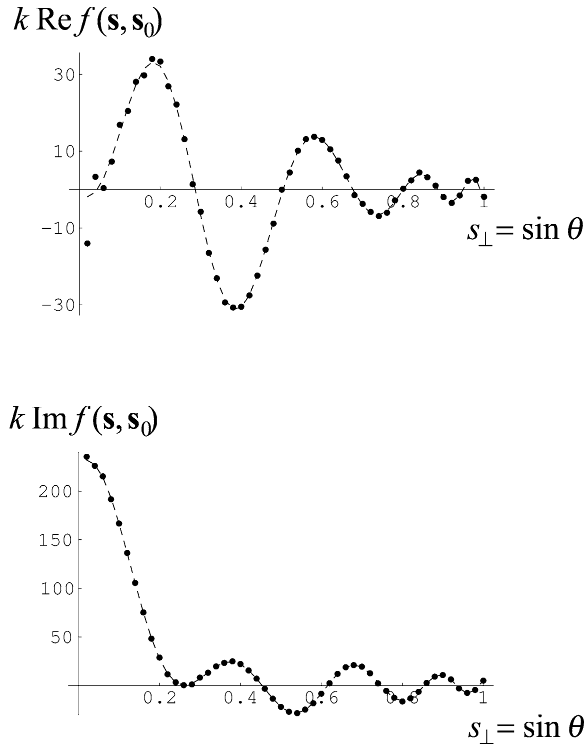

Fig. 2. The real and imaginary parts of the scattering amplitude. The forward measurement plane (in the half-space $\mathcal{R}^+$) was used for the reconstruction. The dots represent the amplitude reconstructed using Eq. (10), and the dashed line represents the actual scattering amplitude determined from the partial wave expansion of the scattered field. Here $s_\perp = \sin\theta$, where θ is the angle between $\mathbf{s}_0$ and $\mathbf{s}$.

We have also determined the extinction cross-section from Eq. (11) for different sphere sizes. The results are shown in Fig. 4. We see that there is good agreement between the true extinction cross-section and the reconstructed form.

The preceding analysis is rigorous within the framework of the theory of potential scattering. Within the accuracy of the first-order Born approximation one has the additional result that (see Ref. [1], p. 713)

$$a^{(\pm)}(s_x, s_y; s_{0x}, s_{0y}) = \frac{ik}{2\pi s_z} \widetilde{F}\big[k(s_x - s_{0x}), k(s_y - s_{0y}), k(\pm s_z - s_{0z})\big], \quad (12)$$

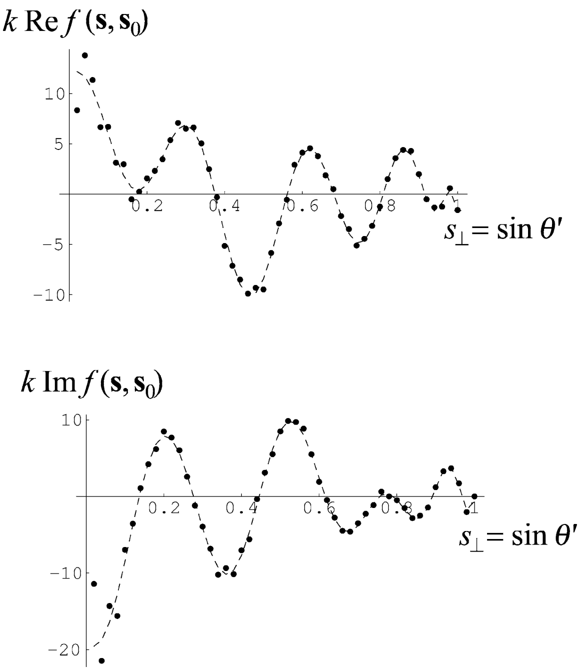

Fig. 3. The real and imaginary parts of the scattering amplitude. The rear measurement plane (in the half-space $\mathcal{R}^-$) was used for the reconstruction. The dots represent the reconstructed amplitude, the dashed line the actual scattering amplitude. Here $s_\perp = \sin\theta'$, where θ' is the angle between $-\mathbf{s}_0$ and $\mathbf{s}$.

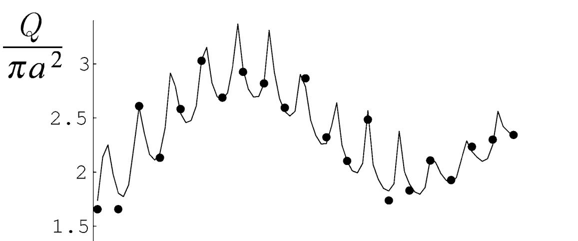

Fig. 4. The extinction cross-section Q normalized to the geometrical cross-section πa^2 of the scattering object for a variety of sphere sizes. The dots represent the cross-section reconstructed by Eq. (11), the solid curve the actual cross-section.

where $\widetilde{F}(K_x, K_y, K_z)$ is the three-dimensional Fourier transform of the scattering potential $F(x, y, z)$, viz.,

$$\widetilde{F}(K_x, K_y, K_z) = \iiint F(x', y', z') \times e^{-\mathrm{i}(K_x x' + K_y y' + K_z z')}\,\mathrm{d}K_x\,\mathrm{d}K_y\,\mathrm{d}K_z. \tag{13}$$

The pair of relations (8) and (12) form the basis of diffraction tomography (see [1], Section 13.2.2).

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