Hybrid diffraction tomography without phase information

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We introduce a hybrid tomographic method, based on recent investigations concerning the connection between computed tomography and diffraction tomography, that allows direct reconstruction of scattering objects from intensity measurements. This technique is noniterative and is intuitively easier to understand and easier to implement than some other methods described in the literature. The manner in which the new method reduces to computed tomography at short wavelengths is discussed. Numerical examples of reconstructions are presented. © 2002 Optical Society of America

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1. INTRODUCTION

The methods of computed tomography (also known as computed axial tomography) and diffraction tomography are well-known techniques used for the reconstruction of absorbing and scattering objects (loosely referred to as three-dimensional imaging). Computed tomography (see, for example, Sec. 4.11 of Ref. 1, or Refs. 2 and 3) is based on a geometrical model of the propagation of radiation. It is applicable when the wavelength of the probing radiation is much smaller than the scale of spatial variation of the object structure that is being reconstructed.

When the probing field has a wavelength that is comparable to the scale of spatial variation of the object, diffraction and scattering effects become significant, and the method of diffraction tomography (Ref. 4 or Sec. 13.2 of Ref. 1) must then be used. The usual theory of diffraction tomography is based on the assumption of weak scattering of the incident field, so that the first Born approximation or the first Rytov approximation may be used.

Both computed tomography and diffraction tomography have found uses in many different fields. Computed tomography is a basic diagnostic technique in medicine, for which it was originally developed.[5,6] Diffraction tomography, however, unlike computed tomography, requires measurement of not only the intensity of the scattered wave but also its phase. Phase measurements generally present considerable practical difficulties. Perhaps because of this, diffraction tomography has not found quite as widespread application as computed tomography, though it is used for some practical applications, as, for example, in oil prospecting.

Numerous methods have been proposed and tested to circumvent the problem of direct phase measurements. Some authors have suggested the use of a cylindrical lens system[7] or an interferometric technique,[8] but such systems are difficult to construct under certain conditions (e.g., when the incident radiation consists of x rays), and they add both experimental and theoretical complications to the inverse problem. Some variants of diffraction tomography that require only intensity measurements have been proposed that involve iterative algorithms[9] or are restricted to reconstructions of objects whose refractive index is real valued.[10] A recently proposed method, called power-extinction diffraction tomography,[11] requires measurements of the power extinguished from a pair of plane waves incident simultaneously on the scatterer. One paper discusses the validity of performing diffraction tomography without using any phase information at all.[12]

Another possibility for circumventing phase measurements are diffraction tomography methods based on the Green's function phase-retrieval technique proposed by Teague.[13] Teague showed that, provided that the field of interest is paraxial, the phase may be determined from intensity measurements by solving a two-dimensional Poisson equation. Several authors have discussed reconstruction of objects using Teague's transport-of-intensity equation.[14,15] However, there are some difficulties relating to the uniqueness of this phase solution when optical vortices are present in the field.[16,17]

In this paper we describe a new technique related to diffraction tomography that requires only intensity measurements motivated by recent research regarding the relationship between computed tomography and diffraction tomography.[18] Our method for determining the phase is inspired by that used by Teague but is appreciably simpler. Numerical examples of reconstructions using this new method are given, and the reduction of it to computed tomography in the short-wavelength limit is discussed.

2. DIFFRACTION TOMOGRAPHY AND THE COMPUTED TOMOGRAPHY LIMIT

We begin by reviewing those elements of scalar scattering theory, diffraction tomography, and computed tomography that are needed for our analysis. We will assume that

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the scatterer is very weak so that the scalar theory may be used (see Ref. 1, Sec. 13.1).

We consider a monochromatic scalar plane wave $U_i(\mathbf{r}, t) = U_i(\mathbf{r})\exp[-\mathrm{i}\omega t]$, of frequency ω and wave number $k = \omega/c$, with spatial dependence $U_i(\mathbf{r}) = \exp[\mathrm{i}k s_0 \cdot \mathbf{r}]$, incident on a scattering object characterized by a potential $F(\mathbf{r})$ occupying a volume V. The arrangement is depicted in Fig. 1. We take $(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_0)$ to be unit vectors along the axes of a right-handed (x, y, z) coordinate system. The time-independent part $U(\mathbf{r})$ of the total field (incident plus scattered) satisfies the equation (Ref. 1, Sec. 13.1)

$$[\nabla^2 + k^2]U(\mathbf{r}) = -4\pi F(\mathbf{r})U(\mathbf{r}), \tag{1}$$

where the scattering potential is given by the expression

$$F(\mathbf{r}) = \frac{k^2}{4\pi}[n^2(\mathbf{r}) - 1], \tag{2}$$

$n(\mathbf{r})$ being the (generally complex) index of refraction. If the scattering potential is sufficiently weak $[n(\mathbf{r}) \approx 1]$, the total field is well represented by the lowest-order term of a perturbation expansion of its complex phase, i.e., as

$$U(\mathbf{r}) \approx U_i(\mathbf{r})\exp[\psi(\mathbf{r})], \tag{3}$$

where

$$\psi(\mathbf{r}) = \frac{1}{U_i(\mathbf{r})}\int_V F(\mathbf{r}')\frac{\exp[\mathrm{i}k|\mathbf{r} - \mathbf{r}'|]}{|\mathbf{r} - \mathbf{r}'|}U_i(\mathbf{r}')\mathrm{d}^3r'. \tag{4}$$

This approximation for the total field is known as the first Rytov approximation (Ref. 1, Sec. 13.5).

Let us determine the total field on a plane $z = d$, taken to be perpendicular to the direction of incidence $\mathbf{s}_0$. In tomographic measurements, this arrangement is often referred to as the classical measurement configuration. Expression (4) may be rewritten in a form more useful for our purposes if we use the Weyl representation of a spherical wave (Ref. 19, Sec. 3.2), which for $z > z'$ takes the form

$$\frac{\exp[\mathrm{i}k|\mathbf{r} - \mathbf{r}'|]}{|\mathbf{r} - \mathbf{r}'|} = \frac{\mathrm{i}}{2\pi}\iint \frac{1}{w}\exp\{\mathrm{i}[u\mathbf{s}_1 + v\mathbf{s}_2 + w\mathbf{s}_0] \cdot (\mathbf{r} - \mathbf{r}')\}\mathrm{d}u\mathrm{d}v, \tag{5}$$

where $\mathbf{r} = x\mathbf{s}_1 + y\mathbf{s}_2 + z\mathbf{s}_0$,

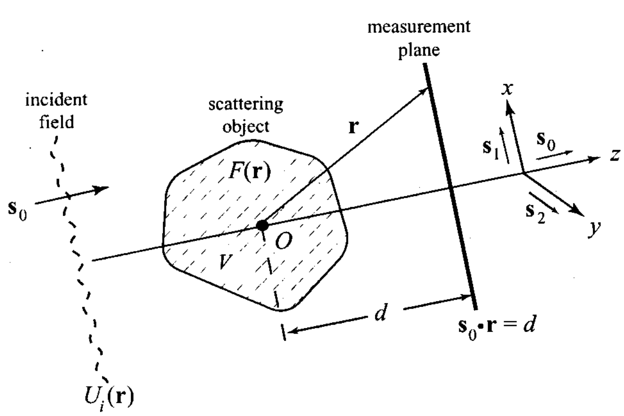

Fig. 1. Depiction of the arrangement and notation.

$$w = \begin{cases} \sqrt{k^2 - u^2 - v^2} & \text{when } u^2 + v^2 \leq k^2 \\ i\sqrt{u^2 + v^2 - k^2} & \text{when } u^2 + v^2 > k^2 \end{cases}, \tag{6}$$

and the integration in Eq. (5) is taken over the entire u, v plane. By substituting from Eq. (5) into Eq. (4), the complex phase $\psi(\mathbf{r})$ on the plane $z = d$ may be expressed in the form

$$\psi(x, y; d) = \frac{\mathrm{i}}{2\pi}\int_V \mathrm{d}^3r' \iint \frac{1}{w}F(\mathbf{r}') \times \exp[\mathrm{i}(w - k)(d - z')] \times \exp\{\mathrm{i}[u(x - x') + v(y - y')]\}\mathrm{d}u\mathrm{d}v. \tag{7}$$

By carrying out the $\mathbf{r}'$ integration, we may express Eq. (7) in the simple form

$$\psi(x, y; d) = (2\pi)^2\mathrm{i}\iint \frac{1}{w}\tilde{F}[u\mathbf{s}_1 + v\mathbf{s}_2 + (w - k)\mathbf{s}_0] \times \exp[\mathrm{i}(w - k)d]\exp[\mathrm{i}(ux + vy)]\mathrm{d}u\mathrm{d}v, \tag{8}$$

where

$$\tilde{F}(\mathbf{K}) = \frac{1}{(2\pi)^3}\int_V F(\mathbf{r}')\exp(-\mathrm{i}\mathbf{K} \cdot \mathbf{r}')\mathrm{d}^3r' \tag{9}$$

is the three-dimensional Fourier transform of the scattering potential. Within the accuracy of the first Rytov approximation, formula (8) may be used to determine the field on any plane $\mathbf{s}_0 \cdot \mathbf{r} = d$ from knowledge of the scattering potential.

We now consider what information about the scattering potential is encoded in the field on such a plane. Knowledge of the complex field in the plane $z = d$ is equivalent to knowledge of the data function

$$D_\psi(x, y; d) \equiv \psi(x, y; d) = \log\left[\frac{U(x, y; d)}{U_i(x, y; d)}\right]. \tag{10}$$

Let us consider the two-dimensional Fourier transform

$$\hat{D}_\psi(u, v; d) = \frac{1}{(2\pi)^2}\iint D_\psi(x, y; d) \times \exp[-i(ux + vy)]\mathrm{d}x\mathrm{d}y \tag{11}$$

of the data function on the plane $z = d$. On substituting from Eq. (8) into Eq. (11), and using the Fourier representation of the Dirac delta function, viz.,

$$\delta(u - u') = \frac{1}{2\pi}\int \exp[\pm\mathrm{i}(u - u')x]\mathrm{d}x, \tag{12}$$

we may express the Fourier transform of the data function D_ψ in the form

$$\hat{D}_\psi(u, v; d) = \frac{(2\pi)^2\mathrm{i}}{w}\tilde{F}[u\mathbf{s}_1 + v\mathbf{s}_2 + (w - k)\mathbf{s}_0] \times \exp[i(w - k)d]. \tag{13}$$

It can be seen from Eq. (13) that the two-dimensional Fourier transform of the data function D_ψ in the plane

$z = d$ is related to the three-dimensional Fourier transform of the scattering potential $F(\mathbf{r})$. In particular, knowledge of the components of $\hat{D}_{\psi}$ for $u^2 + v^2 \leqslant k^2$ is equivalent to knowledge of $\widetilde{F}$ on a half-spherical surface in $\mathbf{K}$ space displaced from the origin [see Fig. 2(a)]. If the direction of propagation $\mathbf{s}_0$ of the incident field takes on all possible directions and measurements are made for each direction in the standard measurement configuration, one can determine all Fourier components of $\widetilde{F}$ such that $|\mathbf{K}| \leqslant \sqrt{2}k$ [see Fig. 2(b)]. Equation (13) forms the basis of diffraction tomography. Algorithms have been described in the literature that may be used to efficiently reconstruct $F(\mathbf{r})$.[20,21] The reconstruction method characterized by Eq. (13) assumes that both the amplitude and the phase of the scattered field are known; we now consider whether intensity measurements alone are sufficient for reconstructing the scattering potential.

We may define the intensity of the field on the plane $z = d$ as

$$\begin{aligned} I(x, y; d) &\equiv |U(x, y; d)|^2 \\ &= \exp[\psi(x, y; d) + \psi^*(x, y; d)], \end{aligned} \tag{14}$$

where the asterisk denotes the complex conjugate. We may also define a data function D_I for the intensity of the field as

$$\begin{aligned} D_I(x, y; d) &= \log[I(x, y; d)] \\ &= \psi(x, y; d) + \psi^*(x, y; d). \end{aligned} \tag{15}$$

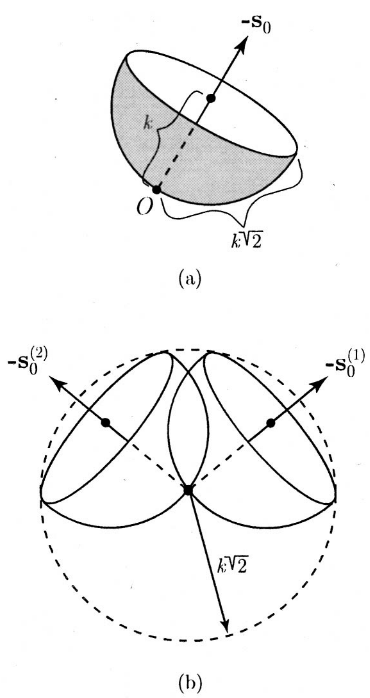

Fig. 2. Accessible Fourier components in the classical measurement configuration of diffraction tomography. (a) Components of $F(\mathbf{r})$ accessible from measurements for one direction $\mathbf{s}_0$ of incidence, (b) components accessible with multiple measurements. The components for directions $-\mathbf{s}_0^{(1)}$ and $-\mathbf{s}_0^{(2)}$ are shown for comparison.

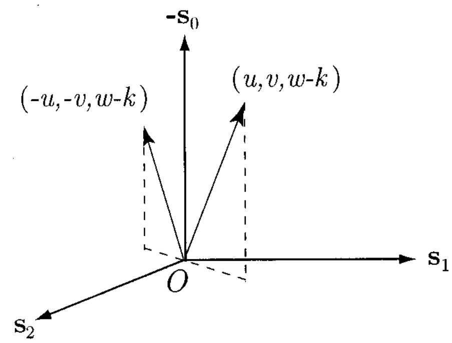

Fig. 3. Vectors $(u, v, w-k)$ and $(-u, -v, w-k)$.

In a manner similar to that used to derive Eq. (13), we can determine the two-dimensional Fourier transform of the data function,

$$\begin{aligned} \hat{D}_I(u, v; d) = \frac{1}{(2\pi)^2} \iint & D_I(x, y; d) \\ & \times \exp[-\mathrm{i}(ux + vy)]\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{16}$$

On substitution from Eqs. (15) and (8) into Eq. (16) and using Eq. (12), one readily finds that

$$\begin{aligned} \hat{D}_I(u, v; d) = \mathrm{i}\frac{(2\pi)^2}{|w|^2}\{ & w^*\widetilde{F}[u\mathbf{s}_1 + v\mathbf{s}_2 + (w - k)\mathbf{s}_0] \\ & \times \exp[\mathrm{i}(w - k)d] - w[\widetilde{F}[-u\mathbf{s}_1 - v\mathbf{s}_2 \\ & + (w - k)\mathbf{s}_0]]^* \exp[-\mathrm{i}(w^* - k)d]\}. \end{aligned} \tag{17}$$

Let us consider values of u, v such that $u^2 + v^2 \leqslant k^2$. It can be seen from Eq. (17) that the Fourier transform of the intensity function contains a combination of the Fourier components of $\widetilde{F}$ specified by the coordinates $(u, v, w - k)$ and $(-u, -v, w - k)$ (see Fig. 3). It seems that only in cases in which the scattering potential is highly symmetric can the individual Fourier components be reconstructed. For evanescent waves $(u^2 + v^2 > k^2)$, w is imaginary and the situation becomes even more complicated. In general, diffraction tomography cannot be used for reconstruction of a three-dimensional object without knowledge of both phase and intensity.

Let us now suppose that the Fourier spectrum $\widetilde{F}(\mathbf{K})$ of the scattering potential is well localized, so that

$$\widetilde{F}[\mathbf{K}] \approx 0, \qquad |\mathbf{K}| \gtrsim 2\pi/\sigma. \tag{18}$$

Such a scattering potential will change slowly with position over distances smaller than or comparable to σ.

As has been noted in our previous work,[18] if $2\pi/\sigma \ll k$, one has, to a good approximation,

$$w - k \approx -\frac{1}{2k}(u^2 + v^2). \tag{19}$$

The data function D_{ψ} may then be expressed in the form

$$D_{\psi}(x, y; d) = (2\pi)^2 \mathrm{i} \int\int \frac{1}{w} \tilde{F}\left[u\mathbf{s}_1 + v\mathbf{s}_2 - \frac{1}{2k}(u^2 + v^2)\mathbf{s}_0\right] \times \exp\left[-\frac{1}{2k}\mathrm{i}(u^2 + v^2)d\right] \times \exp[\mathrm{i}(ux + vy)]\mathrm{d}u\mathrm{d}v. \tag{20}$$

Formally, the integral in Eq. (20) may be expressed as a two-dimensional fractional Fourier transform (see Ref. 18 and the references therein). In the limit of extremely short wavelengths, one can then show, using the properties of fractional Fourier transforms, that

$$D_{\psi}(x, y; d) \sim \frac{2\pi \mathrm{i}}{k} \int_V F(\mathbf{r}')\delta(x - x')\delta(y - y')\mathrm{d}^3 r'. \tag{21}$$

Equation (21) represents a straight-line propagation model of the scattered field, of the type used in computed tomography studies (Ref. 1, Sec. 4.11). By measuring the data function D_{ψ} on planes $z = d$ for all directions of incidence $\mathbf{s}_0$, it can be shown that the function $F(\mathbf{r})$ may be uniquely reconstructed. Similarly, it follows from Eq. (21) that, in the short wavelength limit,

$$D_I(x, y; d) \sim -\frac{4\pi}{k} \int_V \mathrm{Im}[F(\mathbf{r}')]\delta(x - x') \times \delta(y - y')\mathrm{d}^3 r', \tag{22}$$

so that the imaginary part of the scattering potential may be reconstructed from D_I. In this connection we might mention that the ability to reconstruct some structural properties of the object from intensity measurements alone is undoubtedly responsible for computed tomography being a more commonly used technique than diffraction tomography.

3. HYBRID COMPUTED TOMOGRAPHY–DIFFRACTION TOMOGRAPHY METHOD

We have seen that, for extremely short wavelengths, the propagation law for the field reduces to a form in which intensity measurements alone can be used to reconstruct the imaginary part of the scattering potential. We have also seen that, in its most general form, diffraction tomography requires knowledge of both intensity and phase information of the scattered field. We have previously shown [Ref. 18, particularly Eq. (31)], though, that computed tomography and diffraction tomography are closely related, and it seems likely that a hybrid tomographic method might be found that relates the two. In this section we show that this is, in fact, possible.

It is to be noted that, when relation (19) applies, the complex phase $\psi(x, y, z)$ satisfies the differential equation

$$\left(2\mathrm{i}k \frac{\partial}{\partial z} + \nabla_T^2\right)\psi(x, y, z) = 0, \tag{23}$$

where $\nabla_T^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the transverse Laplacian operator. The correctness of this formula can be verified by direct substitution. The complex conjugate of Eq. (23) is

$$\left(-2\mathrm{i}k \frac{\partial}{\partial z} + \nabla_T^2\right)\psi^*(x, y, z) = 0. \tag{24}$$

Taking the sum and difference of Eqs. (23) and (24), we arrive at the pair of equations

$$-2k \frac{\partial}{\partial z}\psi_i(x, y, z) + \nabla_T^2 \psi_r(x, y, z) = 0, \tag{25}$$

$$2k \frac{\partial}{\partial z}\psi_r(x, y, z) + \nabla_T^2 \psi_i(x, y, z) = 0, \tag{26}$$

where ψ_r and ψ_i denote the real and imaginary parts of ψ, respectively. It can be seen from Eq. (14) that

$$\psi_r(x, y, z) = \tfrac{1}{2}\log[I(x, y, z)]. \tag{27}$$

We may therefore express Eq. (26) in the form

$$\nabla_T^2 \psi_i(x, y, z) = -k \frac{1}{I(x, y, z)} \frac{\partial I(x, y, z)}{\partial z}. \tag{28}$$

The imaginary part ψ_i of the complex phase is generally difficult to measure directly. However, Eq. (28) shows that, within the validity of the first Rytov approximation and when $2\pi/\sigma \ll k$, the imaginary part of the complex phase satisfies a two-dimensional Poisson equation, with a source term that depends only on the intensity.

This result is similar to one derived by Teague[13] for determining the phase of a paraxial field from measurements of the intensity. However, Teague assumed that the field itself was paraxial; our approximations in effect assume that the lowest-order perturbation of the phase of the field is paraxial. Our Eq. (28) above is actually much simpler than Teague's equation.

The quantity ψ_i represents the perturbation of the phase of the incident field that is due to the scatterer and, apart from a constant phase shift, is likely to tend to zero for large values of the transverse coordinates. The solution to Eq. (28) with such asymptotic behavior is described in various textbooks (see, for instance, Ref. 22, Sec. 16.6) and is given by the expression

$$\psi_i(\mathbf{r}) = -2k \int\int G(\mathbf{r},\mathbf{r}') \frac{\partial}{\partial z'}\psi_r(\mathbf{r}')\mathrm{d}^2 r', \tag{29}$$

where

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi}\log(|\mathbf{r} - \mathbf{r}'|/\Lambda) \tag{30}$$

is the Green's function of the Laplace equation in two dimensions, Λ being any constant with dimensions of length.

At this point it seems worthwhile to make several observations. First, we note that, according to Eq. (29), ψ_i depends on the derivative of ψ_r in the z direction. Reconstructing ψ_i will therefore require the measurement of ψ_r on at least two different z planes.

On examination of Eq. (22), it is clear that as $\lambda \to 0$, $\partial\psi_r/\partial z \to 0$ and consequently $\psi_i \to 0$. In the limit of small wavelengths, the imaginary part of ψ becomes zero and hence, as expected from computed tomography studies, it is unnecessary to determine it in order to perform a reconstruction.

We note that, in addition to Eq. (26) for ψ_i, we have a second *linear* differential equation (25) relating ψ_i and ψ_r. This circumstance is also different from Teague's phase-reconstruction technique, in which the second equation is a nonlinear one. It may be possible to use Eq. (25) for a consistency check of the correctness of the reconstructed phase.

From Eq. (28), it is clear that when the intensity is zero at any points on the measurement plane, the source term is singular, and the Poisson equation cannot be solved. This observation is in agreement with the known result that the solution to the phase problem is not unique if the field carries optical vortices.[16,17] It is to be noted, however, that under such circumstances the first Rytov approximation is not valid, as it involves the assumption that the transmitted field is weakly perturbed by the scattering object. Conversely, as long as the first Rytov approximation is valid, the phase problem will have a unique solution. There is evidently a strong correlation between the uniqueness of the phase problem and the validity of the first Rytov approximation.

The above arguments suggest that we may define a hybrid computed tomography/diffraction tomography data function of the form

$$D_H(x, y; d) = \psi_r(x, y; d) + \mathrm{i}\psi_i(x, y; d), \tag{31}$$

where ψ_r is one-half of the measured intensity data function D_I and ψ_i is the reconstructed phase, given by Eq. (29). This data function may be used to reconstruct Fourier components of the scattering potential as in Eq. (13). We refer to this method as a hybrid method because it requires only the use of intensity data, as does computed tomography, but it takes into account some scattering effects, as does diffraction tomography. We have also seen that D_H reduces to the computed tomography data function given by Eq. (22) in the limit of extremely short wavelengths.

Equation (31) suggests that the use of this hybrid method requires a somewhat involved two-step inversion process. First one must determine ψ_i by use of Eq. (29), and then one can reconstruct the scattering potential. However, recalling Eq. (13), it is clear that the two-dimensional spatial Fourier transform of the data function is more useful for the inverse problem. The Fourier transform of Eq. (26) may be written as

$$\hat{\psi}_i(u, v; z) = \frac{k}{(u^2 + v^2)} \frac{\partial}{\partial z} \hat{D}_I(u, v; z). \tag{32}$$

We now consider a two-plane measurement scheme, in which the inner plane of measurement is at $z = d$ and the outer plane is at $z = d + \Delta$, where $\Delta > 0$ (see Fig. 4). The two-dimensional Fourier transform of the intensity data function may be written as

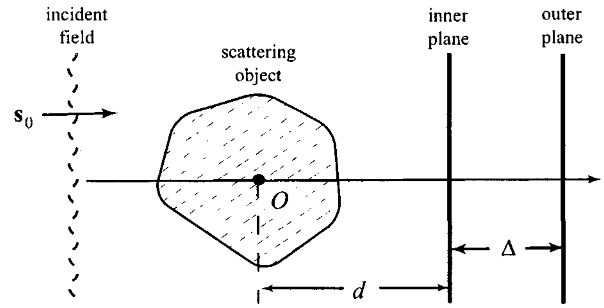

Fig. 4. Two-plane measurement scheme for performing diffraction tomography with only intensity measurements.

$$\hat{D}_I(u, v; z) = \mathrm{i}\frac{(2\pi)^2}{w}\{\tilde{F}[u, v]\exp[-\mathrm{i}z(u^2 + v^2)/2k] - [\tilde{F}[-u, -v]]^* \exp[\mathrm{i}z(u^2 + v^2)/2k]\}, \tag{33}$$

which is just Eq. (17) with approximation (19), and we have written

$$\tilde{F}\left[u\mathbf{s}_1 + v\mathbf{s}_2 - \frac{1}{2k}(u^2 + v^2)\mathbf{s}_0\right] = \tilde{F}[u, v] \tag{34}$$

for brevity. We will approximate the derivative of $\hat{D}_I$ by the expression

$$\frac{\partial}{\partial z}\hat{D}_I(u, v; d) = \frac{\hat{D}_I(u, v; d + \Delta) - \hat{D}_I(u, v; d)}{\Delta}. \tag{35}$$

It then follows on substituting from Eq. (35) into Eq. (32) that

$$\hat{\psi}_i(u, v; d) = \frac{k}{\Delta(u^2 + v^2)}[\hat{D}_I(u, v; d + \Delta) - \hat{D}_I(u, v; d)]. \tag{36}$$

This quantity is undefined when $u^2 + v^2 = 0$, a point to which we will return soon.

On substitution from Eqs. (33) and (36) into the two-dimensional Fourier transform of Eq. (31), it follows after straightforward calculation that

$$\begin{aligned}\hat{D}_H(u, v; d) = \mathrm{i}\frac{(2\pi)^2}{w}\{&\tilde{F}[u, v]\exp[-\mathrm{i}d(u^2 + v^2)/2k] \\ &\times [1 + \exp[-\mathrm{i}\Delta(u^2 + v^2)/4k] \\ &\times j_0[\Delta(u^2 + v^2)/4k]] \\ &- [\tilde{F}[-u, -v]]^* \exp[\mathrm{i}d(u^2 + v^2)/2k] \\ &\times [1 - \exp[-\mathrm{i}\Delta(u^2 + v^2)/4k] \\ &\times j_0[\Delta(u^2 + v^2)/4k]]\}\end{aligned} \tag{37}$$

where j_0 is the spherical Bessel function of order zero.

The first term in the curly brackets, which is proportional to $\tilde{F}[u, v]$, is generally larger than the second term because of the sign difference in the square brackets, provided that the inequality

$$\Delta(u^2 + v^2)/4k < \pi/2 \tag{38}$$

is satisfied. If we note that, according to Eq. (18), the maximum value of $u^2 + v^2$ is $(2\pi/\sigma)^2$, this suggests that Δ must be chosen so that

$$\Delta < \frac{\sigma^2 k}{2\pi}. \tag{39}$$

This inequality defines an upper limit on the distance Δ between the measurement planes. It is to be noted that as $\lambda \to 0$, this upper limit approaches infinity, suggesting that the second measurement plane, which may be arbitrarily far away, is not needed. This is, of course, in agreement with the computed tomography result.

In the limit $\Delta \to 0$, Eq. (37) reduces to

$$\hat{D}_H(u, v; d) = \frac{\mathrm{i}(2\pi)^2}{w}\tilde{F}[u, v]\exp[-\mathrm{i}d(u^2 + v^2)/2k], \tag{40}$$

which is precisely the main formula of diffraction tomography, Eq. (13), subject to approximation (19).

Equation (37) can be used to approximately determine $\tilde{F}[u, v]$. However, it is possible, by using yet another data function, to determine $\tilde{F}[u, v]$ precisely. To see this, let us define D_Δ by the relation

$$\hat{D}_\Delta(u, v; d, \Delta) \equiv \frac{\hat{D}_I(u, v; d) - \hat{D}_I(u, v; d + \Delta)\exp[-\mathrm{i}\Delta(u^2 + v^2)/2k]}{\Delta}. \tag{41}$$

It can be shown in a straightforward manner that

$$\hat{D}_\Delta(u, v; d, \Delta) = \frac{(2\pi)^2\mathrm{i}}{w\Delta}\tilde{F}[u, v]\exp[-\mathrm{i}d(u^2 + v^2)/2k] \times \{1 - \exp[-\mathrm{i}\Delta(u^2 + v^2)/k]\}. \tag{42}$$

Equation (42) is formally similar to Eq. (13), save for the additional term in the curly brackets of Eq. (42), which depends on Δ. It is to be noted that this additional term becomes zero when

$$\frac{\Delta}{k}(u^2 + v^2) = 2n\pi \qquad (n = 0, 1, 2,...). \tag{43}$$

At those points in Fourier space where Eq. (43) is satisfied, the data function vanishes and the Fourier components of the scattering potential cannot be determined. It is therefore necessary, for a good reconstruction, that Δ be chosen so that all components of $\tilde{F}$ have u, v values smaller than those defined by the $n = 1$ term of Eq. (43). When this is done one finds that Δ is constrained again by inequality (39).

Our ability to reconstruct what amounts to the phase of the field from intensity measurements may be understood as follows. At a given spatial frequency of the field (a given u, v value), it can be seen from the Fourier inverse of Eq. (33) that the field consists of a pair of plane waves propagating in the directions (u, v, w) and $(-u, -v, w)$. At any given measurement plane these two plane waves are superposed, and it is not possible to separate their respective contributions. However, each plane wave propagates in a well-defined manner, and at a more distant measurement plane there is a well-defined phase shift between the two waves. Because one can isolate the contribution of each pair (u, v) of plane waves at any measurement plane, one can isolate the amplitude of each plane wave and therefore determine $\tilde{F}[u, v]$. Such a method is similar to that used in so-called power-extinction diffraction tomography,[11] in which phase measurements are circumvented by simultaneously scattering a pair of plane waves off of an object.

A nontrivial difficulty encountered with both hybrid reconstruction techniques described here is the singular behavior of part or all of the Fourier transform of the data function for $u = v = 0$. For D_H, it can be seen from Eq. (32) that $\hat{\psi}_i(0, 0; z)$ is singular, and from Eq. (42) it can be seen that $\hat{D}_\Delta(0, 0; d, \Delta)$ vanishes. The reason for these difficulties can be understood as follows. Physically, the (u, v) component of $\hat{\psi}$ represents a plane wave traveling in the $[u, v, (k^2 - u^2 - v^2)^{1/2}]$ direction. For $u = v = 0$, this is a plane wave propagating in the z direction, which interferes with the incident field. The intensity of this wave, however, does not change with increasing z, and therefore there is no new information about it to be gained from measurements on a more distant measurement plane.

All is not lost, however; it is to be noted from Eq. (33) that

$$\hat{D}_I(0, 0; d) = \frac{(2\pi)^2\mathrm{i}}{k}\{\tilde{F}[0, 0] - [\tilde{F}[0, 0]]^*\} = -\frac{2(2\pi)^2}{k}\tilde{F}_i(0, 0), \tag{44}$$

where $F_i(\mathbf{r})$ is the imaginary part of $F(\mathbf{r})$. This equation shows that the $u = v = 0$ Fourier component of the *imaginary* part of the scattering potential can be determined from the intensity alone.

Furthermore, $\tilde{F}[u, v]$ is the three-dimensional Fourier transform of an object of finite extent. It then follows from a well-known theory of complex analysis (Ref. 23, p. 352) that $\tilde{F}[u, v]$ is the boundary value of an entire analytic function in two complex variables. $\tilde{F}[u, v]$ is therefore continuous in u and v, and presumably $\tilde{F}[0, 0]$ can be found by extrapolating from nearby values.

In concluding this section, we note that, apart from the difficulty connected with the origin $(u = v = 0)$ in Fourier space, these hybrid methods do not require more computation than traditional diffraction tomography. In diffraction tomography, one must determine the Fourier transform of a complex function, ψ, which has a real and imaginary part, whereas in the hybrid methods one must take the Fourier transform of two real functions, the intensity data on two planes. These hybrid methods should therefore not be much more difficult to implement numerically than diffraction tomography itself, but they allow one to circumvent the difficult phase measurements usually required of diffraction tomography.

4. ILLUSTRATIVE EXAMPLE

We consider the scattering of a plane wave from a homogeneous sphere of complex index of refraction n and radius a. We first demonstrate the validity of the new hybrid method using this example and then compare it with conventional computed tomography and with diffraction tomography without phase information.

The scattered field can be determined in series form by the method of partial waves (Ref. 24, p. 385). The solution for the field outside the sphere may be expressed as

$$U_s(r,\ \theta) = \sum_{l=0}^{\infty} \alpha_l h_l^{(1)}(kr) Y_l^0(\theta), \tag{45}$$

where

$$\alpha_l = \mathrm{i}^l \sqrt{4\pi(2l+1)} \times \left\{ \frac{j_l'(ka) j_l(kna) - n j_l(ka) j_l'(kna)}{n h_l^{(1)}(ka) j_l'(kna) - h_l^{(1)\prime}(ka) j_l(kna)} \right\}, \tag{46}$$

Y_l^0 is a spherical harmonic, $h_l^{(1)}$ is the spherical Hankel function of the first kind and order l, and a prime indicates differentiation with respect to the argument.

The total field $U(\mathbf{r})$ is the sum of the incident and scattered fields,

$$U(\mathbf{r}) = U_i(\mathbf{r}) + U_s(\mathbf{r}), \tag{47}$$

and the intensity of the field is given by Eq. (14). We have used the first 75 terms of expression (45) to evaluate the field intensity.

Our reconstruction method assumes weak scattering and is expected to work properly only under such conditions. It is well known (Ref. 25, Chap. 7) that the Rayleigh–Gans theory of weak scattering is applicable when

$$|n - 1| \ll 1, \tag{48}$$

$$2ka|n - 1| \ll 1. \tag{49}$$

To satisfy these inequalities, let $ka = 40$, $n = 1.003 + 0.001\mathrm{i}$.

Our method also requires [recall Eq. (19)] that the scattering is primarily in the forward direction, or that $k\sigma \gg 1$. The spatial Fourier transform of the scattering potential is readily found to be given by the formula

$$\tilde{F}[K] = \frac{k^2 a^3 [n^2 - 1]}{(2\pi)^3} \frac{j_1(Ka)}{Ka}. \tag{50}$$

The value of σ [defined by Eq. (18)], which is the K value beyond which $\tilde{F}$ is negligible, may be regarded as the second zero of j_1; with this choice, $k\sigma = 23.0$, $2\pi/\sigma = 0.27k$.

In solving the inverse problem we assume as prior knowledge the value of σ (which is used to determine the upper limit on Δ) and the spherical symmetry of the scatterer. The latter knowledge greatly simplfies the use of this technique. We proceed to solve the inverse problem as follows. First we determine the spatial Fourier transform of the intensity on two planes $z = d$ and $z = d + \Delta$. For spherically symmetric scatterers, each transform reduces to a one-dimensional integral. Equation (42) is then used to determine the components of $\tilde{F}$ away from the origin ($K = 0$). To determine those components sufficiently close to the origin (at distances $|\mathbf{K}| < 2\pi/6\sigma$) we approximate $\tilde{F}$ as

$$\tilde{F}[K] \approx A + BK^2, \tag{51}$$

where A and B are generally complex numbers. The imaginary part of A can be determined directly from Eq. (44). The real part of A, as well as B, can be determined by matching Eq. (51) to the higher-frequency data. Once the values of $\tilde{F}$ are known, a Fourier inversion to determine $F(r)$ may be performed.

Figure 5 shows the reconstructed real and imaginary parts of the scattering potential obtained from data in the measurement planes $kd = 60$, $k(d + \Delta) = 62$. It can be seen that there is good agreement between the reconstructed and the actual potential. In this reconstruction, use has been made of only intensity data of the field; the phase has been implicitly determined from the intensity.

We also performed a traditional computed tomographic reconstruction of the object using the data, based on the propagation model given by Eq. (22). The results of this reconstruction are shown in Fig. 6. It can be seen in this case that the computed tomographic reconstruction is not satisfactory. This should not be surprising, as our model

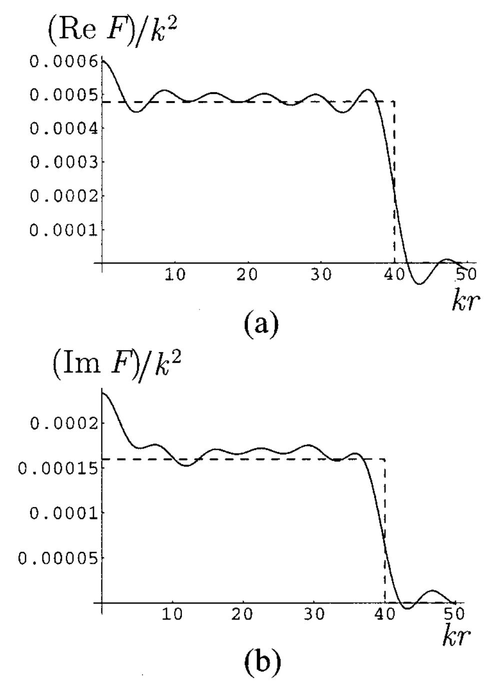

Fig. 5. (a) Real and (b) imaginary parts of the reconstructed and of the true scattering potential, $F(r)$. The true potential is indicated by the dashed lines. The hybrid tomography method is seen to produce a good reconstruction of the scatterer.

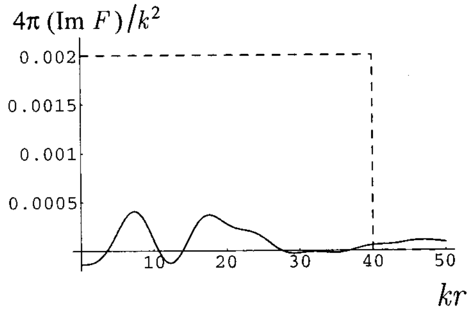

Fig. 6. Reconstruction of the imaginary part of the scattering potential by computed tomography with data from the measurement plane $kd = 60$. The dashed line represents the true scattering potential. Computed tomography evidently is not a good method to use for this model scatterer.

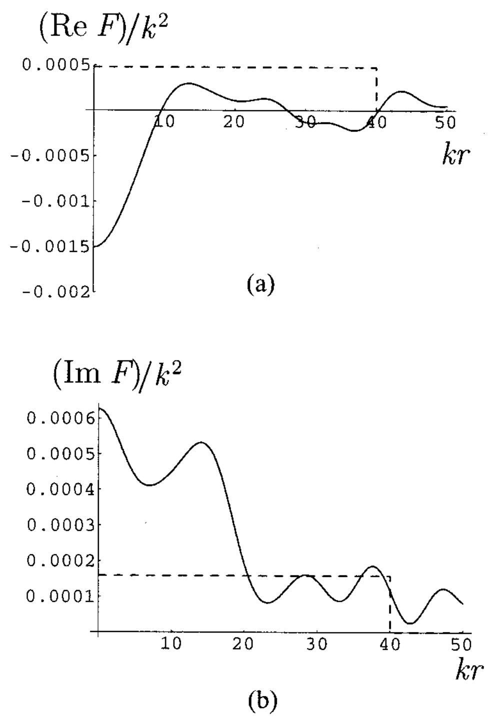

Fig. 7. Reconstruction of the (a) real and (b) imaginary parts of the scattering potential by using the intensity data in the traditional diffraction tomographic reconstruction scheme, Eq. (13). The dashed lines represent the actual scattering potential. It can be seen that this method does not accurately reproduce the scattering object.

scatterer causes appreciable scattering of the incident field, whereas computed tomography assumes straight-line propagation.

Finally, we performed a traditional diffraction tomographic reconstruction of the object using only intensity data. This was done by the use of D_I in place of D_ψ in Eq. (13). The real and imaginary parts of the reconstructed potential are shown in Fig. 7. It can be seen that this method also did not provide a satisfactory reconstruction.

It should be noted that, if the data are corrupted by appreciable noise, the simple extrapolation scheme described by expression (51) must be improved. It seems likely that one could determine the coefficients A and B by using a larger range of measured data values and performing a χ^2 fit of the data to a polynomial. It should also be noted that the hybrid method that we have described requires no phase unwrapping of the field, a well-known difficulty encountered in diffraction tomography. The phase is determined without ambiguity from measurements of the intensity on several planes.

5. CONCLUSIONS

In this paper we have introduced a new diffraction tomography technique that needs no phase information. Instead of phase measurements, the intensity is measured on several planes. We refer to this method as a hybrid tomographic method because it has the advantages of computed tomography and diffraction tomography but is free of certain undesirable features of the two methods. As formulated here, the new method is suitable when the scattering object has a well-localized Fourier spectrum. We have illustrated its validity with a numerical example.

Our analysis is intended only to demonstrate the feasibility of the new method and does not cover the complications that might arise in practice. More work is needed to formulate good algorithms for the reconstruction. In carrying out successful experiments, issues such as sensitivity to noise will also need to be addressed.

The new method might prove useful as a generalization of computed tomography to systems where the object is weakly scattering and as a way to circumvent the need to make phase measurements in diffraction tomography.

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