# Phase singularities of the coherence functions in Young's interference pattern

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We analyze the coherence properties of a partially coherent field emerging from two pinholes in an opaque screen and show that the spectral degree of coherence possesses phase singularities on certain surfaces in the region of superposition. To our knowledge, this is the first illustration of the singular behavior of the spectral degree of coherence, and the results extend the field of singular optics to the study of phase singularities of correlation functions. © 2003 Optical Society of America

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Recent research relating to Young's interference experiment with partially coherent light (Ref. 1, Sec. 7.3.1) has revealed numerous new effects. For example, in Ref. 2 it was predicted that if the pinholes are illuminated with broadband light, substantial spectral changes may occur in the region of superposition. Such spectral changes may in turn be used to determine the spectral degree of coherence of the light at the two pinholes.[3] These predictions have been verified experimentally (see, for example, Refs. 4–8). Furthermore, it has been shown that the spectral degree of coherence of the light also exhibits unexpected behaviors. In a recent article[9] it was predicted that there are regions in the field where the light at a pair of points is fully coherent, regardless of the state of coherence of the light at the pinholes, and the spectral degree of coherence of the light in the region of interference was analyzed in Ref. 10. In the latter article, pairs of points were found at which the spectral degree of coherence has zero value, implying the existence of phase singularities of the coherence function. In this Letter we present a detailed analysis of the spectral degree of coherence and show that it possesses surfaces, defined by pairs of points, on which the phase is singular. The behavior of the field in the vicinity of these phase singularities is investigated.

Suppose that two pinholes located at $Q_1$ and $Q_2$ in a plane opaque screen $\mathcal{A}$ (see Fig. 1) are illuminated by partially coherent light. Under the assumption that the angles of incidence and diffraction are small, it can be shown that the cross-spectral density of the light (Ref. 11, Chap. 2.4.4) at two field points, $P_1$ and $P_2$, at frequency $\omega$ is given by the formula[10]

$$W(P_1, P_2, \omega) = \left(\frac{kA}{2\pi}\right)^2 S(\omega)\,(K_{11}^* K_{12} + K_{21}^* K_{22} + K_{11}^* K_{22}\mu_{12} + K_{12}K_{21}^*\mu_{12}^*)\,. \quad (1)$$

Here $k = \omega/c$ is the wave number associated with frequency $\omega$, $c$ is the speed of light in vacuum, $A$ is the area of each pinhole, $S(\omega)$ is the spectral density at $Q_1$ and $Q_2$, $\mu_{12}$ is the spectral degree of coherence of the light at the two pinholes, and $K_{ij} = \exp(ikR_{ij})/R_{ij}$, where $R_{ij}$ is the distance from the pinhole $Q_i$ to the point $P_j$ ($i, j = 1, 2$).

The spectral density of the light at a point $P_i$ in the region of superposition is given by the diagonal elements of the cross-spectral density, i.e., $S(P_i, \omega) \equiv W(P_i, P_i, \omega)$. In most cases of practical interest $d \ll R_{ij}$, and then $R_{1i} \approx R_{2i}$ and the spectral density has the simple form

$$S(P_i, \omega) = \left(\frac{kA}{2\pi}\right)^2 \frac{2S(\omega)}{R_{1i}^2}\{1 + |\mu_{12}|\cos[\beta + k(R_{2i} - R_{1i})]\}, \quad (2)$$

where $\beta$ is the phase of $\mu_{12}$.

The spectral degree of coherence at frequency $\omega$ of the light at two field points is given by the expression (Ref. 11, Chap. 4.3.2)

$$\mu(P_1, P_2, \omega) = \frac{W(P_1, P_2, \omega)}{\sqrt{S(P_1, \omega)}\sqrt{S(P_2, \omega)}}\,, \quad (3)$$

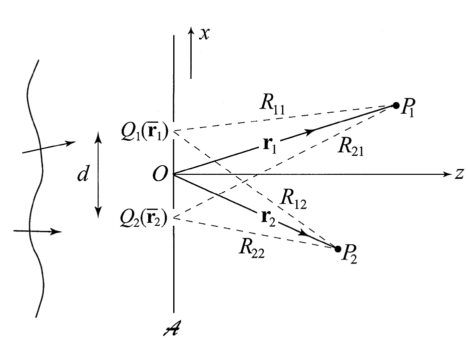


Fig. 1. Illustration of the notation relating to Young's interference experiment with partially coherent light.

 

with $W(P_1, P_2, \omega)$ and $S(P_i, \omega)$ given by Eqs. (1) and (2), respectively.

We now consider pairs of points in the far zone for which the phase of the spectral degree of coherence $\mu(P_1, P_2, \omega)$ becomes singular; this happens at pairs of points for which $\mu(P_1, P_2, \omega) = 0$. One should note that $\mu(P_1, P_2, \omega)$ exhibits an additional singular behavior when $S(P_i, \omega) = 0$. We exclude such cases from our consideration because the approximate form of the spectrum given by Eq. (2) suggests that it will not have zero value provided that the light is not spatially fully coherent at the pinholes, i.e., provided that $|\mu_{12}| < 1$. We take the pinholes to be located on the $x$ axis, situated symmetrically with respect to the origin and separated by a distance $d$. We assume that the use of the far-zone approximation for the factors $K_{ij}$ does not significantly alter the behavior of the singular points, an assumption that is supported below by numerical calculations. In the far zone, the factors $K_{ij}$ have the approximate form

$$K_{ij} \approx \frac{\exp[ik(R_j - \hat{\mathbf{r}}_j \cdot \mathbf{d}_i)]}{R_j}. \tag{4}$$

In expression (4), $R_j$ is the distance from the origin to the observation point, $P_j$; $\hat{\mathbf{r}}_j$ is the unit vector pointing in the direction $OP_j$; and $\mathbf{d}_i = \pm(d/2)\hat{\mathbf{x}}$, where the positive or negative sign is taken accordingly as $i = 1$ or $i = 2$, respectively, and $\hat{\mathbf{x}}$ is the unit vector in the positive $x$ direction. On substituting from expression (4) into Eq. (1), it readily follows that the cross-spectral density may be expressed in the form

$$\begin{aligned} W(P_1, P_2, \omega) = {} & 2\left(\frac{kA}{2\pi}\right)^2 S(\omega) \frac{\exp[ik(R_2 - R_1)]}{R_1 R_2} \\ & \times \left\{\cos\left[\frac{kd}{2}(\cos\theta_1 - \cos\theta_2)\right]\right. \\ & \left. + |\mu_{12}|\cos\left[\frac{kd}{2}(\cos\theta_1 + \cos\theta_2) + \beta\right]\right\}, \end{aligned} \tag{5}$$

where $\theta_i$ is the angle between $\hat{\mathbf{r}}_i$ and the positive $x$ direction. It can readily be seen that Eq. (5) implies the existence of phase singularities, i.e., the existence of pairs of points at which the cross-spectral density, and consequently the spectral degree of coherence, has zero value. In particular, $W(P_1, P_2, \omega) = 0$ at points for which the expression in the braces vanishes. This expression is independent of the distances $R_1$ and $R_2$ and, in fact, depends only on the directions of observation. It follows that a given zero of the cross-spectral density requires that observation points $P_1$ and $P_2$ both lie on conical surfaces $\cos\theta_i =$ constant. A sketch of such surfaces is given in Fig. 2.

One can readily determine the behavior of the phase of $\mu$ in the immediate vicinity of such surfaces by noting that the expression in the braces of Eq. (5) is real valued, so the only possible phase change of this factor on changing angles $\theta_1$ and $\theta_2$ is a change in sign. This change corresponds to a jump in phase of $\pm\pi$, and this is the only possible singular behavior across the singular surfaces.

We have studied the spectral degree of coherence in the region of superposition numerically by using Eqs. (1) and (3). Let $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2)$ specify the position of observation points $P_1(\mathbf{r}_1)$ and $P_2(\mathbf{r}_2)$, respectively. Note that the cross-spectral density is computed from Eq. (1), not the approximate form, Eq. (5). By varying $x_2$ and $y_2$ while keeping $z_2$ and $\mathbf{r}_1$ fixed, we studied the behavior of the phase, $\phi_\mu(P_1, P_2, \omega)$, of the spectral degree of coherence in a plane parallel to the screen containing the apertures. An example is shown in Fig. 3. The vertical line indicates the location of a phase singularity, i.e., a set of points $P_2$ (with $P_1$ fixed) for which $\mu(P_1, P_2, \omega) = 0$ and hence the phase of the spectral

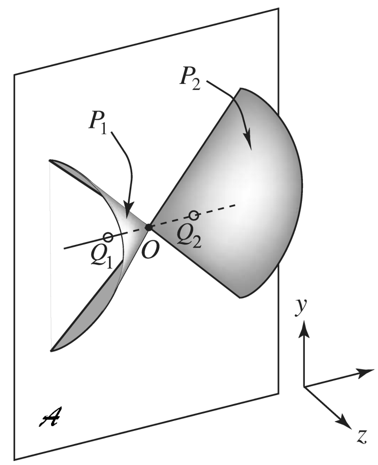


Fig. 2. Schematic illustration of surfaces on which points of observation $P_1$ and $P_2$ in the far zone are located for which $\mu(P_1, P_2, \omega) = 0$, i.e., at which the phase of $\mu(P_1, P_2, \omega)$ is singular. $P_1$ and $P_2$ lie on opposite cones.

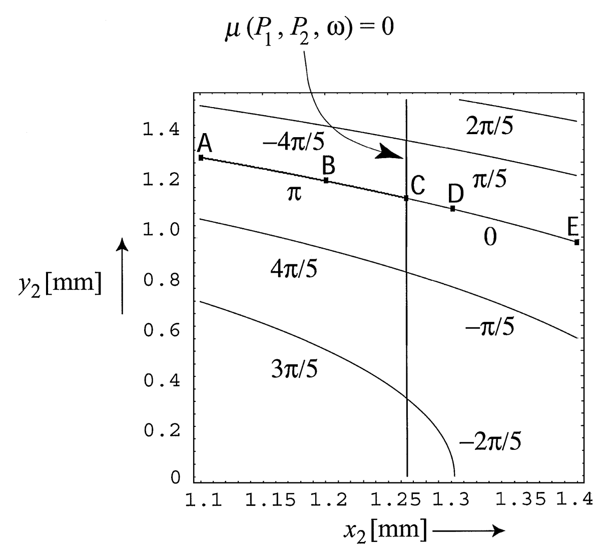


Fig. 3. Contours of equal phase of the spectral degree of coherence $\mu(P_1, P_2, \omega)$ near a singularity of its phase in a plane parallel to the screen. In this example $k = 0.333 \times 10^7\ \mathrm{m}^{-1}$, $d = 0.1$ cm, $\mu_{12} = 0.8 + 0.3i$, $\mathbf{r}_1 = (0, 0, 1.5)$ m, and $z_2 = 1.5$ m. (The significance of the points A–E will be discussed in connection with Fig. 5.)

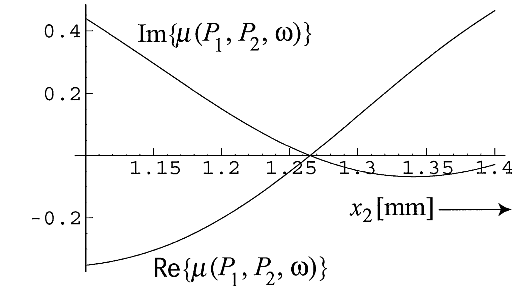


Fig. 4. Real and imaginary parts of the spectral degree of coherence $\mu(P_1, P_2, \omega)$, with $P_1$, $y_2$, and $z_2$ kept fixed while $x_2$ is varied. $y_2 = 0.9$ mm; all other parameters have the same value as in Fig. 3.

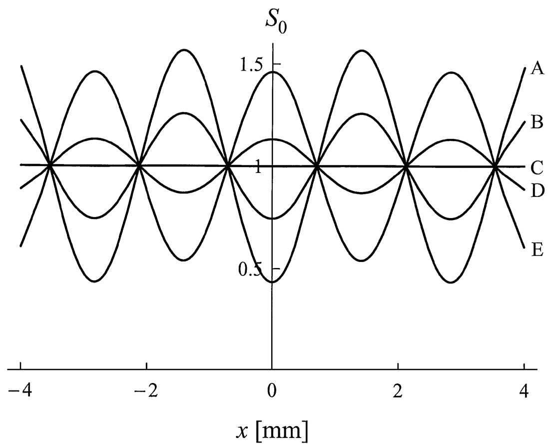


Fig. 5. Spectral interference pattern formed along the $x$ direction by combination of the light from pinholes $P_1$ and $P_2$ in a second Young's interference experiment. The observation plane was taken to be at $z = 1.5$ m, and the spacing of the pinholes was taken to be $d = 0.1$ cm. The positions of points $P_2$ (namely A–E) are illustrated in Fig. 3. $S_0$ is a spectral intensity normalized by the value of the spectral intensity on the curve C. All other parameters are the same as in Fig. 3.

degree of coherence is singular. It can be seen from the figure that the phase has a discontinuity of $\pi$ across the singularity.

A detailed example of the behavior of the spectral degree of coherence is given in Fig. 4. Point $P_1$ and coordinates $y_2$ and $z_2$ are kept fixed while $x_2$ is varied. The real and the imaginary parts of $\mu(P_1, P_2, \omega)$ are seen to change sign at the phase singularity, in accordance with a $\pi$ phase jump.

Observing this phase change requires interfering the light from the vicinity of the pair of points $P_1$ and $P_2$. One can do this, for instance, by bringing together the light from these points by means of another Young's interference experiment and observing the behavior of the interference fringes produced at frequency $\omega$ by this additional experiment as point $P_2$ is moved across the phase singularity. Figure 5 shows the fringe pattern that would be observed in this second experiment for selected points $P_1$, $P_2$. Point $P_1$ was chosen as in Fig. 4, and point $P_2$ was taken along a line of constant phase at several points in the vicinity of the phase singularity. The choices of $P_2$, namely A–E, are illustrated in Fig. 3. It can be seen from Fig. 5 that the $\pi$ phase change results in the minima of the secondary fringe pattern's becoming maxima, and vice versa, in accordance with the spectral interference law (Ref. 11, Sec. 4.3.2).

To summarize, we have demonstrated the existence of phase singularities of the spectral degree of coherence of the field at pairs of points in the region of superposition in Young's interference experiment with partially coherent light. The phase of the spectral degree of coherence is found to make a $\pm\pi$ jump across such singularities. This phase jump can be observed by means of a second Young's interference experiment. To our knowledge, this study is the first extension of the field of singular optics to the realm of correlation functions.

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