Coherence properties of sunlight

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The coherence properties of sunlight were first studied by Verdet around 1869 and were later examined by other scientists. However, all the previous calculations assumed that the Earth is in the far zone of the Sun, an assumption that is incorrect. An investigation of why Verdet's result is nevertheless correct reveals a surprising property of radiation from incoherent sources.

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One of the basic problems related to the theory of optical coherence is the determination of the coherence properties of sunlight that is incident upon the surface of the Earth. The earliest determination was made by Verdet[1] and is considered the first calculation of the coherence properties of light. Here is a key passage from his publication on the subject[2]:

> The points to which all the elements of the source transmit practically identical movements are contained in...a circle whose center is the point P and is of radius
>
> $$\frac{R}{\rho}\, h\lambda$$
>
> It is only in the interior of this circle that the vibrations can be considered as coherent on the sphere S.
>
> ⋮
>
> In applying these results to the case where the luminescent source is the Sun, one is surprised by the smallness of the region in which the movements can be considered coherent.

In the above formula, ρ is the radius of the source, R is the distance from the source to the observation point, λ is the wavelength, and h is a numerical factor "certainly less than 1/4." Because the diameter of the circle of coherent movements is linear with respect to distance from the source, Verdet's calculation suggests, within its scope of validity, that the angular diameter of the region of coherence is constant with respect to this distance.

Since Verdet's time, this calculation of the coherence of sunlight has been performed more quantitatively (for instance, Ref. 3, Sect. 10.4.2, and Ref. 4, Sect. 4.2.2 and 4.4.4). However, such calculations assume that the Earth is in the far zone of the Sun, an assumption that does not hold, even approximately. For instance, the far zone of a source may be defined as the region at a distance r from the source at which the Fresnel number that the source subtends at the observation point is much smaller than unity, i.e.,

$$\frac{\pi a^2}{\lambda r} \ll 1\,, \tag{1}$$

where a is the radius of the source and λ is the wavelength of the radiation. With filtered sunlight at the Earth's surface, $\lambda \approx 500$ nm, $a = 6.96 \times 10^5$ km, and consequently the far zone of the Sun is at distance $r \gg 3 \times 10^{23}$ km, a condition not even remotely satisfied by the distance between the Earth and the Sun, $r = 1.5 \times 10^8$ km. It is therefore of interest to examine whether Verdet's estimate, and other estimates, of the coherence area of sunlight on the Earth's surface are correct.

We consider the properties of a scalar wave field radiated by the surface of a spherical source of radius a centered at the origin (see Fig. 1). The second-order statistical properties of the field at frequency ω outside the source may be characterized by the cross-spectral density function, defined as (Ref. 4, Sect. 4.7.2)

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) \equiv \langle U^*(\mathbf{r}_1, \omega) U(\mathbf{r}_2, \omega)\rangle\,, \tag{2}$$

where $U(\mathbf{r}, \omega)$ is a monochromatic realization of the (statistical) field at frequency ω and the angle brackets denote ensemble averaging over these realizations. The field $U(\mathbf{r}, \omega)$, which is a solution of the scalar Helmholtz equation, may be represented everywhere outside the source domain in a series of the form[5]

$$U(\mathbf{r}, \omega) = \sum_{lm} c_{lm} h_l^{(1)}(kr) Y_{lm}(\theta, \phi)\,, \tag{3}$$

where $h_l^{(1)}$ is the spherical Hankel function of the first kind and order l, Y_{lm} is the spherical harmonic of order l, m, and c_{lm} are random coefficients that depend upon the statistical properties of the field on the surface of the source. On substituting from Eq. (3) into Eq. (2),

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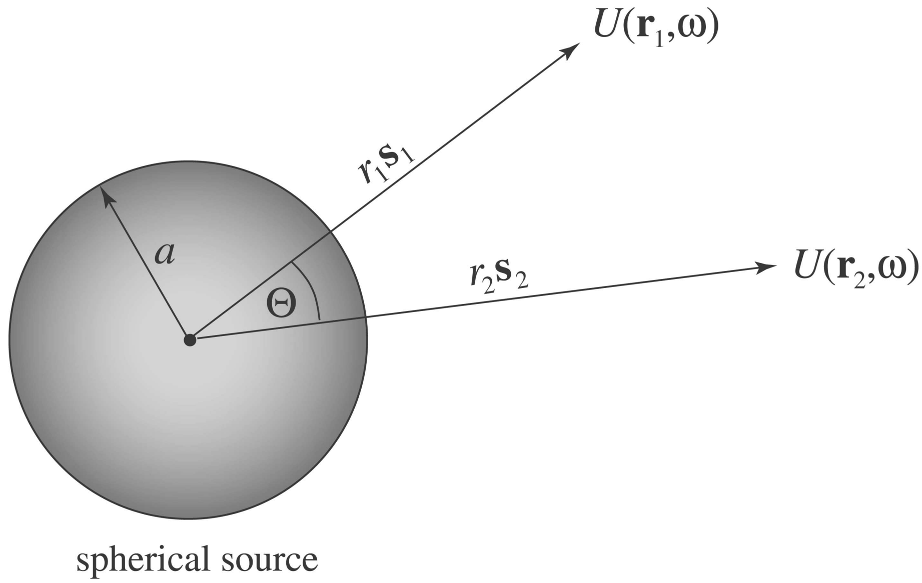

Fig. 1. Notation relating to the radiation from an incoherent spherical source.

it follows that the cross-spectral density of the field outside the sphere is given by the formula

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) = \sum_{lm}\sum_{l'm'} \langle c_{lm}^* c_{l'm'} \rangle h_l^{(1)*}(kr_1) h_{l'}^{(1)} \times (kr_2) Y_{lm}^*(\theta_1, \phi_1) Y_{l'm'}(\theta_2, \phi_2). \quad (4)$$

This is the most general expression for a partially coherent field outside a spherical domain. We may simplify this formula significantly by applying particular boundary conditions at the surface.

Let us assume that the field at points $\mathbf{r}_1 = a\mathbf{s}_1$, $\mathbf{r}_2 = a\mathbf{s}_2 (\mathbf{s}_1^2 = \mathbf{s}_2^2 = 1)$ on the surface of the sphere is incoherent, i.e., its cross-spectral density is delta correlated,

$$W(a\mathbf{s}_1, a\mathbf{s}_2, \omega) = I_0(\omega) \delta^{(2)}(\mathbf{s}_2 - \mathbf{s}_1), \quad (5)$$

where $I_0(\omega)$ is the effective intensity of the field on the surface of the sphere and $\delta^{(2)}$ is the two-dimensional Dirac delta function with respect to the spherical polar coordinates (θ_1, ϕ_1) and (θ_2, ϕ_2) of unit vectors $\mathbf{s}_1$ and $\mathbf{s}_2$, respectively. This delta function may be expanded by use of the spherical harmonic closure relation (Ref. 6, p. 791),

$$\delta^{(2)}(\mathbf{s}_2 - \mathbf{s}_1) = \sum_{lm} Y_{lm}^*(\theta_1, \phi_1) Y_{lm}(\theta_2, \phi_2). \quad (6)$$

On matching Eq. (4) to the boundary condition (5), it is readily found that

$$\langle c_{lm}^* c_{l'm'} \rangle = \delta_{ll'} \delta_{mm'} \frac{I_0(\omega)}{|h_l^{(1)}(ka)|^2}, \quad (7)$$

where $\delta_{ll'}$ is the Kronecker delta. Hence the cross-spectral density of the field is given by

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) = \sum_{lm} \frac{I_0(\omega)}{|h_l^{(1)}(ka)|^2} h_l^{(1)*}(kr_1) h_l^{(1)}(kr_2) \times Y_{lm}^*(\theta_1, \phi_1) Y_{lm}(\theta_2, \phi_2) \quad (8)$$

This expression may be further simplified by using the spherical harmonic addition theorem (Ref. 6, Sect. 12.8), and the cross-spectral density then takes the form

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) = \sum_l \frac{2l+1}{4\pi} \frac{I_0(\omega)}{|h_l^{(1)}(ka)|^2} h_l^{(1)*}(kr_1) h_l^{(1)}(kr_2) P_l(\cos\Theta), \quad (9)$$

where Θ is the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$ and P_l is the Legendre polynomial of order l.

Equation (9) is suitable for performing numerical computation. Before doing so, however, it is worthwhile to examine the asymptotic form of this series. The spherical Hankel functions $h_l^{(1)}(x)$ are well known to take on their asymptotic forms when $x \gg l(l+1)/2$; furthermore, it can be shown that the series (9) will have negligible coefficients for $l > ka$. One might assume that the series will take on its asymptotic form when the highest-order, nonnegligible terms of the series take on their individual asymptotic forms. The highest-order, nonnegligible term is approximately given by $l = ka$, and we then have the asymptotic condition

$$kr \gg \frac{ka(ka+1)}{2}. \quad (10)$$

If we further assume that $ka \gg 1$, we may approximate inequality (10) by the form $kr \gg (ka)^2/2$, or

$$\frac{\pi a^2}{\lambda r} \ll 1, \quad (11)$$

which is the usual Fresnel number requirement mentioned in the introduction.

Numerical computation of the degree of coherence suggests a different transition to the asymptotic limit. Figure 2 shows numerical computations of the spectral degree of coherence on spheres concentric with a source of normalized radius $ka = 100$. It can be seen in the figure that the angular spread of degree of coherence

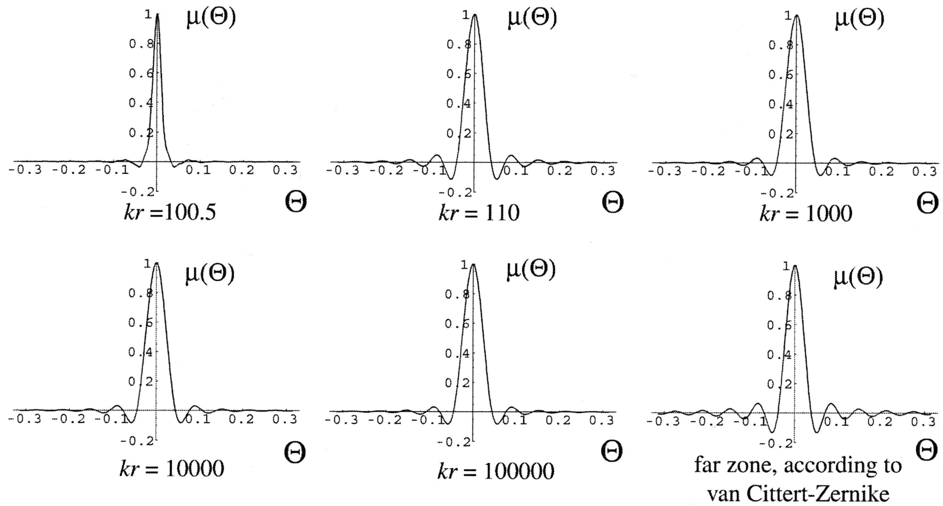

Fig. 2. Form of the spectral degree of coherence at different radial distances from a spherical source of (normalized) radius $ka = 100$. It can be seen that, once the observation point is more than a few wavelengths away from the source domain, the functional form of the spectral degree of coherence is essentially unchanging and roughly equal to the far-zone form, calculated using the van Cittert–Zernike theorem.

is essentially constant once the two field points are more than a few wavelengths away from the source. The boundary of the far zone of the source, defined by Eq. (11), is at a distance $kr \approx 5000$.

It is useful to compare these numerical results with the traditional far-zone calculation of the coherence of the radiated field, based on the van Cittert–Zernike theorem (Ref. 3, Sect. 10.4.2). The degree of coherence of a planar, incoherent circular source of uniform intensity and radius a at points in the far zone is then found to be given by the expression

$$\mu(r\mathbf{s}_1, r\mathbf{s}_2, \omega) = \frac{J_1[2ka\,\sin(\Theta/2)]}{ka\,\sin(\Theta/2)}, \tag{12}$$

where Θ is again the angle between the directions of observation $\mathbf{s}_1$ and $\mathbf{s}_2$ and J_1 is the Bessel function of the first kind and order 1. It can be seen from Fig. 2 that our numerical calculations are in good agreement with the van Cittert–Zernike result.

Our calculations demonstrate that the spectral degree of coherence of the field generated by an incoherent source takes on its far-zone behavior at distances immediately beyond the near zone of the source and well before one reaches the traditionally defined far-zone limit. We have performed calculations of the degree of coherence for sources of various sizes, from very small ($ka = 3$) to quite large ($ka = 1000$), and the distance-independent behavior is found to hold for all cases. It is not possible to use our numerical methods to directly calculate the coherence of sunlight (which would require the summation of a series with $ka \approx 10^{18}$ terms), but our results suggest that the usual van Cittert–Zernike theorem result of a coherence area of sunlight, $\Delta A \approx 3.67 \times 10^{-3}\ \mathrm{mm}^2$ (and consequent angular width $\Delta\Theta = 4.5 \times 10^{-16}$ rad), is correct. It is expected that any experimental measurements of the degree of coherence of sunlight will reproduce not only this coherence area but also the functional form of the degree of coherence given by Eq. (12).

In conclusion, we have shown that the effective far-zone result for the coherence of light from a spatially incoherent spherical source is in fact valid immediately beyond a distance of a few wavelengths from the source surface. This suggests that, at least for incoherent sources, the usual far-zone condition (1) is not appropriate. Similar results have recently been demonstrated for planar incoherent sources,[7] suggesting that this phenomenon is a consequence of the incoherence of the source.

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