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‘Hidden’ singularities in partially coherent wavefields

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Abstract
It is well known that light fields which are partially coherent and/or polychromatic do not typically possess regions of zero intensity and hence do not possess any obvious phase singularities. It is of interest to ask whether or not such fields possess singularities in some ‘hidden’ form, and in this paper we discuss the singular optics of partially coherent fields and the nature of the singularities in such fields.

Keywords: singular optics, spectral changes, coherence theory, coherence vortices

1. Introduction

It is generally accepted that fixed points of complete destructive interference of a wavefield, i.e. fixed points of zero intensity at which the phase is necessarily singular, typically occur only in monochromatic, spatially coherent wavefields. Most of the research in singular optics [1] concerns fields of this type. When a wavefield is partially coherent[4] and/or polychromatic, its random fluctuations will tend to move its singular points, leaving no zeros in the average intensity. The disappearance of zeros as the coherence of a system is decreased has been demonstrated for a number of systems [2, 3]. A natural question which then arises is this: are there any generic[5] singularities in such partially coherent wavefields and, if so, what are their physical characteristics? To answer this question we will examine two cases: fields which are spatially coherent but temporally partially coherent (polychromatic), e.g. the output of a laser operating in a single transverse mode but in multiple longitudinal modes, and fields which are spatially partially coherent but quasi-monochromatic, e.g. the output of a laser operating in a single longitudinal mode but in multiple transverse modes. It is demonstrated in both cases that singular points of the field do exist, and their connection with the phase singularities of monochromatic, spatially coherent fields is discussed.

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[4] The term ‘partially coherent’ is often used to refer both to the state of temporal coherence and the state of spatial coherence of a wavefield. In this paper we use it to refer exclusively to the state of spatial coherence, reserving the term ‘polychromatic’ for fields which lack appreciable temporal coherence.
[5] ‘Generic’ features of a wavefield are loosely defined as those typical features that appear naturally in a wavefield. Genericity is discussed in more detail in chapter 1 of [4].

2. Singularities in spatially coherent, polychromatic wavefields

The singular behaviour of spatially coherent but polychromatic fields has been studied recently in a number of papers (e.g. [5–8]; see also [9, 10]) which have demonstrated that anomalous behaviour of the field spectrum is related to the phase singularities which are present in individual spectral components of the field.

We consider as a simple illustration of such effects the coherent superposition of three polychromatic plane waves with propagation directions $\mathbf{s}_1 = \hat{z}$, $\mathbf{s}_2 = \hat{z}\cos\theta_0 + \hat{x}\sin\theta_0$ and $\mathbf{s}_3 = \hat{z}\cos\theta_0 - \hat{x}\sin\theta_0$, where $\hat{x}$ and $\hat{z}$ are unit vectors. The cross-spectral density of a spatially coherent field can be written in the factorized form [11, section 4.5.3]

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) = \psi^*(\mathbf{r}_1, \omega)\psi(\mathbf{r}_2, \omega), \tag{1}$$

where $\psi(\mathbf{r}, \omega)$ is an average monochromatic realization of the field at frequency ω (this is discussed in more detail in [12]). For our example,

$$\psi(\mathbf{r}, \omega) \equiv \sqrt{S_0(\omega)}\left[e^{ik\mathbf{s}_1\cdot\mathbf{r}} + e^{ik\mathbf{s}_2\cdot\mathbf{r}} + e^{ik\mathbf{s}_3\cdot\mathbf{r}}\right] \tag{2}$$

where $k = \omega/c$ is the wavenumber associated with frequency ω, c is the speed of light in vacuum, and $S_0(\omega)$ is the spectrum of

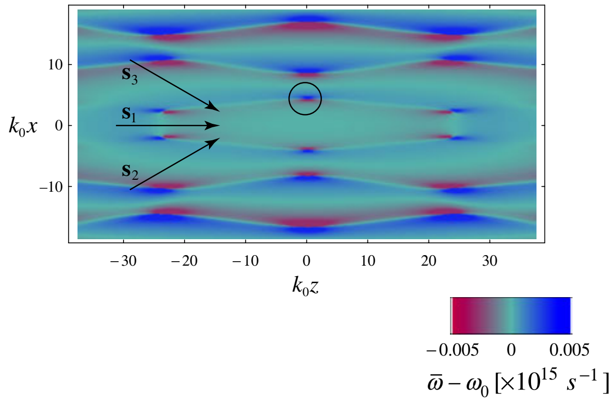

Figure 1. Colour plot of the mean frequency $\bar{\omega}$ of the spectrum of the total field. Here $k_0 = \omega_0/c$. For this example $\omega_0 = 10^{15}$ s^{-1}, $\sigma_0/\omega_0 = 0.01$, and $\theta_0 = \pi/6$. The colour is more red or blue as the mean frequency is more redshifted or blueshifted, respectively. The spectrum at points within the encircled region is shown in figure 2. (This figure is in colour only in the electronic version)

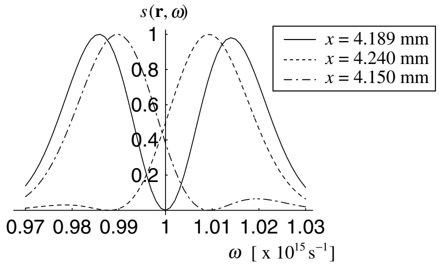

Figure 2. Detail of the (normalized) spectrum $s(\mathbf{r}, \omega)$ at selected points in the neighbourhood of the first axial singular region, with $k_0z = 0$. The spectra are normalized to have their peak values equal to unity, i.e. $s(\mathbf{r}, \omega) \equiv S(\mathbf{r}, \omega)/S(\mathbf{r}, \omega_{\max})$, where $\omega_{\max}$ is the frequency at which the spectrum at position $\mathbf{r}$ attains its maximum value. All other parameters are as in figure 1.

the individual plane waves, taken to be a Gaussian line of centre frequency ω_0 and rms width σ_0. The spectrum of the total field is given by the diagonal element of the cross-spectral density, i.e. $S(\mathbf{r}, \omega) = W(\mathbf{r}, \mathbf{r}, \omega)$, and it follows on substitution from equation (2) into (1) that it takes on the simple form

$$S(\mathbf{r}, \omega) = S_0(\omega)\{3 + 2\cos[kz(1 - \cos\theta_0) - kx\sin\theta_0] + 2\cos[kz(1 - \cos\theta_0) + kx\sin\theta_0] + 2\cos[2kx\sin\theta_0]\}. \quad (3)$$

It is clear from equation (3) that, in the xz-plane and at each frequency ω, the field possesses numerous points at which the spectral density is zero and at which the phase of the field (i.e., the phase of $\psi(\mathbf{r}, \omega)$) at that frequency is therefore singular. However, because of the k-dependence of the different terms of this equation, the positions of these phase singularities depend on frequency and in general no point in space will have a zero spectral density for all frequencies of $S_0(\omega)$. The total average intensity $I(\mathbf{r})$ of this field,

$$I(\mathbf{r}) = \int_0^\infty S(\mathbf{r}, \omega')\,\mathrm{d}\omega', \quad (4)$$

will be nonzero throughout space, and the phase singularities at a given frequency are therefore 'hidden' by the contributions of other frequency components.

These singularities still manifest themselves in the spectrum of the field, however, as can be readily shown. We first consider the mean frequency $\bar{\omega}$ of the spectrum at different points in space. A colour-coded plot of the mean is shown in figure 1, where the colour is more red or more blue as the mean frequency of the spectrum is more redshifted or more blueshifted, respectively. The directions of propagation of the three plane waves are shown on the figure for illustrative purposes. It can be seen that, although the mean frequency throughout most of the region is essentially the same as the mean frequency of each of the individual plane waves $S_0(\omega)$, there exist isolated regions where the mean frequency changes rapidly, for instance at the encircled location along the $z = 0$ axis. A detail of the spectrum at selected points within this region is shown in figure 2. It can be seen that the changes in the spectrum result from the presence of a zero in the spectrum of the total field, corresponding to a phase singularity at that frequency. The presence of this zero causes the spectrum to be effectively redshifted, blueshifted, or even split into two lines, depending on its position within the spectrum.

For light of narrow bandwidth, the region of space within which the spectrum changes rapidly is quite small, and the intensity within such a region is quite low, and hence would be difficult to measure. Such measurements have already been successfully carried out, however [13]. The spectral changes described here have been shown to be a generic feature of wavefields which possess phase singularities [7]. It is to be noted that a decrease in spatial coherence of the wavefield tends to remove the zeros of the spectrum and consequently reduce the spectral changes of the field [14].

3. Singularities in quasi-monochromatic, partially coherent wavefields

For a spatially coherent field as considered in the previous section, it is still reasonable to speak about phase singularities at a given frequency of the field because the field has a well defined phase at each frequency, i.e. the phase of $\psi(\mathbf{r}, \omega)$. When a field is partially coherent, however, its phase itself is random and is no longer well defined, even if the field is quasi-monochromatic. This can be seen using a heuristic argument based on the coherent mode representation [11, section 4.7] of the cross-spectral density of a partially coherent field within a finite volume V, i.e.

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) = \sum_{n=1}^{N} \lambda_n(\omega)\psi_n^*(\mathbf{r}_1, \omega)\psi_n(\mathbf{r}_2, \omega), \quad (5)$$

where $\psi_n(\mathbf{r}, \omega)$ are the coherent modes of the field, mutually orthogonal within the volume V, and the $\lambda_n(\omega)$ are real and positive. The index n generally represents multiple indices, two indices in a two-dimensional domain, three indices in a three-dimensional domain, and for a partially coherent field $N > 1$ and is possibly infinite. Because of the orthogonality of the modes, the cross-spectral density cannot be factorized as in equation (1), and hence there is no well defined phase of the field. Furthermore, it can be readily shown from equation (5) that zeros of the spectral density are not generic: a zero of the spectral density would require the real and imaginary parts of each mode to vanish at the same point, which requires that

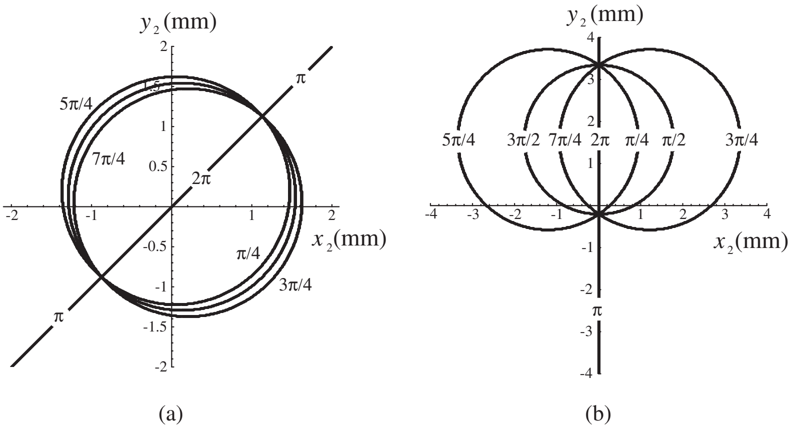

Figure 3. Equiphase contours of the cross-spectral density in the neighbourhood of coherence vortices. In both examples, $w_0 = 1.0$ mm. In (a), $\delta = 1$ mm, $x_1 = 0.1$ mm, $y_1 = 0.1$ mm. In (b), $\delta = 0.75$ mm, $x_1 = 0$ mm, $y_1 = 1$ mm.

$2N$ homogeneous equations must be solved simultaneously. In three-dimensional space, and with $N > 1$, this is an overspecified set of equations which generally has no solution.

However, one can produce model random fields for which every realization of the field possesses phase singularities but for which the average intensity has no zeros. An example of this is a Laguerre–Gauss beam containing an optical vortex which passes through weak atmospheric turbulence. Because an optical vortex is stable under small perturbations of the field, it will generally be present even after passing through the turbulent region. Its position, however, will change as the atmosphere fluctuates, so that on average no point in space will possess a vortex structure. One might wonder if the presence of this vortex is expressed in another property of the field, in some sort of 'hidden' vortex.

A good candidate for such hidden vortices is the singularities of two-point correlation functions described recently, so-called *coherence vortices* [15]. Such vortices are pairs of points at which the spectral degree of coherence of the field vanishes, i.e. where

$$\mu(\mathbf{r}_1, \mathbf{r}_2, \omega) \equiv \frac{W(\mathbf{r}_1, \mathbf{r}_2, \omega)}{\sqrt{S(\mathbf{r}_1, \omega) S(\mathbf{r}_2, \omega)}} = 0, \tag{6}$$

but at which the spectrum $S(\mathbf{r}_i, \omega)$ $(i = 1, 2)$ of the field is nonzero. Coherence vortices have been shown to be a generic feature of partially coherent wavefields [15].

To investigate the relation between coherence vortices of a partially coherent field and traditional optical vortices, we consider as an example a monochromatic field which consists of a low-order Laguerre–Gaussian beam propagating in the z-direction whose central axis is a slowly varying random function of position. Such a field may be considered a simple model of so-called 'beam wander' in atmospheric turbulence (see for instance, [16, section 6.5.3]). The cross-spectral density of such a field in the plane $z = 0$ can be written as

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) = \int f(\mathbf{r}_0) U^*(\mathbf{r}_1 - \mathbf{r}_0, \omega) U(\mathbf{r}_2 - \mathbf{r}_0, \omega)\, \mathrm{d}^2 r_0, \tag{7}$$

where $f(\mathbf{r}_0)$ is the probability density for the position of the axis, and

$$U(\mathbf{r}, \omega) \equiv \sqrt{2} U_0(\omega) \mathrm{e}^{-\mathrm{i}\phi} \frac{r}{w_0} \mathrm{e}^{-r^2/w_0^2} \tag{8}$$

is the transverse profile of a Laguerre–Gaussian beam of order $l = 1$, $n = 0$ which possesses a vortex at the origin, ϕ being the azimuthal angle, and U_0 represents the field amplitude of the beam. We take the probability density to be a Gaussian function,

$$f(\mathbf{r}_0) = \frac{1}{\sqrt{\pi}\delta} \mathrm{e}^{-r_0^2/\delta^2}. \tag{9}$$

In the limit $\delta \to 0$, the position of the beam axis is fixed and the field is spatially coherent. An increase in δ corresponds to a decrease in the spatial coherence.

The integral (7) can be evaluated by straightforward but tedious calculation, and the result is given by

$$\begin{aligned} W(\mathbf{r}_1, \mathbf{r}_2, \omega) = {} & \frac{2\sqrt{\pi} |U_0(\omega)|^2}{w_0^6 A^3 \delta} \mathrm{e}^{-(\mathbf{r}_1 - \mathbf{r}_2)^2/w_0^4 A} \mathrm{e}^{-(r_1^2 + r_2^2)/\delta^2 w_0^2 A} \\ & \times \left\{ \left[\gamma^2 (x_1 + \mathrm{i} y_1) + (x_1 - x_2) + \mathrm{i}(y_1 - y_2) \right] \right. \\ & \times \left. \left[\gamma^2 (x_2 - \mathrm{i} y_2) - (x_1 - x_2) + \mathrm{i}(y_1 - y_2) \right] + w_0^4 A \right\}, \end{aligned} \tag{10}$$

where $\gamma \equiv w_0/\delta$, $\mathbf{r} \equiv (x, y)$, and

$$A \equiv \left(\frac{2}{w_0^2} + \frac{1}{\delta^2} \right). \tag{11}$$

The zero points of the cross-spectral density are determined by the zeros of the factor in the curly brackets of equation (10). It can be seen immediately by setting $\mathbf{r}_1 = \mathbf{r}_2$ that no zeros of the spectral density exist when the field is partially coherent—the phase singularity of the Laguerre–Gauss beam does not appear in the average intensity due to the wandering of the beam axis.

For a given value of $\mathbf{r}_1$, however, it can be shown that there exists a pair of coherence vortices in the $z = 0$ plane which are collinear with the $x = y = 0$ axis. For positive r_1, the radial positions of these coherence vortices are given by the formula

$$r_2 = \frac{(\gamma^4 + 2\gamma^2 + 2) r_1 \pm \sqrt{[\gamma^8 + 4\gamma^6 + 4\gamma^4] r_1^2 + 4(\gamma^2 + 1) w_0^4 A}}{2(\gamma^2 + 1)}. \tag{12}$$

Two examples of these coherence vortices are shown in figure 3. It can be seen from the example that the location

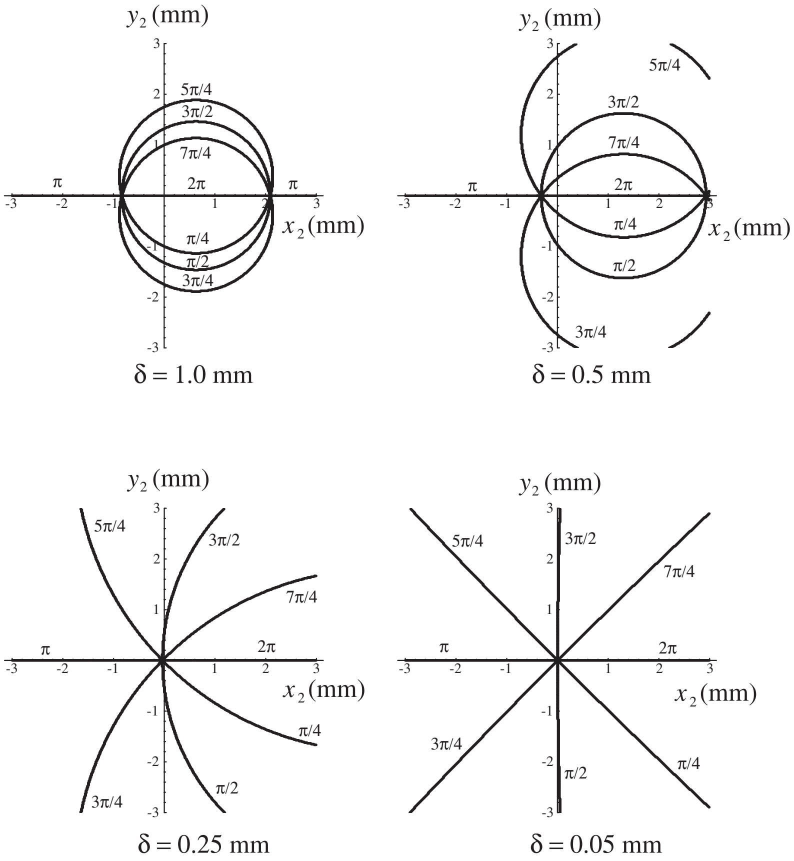

Figure 4. Illustration of the evolution of a coherence vortex into an intensity vortex. The contour lines of the phase of $\mu(\mathbf{r}_1, \mathbf{r}_2, \omega)$ are shown for selected values. For this example $x_1 = 0.5$ mm, $y_1 = 0$, $w_0 = 1.0$ mm. It can be seen that as the field is made more coherent (δ is decreased) the leftmost coherence vortex moves to the origin and evolves into the usual intensity vortex of the Laguerre–Gauss mode.

of the vortices depends on the choice of the position variable $\mathbf{r}_1$, and therefore cannot be assigned to any definite location in space.

To investigate further the connection between the coherence vortices and the intensity vortex of the Laguerre–Gauss beam, we examine the behaviour of the coherence vortices for a fixed value of $\mathbf{r}_1$ as the coherence of the field is continuously increased. Phase maps of the cross-spectral density are shown in figure 4 for several values of δ. It can be seen that as the coherence is increased, the rightmost vortex moves off towards infinity, whilst the left one moves to the origin. In the coherent limit, this leftmost coherence vortex becomes the vortex of intensity of the coherent Laguerre–Gauss beam. As is to be expected, although the position of the coherence vortex depends on the position $\mathbf{r}_1$, as it becomes an intensity vortex, it in fact becomes independent of this position.

This example suggests that coherence vortices are the manifestations of intensity vortices in partially coherent fields. In this sense, the traditional intensity vortices might be considered a special case of the broader class of singularities of two-point coherence functions of a field. More study is needed to elucidate the connection between these 'hidden' coherence vortices and their fully coherent intensity counterparts.

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