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# Complete destructive interference of partially coherent fields

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## Abstract

A three-point source model is used to study the interference of wavefields which are mutually partially coherent. It is shown that complete destructive interference of the fields is possible in such a "three-pinhole interferometer" even if the sources are not fully coherent with respect to each other. An explanation of this surprising effect is given, and conditions necessary for complete destructive interference are stated.
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## 1. Introduction

In recent years there has been much interest in the properties of coherent wavefields in the neighborhood of regions in which the field amplitude is zero and hence the phase is singular. The rapid growth of this subject, referred to as singular optics [1], has encouraged a number of authors to extend the subject to wavefields which are partially coherent [2,3]. However, it is generally accepted, and has been shown for a number of systems [4–7], that a decrease in spatial coherence generally reduces the destructive interference of wavefields, invariably erasing the singular points. Indeed, it was recently demonstrated [8] by means of the coherent mode expansion of the cross-spectral density of a partially coherent field that the intensity of such a field generally does not possess zeros at all.

In view of these observations, it is of interest to determine under what conditions, if any, partially

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coherent fields may give rise to complete destructive interference. We investigate this possibility by considering the interference of fields produced by three or more point sources in a "Young's multiple-pinhole interferometer"; such interferometers have been shown in recent years to possess a number of intriguing diffraction properties [9–11]. We find necessary conditions for destructive interference in such a system which turn out to be related to the non-negative definiteness conditions for the spectral degree of coherence of a wavefield. Surprisingly, the fields from several point sources may exhibit complete destructive interference in isolated regions even if the sources are not fully coherent with respect to each other. These results suggest new possibilities for the study of singular optics with partially coherent light.

In Section 2 we briefly review the behavior of the classic Young's two-pinhole interferometer with partially coherent light, establishing the concepts and notation needed in the later sections. In Section 3 we discuss a three-pinhole interferometer and the necessary conditions for it to generate complete destructive interference. We also discuss how these results extend to an interferometer with more than three pinholes. In Section 4 we suggest a physical explanation for the surprising interference properties of the three-pinhole interferometer and a method of producing the correlations necessary for complete destructive interference.

## 2. Coherence and Young's two-pinhole interferometer

The two-pinhole interference experiment of Thomas Young and its relationship to the spatial coherence of the incident field has been discussed in great detail elsewhere (see, for example, [12, Sec. 4.3]); here we briefly review those results which are necessary for our analysis.

The system of interest is illustrated in Fig. 1. A partially coherent field is incident on an opaque screen $\mathscr{A}$ containing two pinholes, $Q_1$ and $Q_2$, separated by a distance $d$. The interference pattern is examined in the neighborhood of a point $P$ on a screen $\mathscr{B}$ placed a distance $L$ from and parallel to $\mathscr{A}$. It is assumed that the area $a$ of the individ-
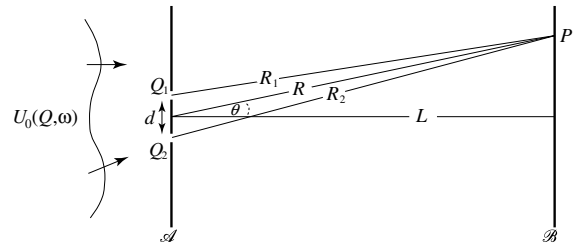


Fig. 1. Illustrating the notation for Young's two-pinhole interferometer.

ual pinholes is small enough that they can be treated as point sources, and it is also assumed that $d \ll L$.

The field incident on the pinholes may in general be partially coherent and polychromatic. We will consider its statistical properties at a single frequency $\omega$, employing the space–frequency representation of a partially coherent field [12, Section 4.7]. Such a treatment is mathematically similar in form to most studies of singular optics, which deal primarily with monochromatic fields. [2] Furthermore, if the field is quasi-monochromatic with center frequency $\omega$, its overall behavior is well approximated by its behavior at the center frequency. Under the assumption that the angles of incidence and diffraction are small, the field beyond the screen $\mathscr{A}$ is then given by the sum of contributions from the individual pinholes, and the spectral density [3] is given by the expression

$$S(P,\omega) = \langle |U_1(P,\omega) + U_2(P,\omega)|^2 \rangle, \qquad (1)$$

where

$$U_j(P,\omega) = -\mathrm{i}\frac{ka^2}{2\pi}U_0(Q_j,\omega)\frac{\mathrm{e}^{\mathrm{i}kR_j}}{R_j} \quad (j=1,2) \qquad (2)$$

is the field produced by the $j$th pinhole, $U_0(Q_j,\omega)$ is the value of the incident field at the $j$th pinhole, $R_j$

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[2] Some notable exceptions are the recent publications [13–17].

[3] The spectral density may be identified with the field intensity at frequency $\omega$. For a quasi-monochromatic field of center frequency $\omega$, the spectral density is roughly proportional to the total intensity of the field.

is the distance from the $j$th pinhole to the observation point $P$, and $k = \omega/c$ is the wavenumber of the light, $c$ being the speed of light. In Eq. (1) the angular brackets denote averaging over an ensemble of space–frequency realizations of the field ([18]; see also [12, section 4.7]). On substituting from Eq. (2) into Eq. (1), the spectral density of the light at $P$ is found to be given by the expression [12, Eq. (4.3–54)]

$$
\begin{aligned}
S(P, \omega) = {} & S^{(1)}(P, \omega) + S^{(2)}(P, \omega) \\
& + \sqrt{S^{(1)}(P, \omega)S^{(2)}(P, \omega)} \\
& \times (\mu_{12}e^{ik(R_2 - R_1)} + \mu_{12}^* e^{-ik(R_2 - R_1)}),
\end{aligned} \tag{3}
$$

where

$$
S^{(j)}(P, \omega) \equiv \left(\frac{ka^2}{2\pi}\right)^2 \frac{S_j(\omega)}{R_j^2} \tag{4a}
$$

is the spectral density of light at point $P$ if only the $j$th pinhole is open, $S_j(\omega) = \langle |U_0(Q_j, \omega)|^2 \rangle$ is the spectral density of light at the $j$th pinhole,

$$
\mu_{12} = \langle U_0^*(Q_1, \omega)U_0(Q_2, \omega) \rangle / \sqrt{S_1(\omega)S_2(\omega)} \tag{4b}
$$

is the spectral degree of coherence of the light at the two pinholes, and the asterisk denotes the complex conjugate. Eq. (3) is known as the *spectral interference law* for partially coherent light, because it describes how the spectrum of the diffracted field depends on the interference between the fields from the two pinholes. This equation may be simplified by noting that for $L \gg d$, $R_1 \approx R_2 \approx R$ in the denominator and hence

$$
\begin{aligned}
\left(\frac{2\pi R}{ka^2}\right)^2 S(P, \omega) = {} & S_1(\omega) + S_2(\omega) + \sqrt{S_1(\omega)S_2(\omega)} \\
& \times (\mu_{12}e^{ik(R_2 - R_1)} + \mu_{12}^* e^{-ik(R_2 - R_1)}).
\end{aligned} \tag{5}
$$

This equation may be rewritten in a compact matrix form as

$$
\left(\frac{2\pi R}{ka^2}\right)^2 S(P, \omega) = \mathbf{x}^{(2)\dagger}\mathbf{M}^{(2)}\mathbf{x}^{(2)}, \tag{6}
$$

where the matrix $\mathbf{M}^{(2)}$ is defined as

$$
\mathbf{M}^{(2)} \equiv \begin{bmatrix} 1 & \mu_{12} \\ \mu_{12}^* & 1 \end{bmatrix}, \tag{7}
$$

and the vector $\mathbf{x}^{(2)}$ is given by

$$
\mathbf{x}^{(2)} \equiv \begin{pmatrix} y_1 e^{i\phi_1} \\ y_2 e^{i\phi_2} \end{pmatrix} \tag{8}
$$

with

$$
y_j \equiv \sqrt{S_j(\omega)}, \tag{9}
$$

and

$$
\phi_j \equiv kR_j, \tag{10}
$$

and $j = 1,2$. It is to be noted from Eq. (7) that $\mathbf{M}^{(2)}$ does not depend upon the spectral density of the light at the two pinholes, nor on the position of the point of observation; it depends solely upon the correlation properties of light with respect to the two pinholes, represented by $\mu_{12}$.

We are now in a position to consider zeros of the spectral density, i.e. points $P$ such that

$$
S(P, \omega) = 0. \tag{11}
$$

It is shown in Appendix A that a *necessary* condition for the spectral density to take on zero value is that $\mathbf{M}^{(2)}$ possesses a zero eigenvalue. It follows from elementary linear algebra that it has such an eigenvalue if and only if its determinant vanishes, i.e. if

$$
1 - |\mu_{12}|^2 = 0, \tag{12}
$$

which implies that

$$
|\mu_{12}| = 1. \tag{13}
$$

In words, the fields from the two pinholes can create complete destructive interference only if the fields at the two pinholes are completely spatially coherent.

It is interesting to note that this necessary condition for complete destructive interference is equivalent to the extreme value of the second-order non-negative definiteness condition discussed in Appendix B. The requirement for a field to be non-negative definite to second order at two points is that

$$
|\mu_{12}|^2 \leqslant 1, \tag{14}
$$

and the extreme value is given by Eq. (13). We will see later in the discussion of $N$-pinhole interferometers that the necessary condition for complete destructive interference is that the $N$th-order
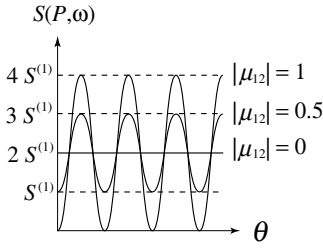
Fig. 2. Typical interference patterns in Young's two-pinhole interferometer, for several values of $|\mu_{12}|$. Adapted from [12, Fig. 4.7]. $\theta$ is defined in Fig. 1.



Fig. 3. Illustrating the notation for a three-pinhole interferometer.

non-negative definiteness condition takes on its extreme value.

Some typical spectral densities observed at the screen $\mathscr{B}$ are shown in Fig. 2 for a variety of values of $|\mu_{12}|$, and with $S_1(\omega) = S_2(\omega)$. We see, as expected, that only when $|\mu_{12}|$ takes on the extreme value of unity does the interference pattern have zeros.

It is important to note that Eq. (13) is only a necessary condition for complete destructive interference of the field. Sufficiency requires that, in addition, the spectral densities at the two pinholes be equal, i.e. that $S_1(\omega) = S_2(\omega)$. Generating true zeros in such an interferometer therefore requires a careful tuning of both the spectral degree of coherence and the spectra at the two pinholes. We will encounter similar requirements in interferometers with more than two pinholes.

## 3. The three-pinhole interferometer and complete destructive interference

We now consider a screen $\mathscr{A}$ which has three pinholes (see Fig. 3), located at points $Q_1$, $Q_2$ and $Q_3$. We take all three pinholes to lie within a circle of radius $d$ centered on the $z$-axis such that $d \ll L$. In a manner similar to that used in the previous section, we may express the spectral density of the field at the point $P$ beyond the screen in the form

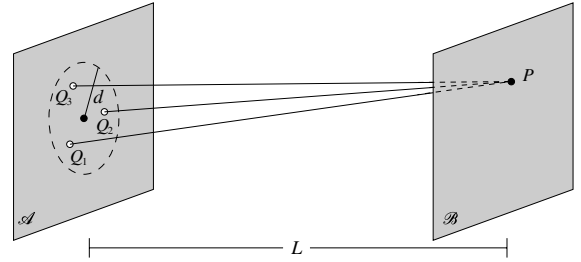$$S(P,\omega) = \langle |U_1(P,\omega) + U_2(P,\omega) + U_3(P,\omega)|^2 \rangle, \tag{15}$$

where $U_j(P,\omega)$ again represents the field produced by the $j$th pinhole and is given by formula (2). On substituting from Eq. (2) into Eq. (15) and taking the ensemble average, it follows that the spectral density may be written as

$$S(P,\omega) = \sum_{i=1}^{3} S^{(i)}(P,\omega) + \sum_{i<j\leqslant 3} \sqrt{S^{(i)}(P,\omega)S^{(j)}(P,\omega)}$$
$$\times (\mu_{ij} e^{ik(R_j - R_i)} + \mu_{ij}^* e^{-ik(R_j - R_i)}). \tag{16}$$

Assuming that $R_1 \approx R_2 \approx R_3 \approx R$, this expression may also be written in a simple matrix form as

$$\left(\frac{2\pi R}{ka^2}\right)^2 S(P,\omega) = \mathbf{x}^{(3)\dagger} \mathbf{M}^{(3)} \mathbf{x}^{(3)}, \tag{17}$$

where

$$\mathbf{M}^{(3)} \equiv \begin{bmatrix} 1 & \mu_{12} & \mu_{13} \\ \mu_{12}^* & 1 & \mu_{23} \\ \mu_{13}^* & \mu_{23}^* & 1 \end{bmatrix}, \tag{18}$$

and

$$\mathbf{x}^{(3)} \equiv \begin{pmatrix} y_1 e^{i\phi_1} \\ y_2 e^{i\phi_2} \\ y_3 e^{i\phi_3} \end{pmatrix}. \tag{19}$$

Here $y_j$ and $\phi_j$ are again given by Eqs. (9) and (10), respectively, but now $j = 1,2,3$. As in the two-pinhole case, a necessary condition for the spectral density to have zero value is that the matrix $\mathbf{M}^{(3)}$ possess a zero eigenvalue, which is equivalent to the matrix possessing a zero determinant, i.e.

$$1 - |\mu_{12}|^2 - |\mu_{23}|^2 - |\mu_{13}|^2 + \mu_{12}\mu_{23}\mu_{13}^* + \mu_{12}^*\mu_{23}^*\mu_{13}$$
$$= 0. \tag{20}$$

Analogous to the two-pinhole case, this condition is the extreme value of the third-order non-negative definiteness condition. Indeed, it follows by extending the preceding analysis to an $N$-pinhole interferometer that a necessary condition for such an interferometer to give rise to a field which has zeros of spectral density is that the $N$th-order non-negative definiteness condition take on its extreme value, i.e. that

$$\det\left[\mathbf{M}^{(N)}\right] = 0, \tag{21}$$

where $\mathbf{M}^{(N)}$ is defined as

$$\mathbf{M}_{ij}^{(N)} = \begin{cases} \mu_{ij}, & i \neq j, \\ 1, & i = j, \end{cases} \tag{22}$$

and $\mu_{ij}^* = \mu_{ji}$.

As a simple example of the three-point source system, let us consider the case when $\mu_{12} = \mu_{23} = \mu_{13} \equiv \mu_0$, and $\mu_0$ is real-valued. Condition (20) then takes on the simple form

$$1 - 3\mu_0^2 + 2\mu_0^3 = 0. \tag{23}$$

Two of the roots of this cubic equation are unity, as might be expected – if the fields at the pinholes are fully coherent with respect to each other, destructive interference is possible. However, the third root of the equation is $\mu_0 = -1/2$. We therefore have the surprising result that in a three-pinhole interferometer, complete destructive interference is possible even if the field fluctuations at the three pinholes are not fully coherent with respect to each other.

We now present several examples of such an interferometer with $\mu_{12} = \mu_{13} = \mu_{23} = -1/2$ and $S_1(\omega) = S_2(\omega) = S_3(\omega) \equiv S_0(\omega)$. As we will see, different configurations of the pinholes in the plane $\mathscr{A}$ may result in significantly different behaviors.

When the positions of the pinholes form an equilateral triangle of side length $a$ (Fig. 4(a)), it can be readily seen from Eq. (17) that the spectral density is zero along the $z$-axis. The spectral density at a typical plane of constant $z$ is shown in Fig. 4(b). The zero of spectral density on the $z$-axis is clearly shown, as are a number of other points
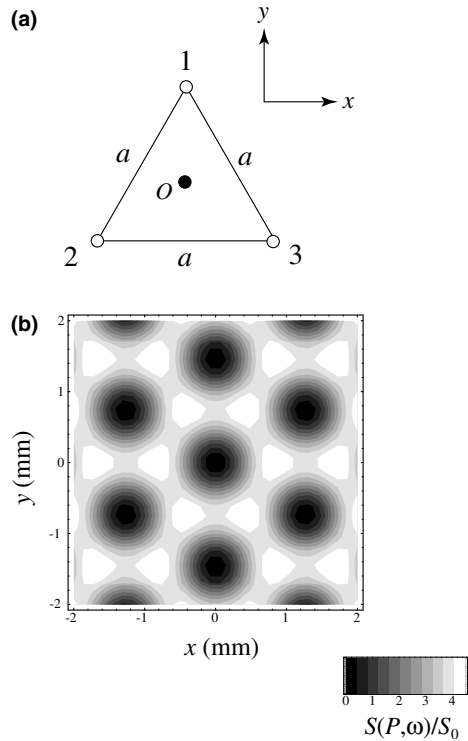


Fig. 4. The (a) geometry of a three-pinhole system arranged as an equilateral triangle, and (b) the spectral density produced by such a system, with $a = 1$ mm, $k = 9921$ mm$^{-1}$ and $z = 2$ m.

which are strong minima, and possibly zeros as well.

When the pinholes are located along a line, as in Fig. 5(a), we obtain a significantly different pattern of zeros in the spectral density. We can determine the pattern of zeros analytically by assuming that the distance $L$ to the observation plane is significantly greater than the transverse separations of the pinholes and the transverse distance of the point of observation from the $z$-axis. The distances $R_i$ may then be expressed in the approximate form

$$R_1 \approx L\left[1 + \frac{1}{2}\frac{(x+a)^2 + y^2}{L^2}\right], \tag{24}$$

$$R_2 \approx L\left[1 + \frac{1}{2}\frac{x^2 + y^2}{L^2}\right], \tag{25}$$

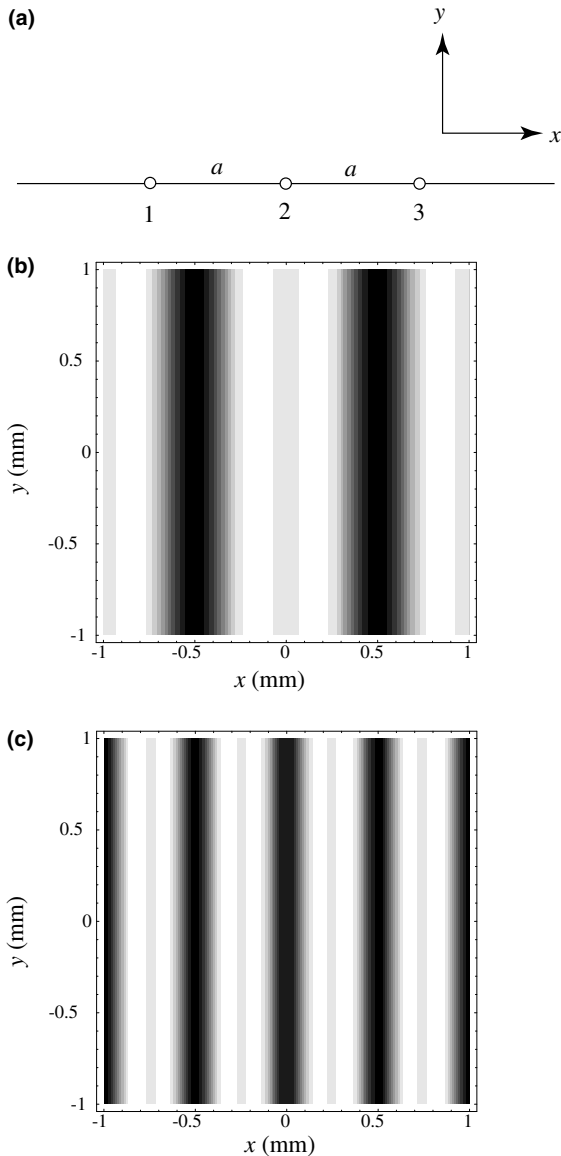**(a)**



**(b)**



**(c)**



Fig. 5. The (a) geometry of a three-pinhole system with the pinholes arranged in a line along the *x*-axis. In (b) the spectral density produced by such a configuration at a distance $L = ka^2/2\pi$ is shown, and in (c) the spectral density produced by such a configuration at a distance $L = ka^2/4\pi$ is shown, with $a = 1$ mm, $k = 9921$ mm$^{-1}$.

$$R_3 \approx L\left[1 + \frac{1}{2}\frac{(x-a)^2 + y^2}{L^2}\right]. \qquad (26)$$

It is readily found on substitution from these expressions into Eq. (17) and the use of simple trig-

onometric identities that the spectral density at *P* is given by

$$\left(\frac{2\pi R}{ka^2}\right)^2 S(P, \omega)$$

$$= S_0(\omega)\left\{3 - 2\cos\frac{ka^2}{2L}\cos\frac{kax}{L} - \cos\frac{2kax}{L}\right\}. \qquad (27)$$

This expression can only vanish when the product of the first two cosine terms is equal to unity and the third cosine term is also equal to unity. Because the arguments of the latter two cosine terms only differ by a factor of 2, we have, in fact, only two equations that must be solved to obtain the locations of a zero, namely

$$\frac{ka^2}{2L} = \pi n, \qquad (28)$$

$$\frac{kax}{L} = \pi m, \qquad (29)$$

where *n* and *m* are integers, either both odd or both even. It is to be noted that Eq. (28) shows that the system may only have zeros of the spectral density at distances where the Fresnel number $N = a^2/\lambda L$ of the system [19, p. 417] takes on integer value. Because these two equations isolate particular values of *x* and *L* in space, zeros of the spectral density lie upon lines along the *y*-direction. Fig. 5(b) shows the result of numerical calculations of Eq. (27) for the case when $ka^2/2L = \pi$. It follows from Eq. (29) that the first zeros will occur at a distance $x = \pm 0.5$ mm, which is confirmed by the calculations. Fig. 5(c) shows the spectral density on the screen when $ka^2/2L = 2\pi$. For this case it follows that there is a zero at $x = 0$, with the next closest zeros at $x = \pm 0.5$ mm, a prediction again confirmed by the figure.

It is to be noted that the requirement (28) predicts that there are no zeros of spectral density at distances greater than $L = ka^2/2\pi$ from the screen.

## 4. Physical interpretation and conclusions

We have demonstrated that it is possible to produce complete destructive interference in partially

coherent fields even if the fields are not completely correlated with each other. An example of such a situation is a three-pinhole interferometer with the spectrum at the three pinholes being equal and the spectral degree of coherence of the light at each pair of pinholes having the value $\mu_0 = -1/2$. The existence of such an effect with partially coherent fields is surprising, but has a clear physical explanation, as we now show.

It is to be noted that although the fields from a pair of point sources may be individually partially coherent with respect to each other and with respect to the field from a third point source, it is possible for the *sum* of the fields from the pair of point sources to be fully correlated with the field of the third. In this case, the sum of the fields from the pair of point sources can destructively interfere with the field of the third source. To illustrate this, we first consider a point source $P$ which is fully correlated with a "black box" source $\beta$, as illustrated in Fig. 6(a). Because the fields are fully correlated, we have



Fig. 6. A physical explanation for complete destructive interference with partially coherent fields. If (a) a point source $P$ is fully correlated with a "black box" source $\beta$, the two sources can generate complete destructive interference. The black box may consist of (b) two other point sources. It is shown that the black box source is fully correlated with point source $P$ if $\mu_0 = 1, -1/2$. By (c) moving the point sources apart, there remains a plane on which the field produced by two of the sources is completely correlated with the third.

$$|\mu_{P\beta}|^2 = 1, \tag{30}$$

where $\mu_{P\beta}$ is the spectral degree of coherence with respect to the point source and the "black box". From the definition

$$\mu_{P\beta} \equiv \frac{W_{P\beta}}{\sqrt{S_P}\sqrt{S_\beta}}, \tag{31}$$

of the spectral degree of coherence, it follows on substitution into Eq. (30) that

$$|W_{P\beta}|^2 = S_P S_\beta. \tag{32}$$

Now let us assume that the "black box" $\beta$ consists of a pair of closely spaced point sources $Q$ and $R$, as illustrated in Fig. 6(b). We further assume that the sources $Q$ and $R$ are separated by a distance smaller than the wavelength, so that the total field produced by them is approximately given by their sum. It then follows that

$$W_{P\beta} = W_{PQ} + W_{PR}, \tag{33}$$

and that

$$S_\beta = S_Q + S_R + 2\sqrt{S_Q S_R} \operatorname{Re}\{\mu_{QR}\}, \tag{34}$$

Re denoting the real part. The latter equation follows from the spectral interference law, Eq. (3).

Let us further assume that the spectral degree of coherence of light with respect to all pairs of point sources is real and equal to each other, i.e. that $\mu_{PQ} = \mu_{PR} = \mu_{QR} = \mu_0$, and that the spectra of the light at the three-point sources are equal to $S_0$. After substitution from Eqs. (33) and (34) into Eq. (32), we arrive at the equation

$$4\mu_0^2 - 2\mu_0 - 2 = 0. \tag{35}$$

We obtained this equation by assuming that one point source was perfectly correlated with the sum of the other two. The roots of this equation are $\mu_0 = 1$, as might be expected, but also $\mu_0 = -1/2$, the result from Section 3. Our explanation is thus in agreement with our earlier results.

If the point sources $Q$ and $R$ are now moved away from each other, the field on the plane bisecting the distance between them will be given by the sum of the fields produced by each source (Fig. 6(c)). The field produced by the third source is completely correlated with the field on this bisecting plane. In this way, we may move the pair of
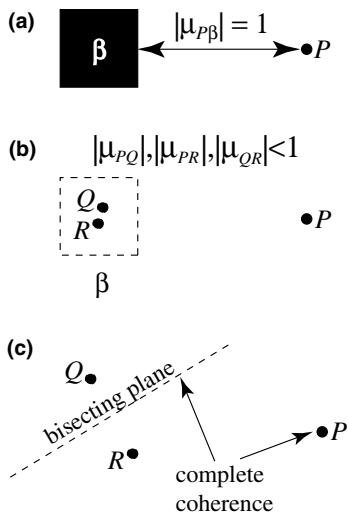
sources $Q$ and $R$ to produce the equilateral triangle geometry of Fig. 4(a), or the linear geometry of Fig. 5(a), and we can see that in both these systems there is a correlation between the field produced by two of the sources and by the third source.

The correlations described in this article obviously represent a rather unique state of coherence of the incident field, and it is not immediately obvious how to produce such a field. One possibility is to generate them by use of an incoherent superposition of Laguerre–Gauss modes, as we will now show.

In the waist plane of a Laguerre–Gauss beam of radial order $p = 0$ and helical order $l = \pm 1$, the field amplitude is [20, p. 87]

$$U_{\pm 1}^0(\mathbf{r}) = \sqrt{2} A \mathrm{e}^{\pm i\phi} \frac{r}{w_0} \mathrm{e}^{-r^2/w_0^2}, \tag{36}$$

where $A$ is the amplitude of the beam and $w_0$ is the beam width in the waist plane. If we take an incoherent superposition of two such beams with equal amplitude but $l$-values of opposite sign, the cross-spectral density of the resulting field is given by the expression

$$W(\mathbf{r}_1, \mathbf{r}_2) = \frac{2|A|^2}{w_0^2} r_1 r_2 \mathrm{e}^{-(r_1^2 + r_2^2)/w_0^2} \{ \mathrm{e}^{-i(\phi_2 - \phi_1)} + \mathrm{e}^{i(\phi_2 - \phi_1)} \}. \tag{37}$$

Such a beam might be produced by superimposing a Laguerre–Gauss beam of $l = +1$ with a beam of $l = -1$ originating from two independent lasers. It is straightforward to show that the spectral degree of coherence of such a field is given by the expression

$$\mu(\mathbf{r}_1, \mathbf{r}_2) = \cos(\phi_2 - \phi_1). \tag{38}$$

When the azimuthal coordinates of $\mathbf{r}_1$, $\mathbf{r}_2$ differ by $2\pi/3$, it immediately follows that $\mu = -1/2$. If this partially coherent beam is allowed to impinge on a screen with pinholes arranged in an equilateral triangle (as in Fig. 4(a)), the desired state of coherence is achieved.

### Acknowledgement

### Appendix A. Zeros of the spectral density

The condition for the spectral density of an $N$-slit interferometer to possess zeros may be written in a matrix form as

$$\left(\frac{2\pi R}{ka}\right)^2 S(P, \omega) = \mathbf{x}^{(N)\dagger} \mathbf{M}^{(N)} \mathbf{x}^{(N)} = 0, \tag{A.1}$$

where the components of the matrix are given by Eq. (22), and the components of the vector are given by the expression $\mathbf{x}_i^{(N)} = y_i \mathrm{e}^{-i\phi_i}$. It is assumed that all the $y_i$ are non-zero (otherwise we would be dealing with a "less-than-$N$-pinhole" interferometer). There are two possible ways for a matrix equation of the form (A.1) to be satisfied: either the matrix possesses a zero eigenvalue, or the matrix rotates the vector $\mathbf{x}^{(N)}$ to a position orthogonal to itself.

To investigate these possibilities, it is useful to use Dirac's "bra–ket" notation. We may write the matrix $\mathbf{M}^{(N)}$ in the form

$$\mathbf{M}^{(N)} = \sum_{n=1}^{N} \lambda_n |n\rangle \langle n|, \tag{A.2}$$

where $|n\rangle$ is the $n$th eigenvector of the matrix, and $\lambda_n \geqslant 0$ is the $n$th eigenvalue. Such an expansion is possible because the matrix $\mathbf{M}^{(N)}$ is Hermitian and non-negative definite. We may expand the vector $\mathbf{x}^{(N)}$ in the form

$$\mathbf{x}^{(N)} = \sum_{n=1}^{N} x_n |n\rangle, \tag{A.3}$$

where $x_n$ is the component of $\mathbf{x}^{(N)}$ in the $n$-direction. On substituting from Eqs. (A.3) and (A.2) into Eq. (A.1), it follows that

$$\mathbf{x}^{(N)\dagger} \mathbf{M}^{(N)} \mathbf{x}^{(N)} = \sum_{n=1}^{N} \lambda_n |x_n|^2. \tag{A.4}$$

The only way that this quantity can have zero value without all the $x_n$ being zero is if $\lambda_n = 0$ for one or more values of $n$. Therefore a necessary condition for the spectral to possess zeros is that the matrix $\mathbf{M}^{(N)}$ possesses one or more zero eigenvalues. It follows from elementary considerations based on linear algebra that a necessary and sufficient condition for the matrix to possess a zero eigenvalue is that the determinant of the matrix vanishes.

## Appendix B. Non-negative definiteness of correlation functions

Here we briefly review the concept of non-negative definiteness of a correlation function; further description can be found in [19, Appendix VIII] and the reference therein.

We consider a field measured at $N$ distinct points, and consider the following superposition $U_s(\omega)$ of the fields at those points:

$$U_s(\omega) \equiv \sum_{n=1}^{N} a_n U(Q_n, \omega), \tag{B.1}$$

where $Q_n$ are the $N$ distinct points and the $a_n$ are arbitrary real or complex constants. It is obvious that the squared modulus of this quantity is non-negative, and so is the ensemble average of the squared modulus, i.e.

$$\langle |U_s(\omega)|^2 \rangle = \left\langle \left| \sum_{n=1}^{N} a_n U(Q_n, \omega) \right|^2 \right\rangle \geqslant 0. \tag{B.2}$$

On taking the average of the appropriate quantities, this formula may be expressed in the form

$$\sum_{i,j=1}^{N} a_i^* a_j \sqrt{S_i(\omega)} \sqrt{S_j(\omega)} \mu_{ij} \geqslant 0. \tag{B.3}$$

We assume that the spectra $S_i(\omega)$ are non-zero. We may then define new constants $b_i \equiv a_i \sqrt{S_i(\omega)}$, so that condition (B.3) becomes

$$\sum_{i,j=1}^{N} b_i^* b_j \mu_{ij} \geqslant 0. \tag{B.4}$$

This equation is the non-negative definiteness condition for a field at $N$ points. It may be expressed in a matrix form as

$$\mathbf{b}^{(N)\dagger} \mathbf{M}^{(N)} \mathbf{b}^{(N)} \geqslant 0, \tag{B.5}$$

where $\mathbf{M}^{(N)}$ is given by Eq. (22), $\mathbf{b}_j = b_j$, and $\mu_{ji} = \mu_{ij}^*$. The matrix $\mathbf{M}^{(N)}$ that appears in the non-negative definiteness condition is, therefore, the same that appears in the interference problem. A necessary condition that inequality (B.5) be satisfied is that the determinant of $\mathbf{M}^{(N)}$ be non-negative. For $N = 2$ this condition takes on the form

$$|\mu_{12}|^2 \leqslant 1, \tag{B.6}$$

and for $N = 3$ this condition takes on the form

$$\begin{aligned} 1 - |\mu_{12}|^2 - |\mu_{23}|^2 - |\mu_{13}|^2 + \mu_{12}\mu_{23}\mu_{13}^* \\ + \mu_{12}^*\mu_{23}^*\mu_{13} \geqslant 0. \end{aligned} \tag{B.7}$$

## References

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