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Phase singularities and coherence vortices in linear optical systems

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Abstract

It is demonstrated for a time-invariant linear optical system that there exists a definite connection between the optical vortices (phase singularities of the field amplitude) which appear when it is illuminated by spatially coherent light and the coherence vortices (phase singularities of the field correlation function) which appear when it is illuminated by partially coherent light. Optical vortices are shown to evolve into coherence vortices when the state of coherence of the field is decreased. Examples of the connection are given. Furthermore, the generic behavior of coherence vortices in linear optical systems is described.

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1. Introduction

There has been much interest in recent years in describing the behavior of wavefields in the neighborhood of regions where the field amplitude is zero and consequently the phase of the field is singular. The regions of zero amplitude are typically lines in three-dimensional space and their intersections with a plane are typically isolated points. The phase has a circular flow around such singular points (called phase singularities), and such a point is generally referred to as an 'optical vortex'. The study of the behavior of optical vortices has spawned a subfield of optical research, now known as singular optics [1].

Following the success of singular optics in classifying the phase singularities of coherent wavefields, a number of authors have extended this research to the study of singularities of two-point correlation functions such as the cross-spectral density [2–5]. Although zeros of intensity are not typically present in partially coherent wavefields, it has been shown that zeros of the cross-spectral density are common [3], and their study is therefore of interest.

One curious observation concerning such 'coherence vortices' (also referred to as 'spatial correlation vortices' [5,6]) is their relation to ordinary optical vortices of spatially coherent fields. For certain systems, it has been observed that optical vortices evolve into coherence vortices when the spatial coherence of the input light field is decreased. These observations have been made both theoretically [4,7] and experimentally [5], and it seems that such a transformation is common; however, no general proof of this has been given. Furthermore, although the typical or 'generic' properties of optical vortices are well-understood, no study of the generic behavior of coherence vortices has been undertaken.

The relation between the two types of vortices is of some significance, especially in light of experimental observations of beams which possess phase singularities of both the field amplitude and the correlation function [8]. Furthermore, there has been some interest in the study of the propagation of vortex fields through atmospheric turbulence [9], a process which inevitably reduces the spatial coherence of the propagating field.

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In this paper, we demonstrate that optical vortices which appear in the output field of a time-invariant linear optical system which is illuminated by spatially coherent light always evolve into coherence vortices as the spatial coherence of the illuminating field is gradually decreased. From this demonstration, the generic behavior of coherence vortices in partially coherent optical fields is determined. This behavior is illustrated by considering several examples.

2. Partially coherent fields in linear optical systems

We consider the propagation of scalar wavefields through an arbitrary time-invariant linear optical system. We study the response of the system at a single frequency ω, and this frequency dependence will be suppressed in the equations which follow. If the input field at the plane $z = z_0$ is given by $U_0(\mathbf{r}_\perp)$, and the output field in the half-space $z > 0$ is given by $U(\mathbf{r})$, then these fields are related by (see Fig. 1) [10]

$$U(\mathbf{r}) = \int U_0(\mathbf{r}'_\perp) f(\mathbf{r}; \mathbf{r}'_\perp)\, \mathrm{d}^2 r'_\perp, \tag{1}$$

where $f(\mathbf{r}; \mathbf{r}'_\perp)$ is the *kernel* of the system and represents its response to a Dirac delta function input at position $\mathbf{r}'_\perp$ and frequency ω, and $\mathbf{r} = (x, y, z)$, $\mathbf{r}'_\perp = (x', y')$. Throughout the paper, a single integral sign will be used to refer to a two-dimensional integral over an infinite plane, either in spatial variables (x, y) or spatial frequency variables (K_x, K_y), unless otherwise specified. It is to be noted that $U(\mathbf{r})$, and therefore $f(\mathbf{r}; \mathbf{r}'_\perp)$, is a solution of the homogeneous Helmholtz equation,

$$(\nabla^2 + k^2) U(\mathbf{r}) = 0, \tag{2}$$

where $k = \omega/c$ is the wave number of the field, c being the speed of light. Eq. (1) can be used to represent a broad class of optical systems, such as focusing systems, aperture diffraction systems, and weak scattering systems. For instance, the kernel for Rayleigh–Sommerfeld aperture diffraction is given by [11, Section 8.11.2, Eq. (14)]

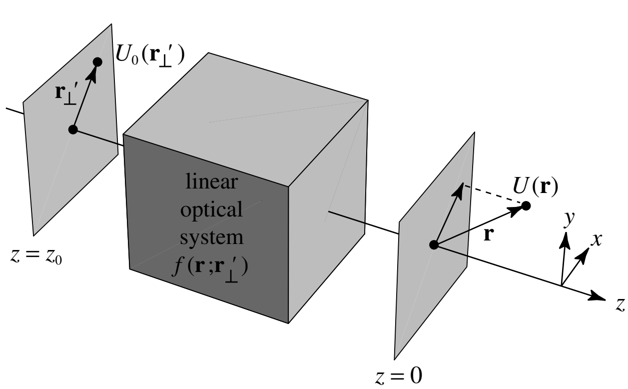

Fig. 1. Illustration of the notation used in describing a general, time-invariant linear optical system.

$$f(\mathbf{r}, \mathbf{r}'_\perp) = \frac{1}{2\pi} B(\mathbf{r}'_\perp) \frac{\partial}{\partial z'} \left(\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right)_{z'=0}, \tag{3}$$

where $\mathbf{r}' = (x', y', z')$ and $B(\mathbf{r}'_\perp)$ is an aperture function, having value unity when $\mathbf{r}'_\perp$ lies within the aperture, and value zero elsewhere.

For input fields which are partially coherent, it is necessary to work with the *cross-spectral densities* of the input and output fields instead of the fields themselves. According to the theory of optical coherence in the space–frequency domain [12, Section 4.7], we may represent the fields in these regions by ensembles $\{U_0(\mathbf{r}'_\perp)\}$ and $\{U(\mathbf{r})\}$ of monochromatic fields. The cross-spectral density of the input field in the plane $z' = z_0$ is given by

$$W_0(\mathbf{r}'_\perp, \mathbf{r}''_\perp) = \langle U_0^*(\mathbf{r}'_\perp) U_0(\mathbf{r}''_\perp) \rangle \tag{4}$$

and the cross-spectral density of the field output from the linear system is given by

$$W(\mathbf{r}_1, \mathbf{r}_2) = \langle U^*(\mathbf{r}_1) U(\mathbf{r}_2) \rangle, \tag{5}$$

where the angular brackets denote ensemble averaging. The diagonal element of the cross-spectral density gives the power spectrum of the field (the intensity at frequency ω), i.e.

$$S(\mathbf{r}) = W(\mathbf{r}, \mathbf{r}). \tag{6}$$

We may also define the *spectral degree of coherence* by the relation [12, Section 4.3.2],

$$\mu(\mathbf{r}_1, \mathbf{r}_2) \equiv \frac{W(\mathbf{r}_1, \mathbf{r}_2)}{\sqrt{S(\mathbf{r}_1)}\sqrt{S(\mathbf{r}_2)}}. \tag{7}$$

The spectral degree of coherence is a measure of the spatial coherence of the wavefield and its modulus is confined to the values

$$0 \leqslant |\mu(\mathbf{r}_1, \mathbf{r}_2)| \leqslant 1, \tag{8}$$

zero indicating complete incoherence, unity indicating complete coherence.

We consider partially coherent input fields such that the plane $z = z_0$ is the waist plane of the input field and the field therefore has a constant average phase on this plane. It is to be noted that this condition does not reduce the generality of the analysis, for optical elements can be added to our arbitrary linear optical system to produce any average phase profile that is desired (e.g. a lens could be added to the entrance of the system to produce an input field which is a converging spherical wave). We further assume that the correlations of the field in the input plane are homogeneous and isotropic, i.e. that the spectral degree of coherence $\mu_0(\mathbf{r}'_\perp, \mathbf{r}''_\perp)$ in the input plane depends only upon the magnitude of the difference in the position variables,

$$\mu_0(\mathbf{r}'_\perp, \mathbf{r}''_\perp) = \mu_0(|\mathbf{r}'_\perp - \mathbf{r}''_\perp|). \tag{9}$$

We may then write the cross-spectral density of the input field in the form

$$W_0(\mathbf{r}'_\perp, \mathbf{r}''_\perp) = \mu_0(|\mathbf{r}'_\perp - \mathbf{r}''_\perp|) \sqrt{S_0(\mathbf{r}'_\perp)} \sqrt{S_0(\mathbf{r}''_\perp)}, \tag{10}$$

where S_0 is the spectral density of the field in the input plane. Fields with a cross-spectral density in the form of Eq. (10) are referred to as Schell-model fields [12, Section 5.2.2], and represent a broad class of physically realizable partially coherent fields.

The effect of the linear system on the cross-spectral density of the field can be found by substituting from Eq. (1) into Eq. (5) and taking the ensemble average, leading to the relation

$$W(\mathbf{r}_1,\mathbf{r}_2) = \int\int W_0(\mathbf{r}'_\perp,\mathbf{r}''_\perp) f^*(\mathbf{r}_1;\mathbf{r}'_\perp) f(\mathbf{r}_2;\mathbf{r}''_\perp)\, \mathrm{d}^2r'_\perp\, \mathrm{d}^2r''_\perp. \tag{11}$$

In analyzing the evolution of the input field from complete coherence to partial coherence, it is to be noted that there exists no unique way of making such a transition. The coherent limit, assuming a constant average phase in the plane $z = z_0$, is simply given by

$$\mu_0^{(\mathrm{coh})}(|\mathbf{r}'_\perp - \mathbf{r}''_\perp|) \equiv 1. \tag{12}$$

For partially coherent fields, the spectral degree of coherence has the value unity, when $\mathbf{r}'_\perp = \mathbf{r}''_\perp$, and generally decreases to zero as the points $\mathbf{r}'_\perp$ and $\mathbf{r}''_\perp$ become more separated. To define the transition more clearly, we further assume that the spectral degree of coherence has the following functional dependence, viz:

$$\mu_0(|\mathbf{r}'_\perp - \mathbf{r}''_\perp|) \equiv \xi_0[|\mathbf{r}'_\perp - \mathbf{r}''_\perp|/\Delta], \tag{13}$$

where Δ is the correlation length of the input field, i.e. the distance over which the field remains essentially correlated (see Fig. 2). In words, Eq. (13) indicates that a change in Δ scales the width of the function μ_0, while leaving its shape intact. It is reasonable to assume that μ_0 has a Fourier representation, so that we may express it in the form

$$\mu_0(\mathbf{R}_\perp) = \int \tilde{\mu}_0(\mathbf{K}_\perp) \mathrm{e}^{\mathrm{i}\mathbf{K}_\perp\cdot\mathbf{R}_\perp}\, \mathrm{d}^2K_\perp, \tag{14}$$

with a corresponding inverse Fourier relation. It can be shown from general properties of Fourier transforms that in this conjugate space, as the coherence of the input field is increased, the function $\tilde{\mu}_0$ becomes narrower. Defining a modified system kernel

$$F(\mathbf{r};\mathbf{r}'_\perp) \equiv f(\mathbf{r};\mathbf{r}'_\perp)\sqrt{S_0(\mathbf{r}'_\perp)}, \tag{15}$$

we may rewrite Eq. (11) in the form

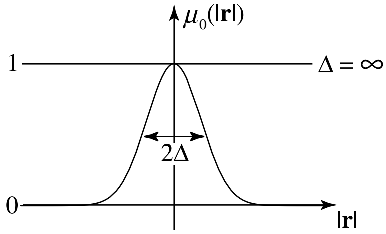

Fig. 2. Dependence of the spectral degree of coherence, $\mu(|\mathbf{r}|)$, on the correlation length Δ. For $\Delta \to \infty$, the field is spatially coherent.

$$W(\mathbf{r}_1,\mathbf{r}_2) = (2\pi)^4 \int \tilde{\mu}_0(\mathbf{K}_\perp)\tilde{F}^*(\mathbf{r}_1;\mathbf{K}_\perp)\tilde{F}(\mathbf{r}_2;\mathbf{K}_\perp)\, \mathrm{d}^2K_\perp, \tag{16}$$

where

$$\tilde{F}(\mathbf{r};\mathbf{K}_\perp) \equiv \frac{1}{(2\pi)^2}\int F(\mathbf{r};\mathbf{r}'_\perp)\mathrm{e}^{-\mathrm{i}\mathbf{K}_\perp\cdot\mathbf{r}'_\perp}\, \mathrm{d}^2r'_\perp \tag{17}$$

is the two-dimensional Fourier transform of the modified system kernel. The form of this equation in the coherent limit can be found first by substituting from Eq. (12) into Eq. (14) and taking the inverse Fourier transform. We find that $\tilde{\mu}_0^{(\mathrm{coh})}(\mathbf{K}_\perp) = \delta^{(2)}(\mathbf{K}_\perp)$, with $\delta^{(2)}$ being the two-dimensional Dirac delta function, and the cross-spectral density output from the system takes on the form

$$W^{(\mathrm{coh})}(\mathbf{r}_1,\mathbf{r}_2) = (2\pi)^4\tilde{F}^*(\mathbf{r}_1;0,0)\tilde{F}(\mathbf{r}_2;0,0). \tag{18}$$

It is seen from Eq. (18) that the cross-spectral density factorizes and the field output from the system may, in the coherent limit, be written as

$$U^{(\mathrm{coh})}(\mathbf{r}) = (2\pi)^2\tilde{F}(\mathbf{r};0,0). \tag{19}$$

3. Coherence vortices in linear optical systems

We are interested in studying the behavior of optical vortices in a time-invariant linear optical system when the spatial coherence of the input field is decreased. To begin, we consider a partially coherent field input into the optical system, but one which is still highly coherent, i.e. $\tilde{\mu}_0$ is assumed to be extremely localized about the origin in $K_\perp$-space. In this case, we are justified in approximating the two occurrences of $\tilde{F}$ in Eq. (16) by their lowest-order Taylor series expansions in spatial-frequency coordinates $\mathbf{K}_\perp = (K_x, K_y)$,

$$\tilde{F}(\mathbf{r}_i;\mathbf{K}_\perp) \approx \tilde{F}(\mathbf{r}_i;0,0) + \boldsymbol{\gamma}(\mathbf{r}_i)\cdot\mathbf{K}_\perp \quad (i = 1,2), \tag{20}$$

where $\boldsymbol{\gamma}(\mathbf{r}_i) \equiv (\gamma_x(\mathbf{r}_i), \gamma_y(\mathbf{r}_i))$ and

$$\gamma_j(\mathbf{r}_i) \equiv \frac{\partial}{\partial K_j}\tilde{F}(\mathbf{r}_i;K_x,K_y)\Big|_{K_x=K_y=0}, \tag{21}$$

with $j = x, y$. We immediately note from Eq. (19) that the expression for $\tilde{F}$ may be written in terms of the response of the system to a coherent field, i.e.

$$\tilde{F}(\mathbf{r}_i;\mathbf{K}_\perp) \approx \frac{U^{(\mathrm{coh})}(\mathbf{r}_i)}{(2\pi)^2} + \boldsymbol{\gamma}(\mathbf{r}_i)\cdot\mathbf{K}_\perp. \tag{22}$$

On substituting from Eq. (22) into Eq. (16), we may evaluate the integrals over K_x and K_y; using elementary Fourier analysis, it follows that

$$\int \tilde{\mu}_0(K_x,K_y)\, \mathrm{d}K_x\, \mathrm{d}K_y = \mu_0(0,0) = 1, \tag{23}$$

$$\int \tilde{\mu}_0(K_x,K_y)K_x^2\, \mathrm{d}K_x\, \mathrm{d}K_y = -\frac{\partial^2}{\partial X^2}\mu_0(X,Y)\Big|_{X=Y=0}, \tag{24}$$

$$\int \tilde{\mu}_0(K_x,K_y)K_y^2\, \mathrm{d}K_x\, \mathrm{d}K_y = -\frac{\partial^2}{\partial Y^2}\mu_0(X,Y)\Big|_{X=Y=0}, \tag{25}$$

$$\int \tilde{\mu}_0(K_x, K_y) K_x K_y \, \mathrm{d}K_x \, \mathrm{d}K_y = -\frac{\partial^2}{\partial X \partial Y} \mu_0(X, Y) \Big|_{X=Y=0} = 0, \tag{26}$$

$$\int \tilde{\mu}_0(K_x, K_y) K_x \, \mathrm{d}K_x \, \mathrm{d}K_y = \int \tilde{\mu}_0(K_x, K_y) K_y \, \mathrm{d}K_x \, \mathrm{d}K_y = 0. \tag{27}$$

The values of the last three integrals follow from the isotropy of the spectral degree of coherence, as expressed by Eq. (9). We may further define the quantity

$$D_0 \equiv -\frac{\partial^2}{\partial (X_i/\Delta)^2} \mu_0(X, Y) \Big|_{X=Y=0}, \tag{28}$$

where X_i can be either X or Y. D_0 can be seen by the use of Eq. (13) to be a quantity independent of Δ and, because of isotropy, independent of i. On substitution of these results into Eq. (16), it follows that

$$W(\mathbf{r}_1, \mathbf{r}_2) = U^{(\mathrm{coh})*}(\mathbf{r}_1) U^{(\mathrm{coh})}(\mathbf{r}_2) + (2\pi)^4 \boldsymbol{\gamma}^*(\mathbf{r}_1) \cdot \boldsymbol{\gamma}(\mathbf{r}_2) D_0/\Delta^2. \tag{29}$$

This expression indicates that when the coherence of the input field is decreased, the immediate effect on the cross-spectral density of the output field is the addition of a term which is inversely proportional to the square of the correlation length Δ. The quantity $\boldsymbol{\gamma}$ depends only on the modified system kernel $\tilde{F}$ and describes the response of the system to the coherence properties of the input field.

We now consider a linear optical system whose output field contains optical vortices when it is illuminated by a spatially coherent input field. Let us assume that the system possesses an optical vortex at the point $\mathbf{r}^{(1)}$, i.e. that when a spatially coherent field is input into the optical system, the field amplitude takes on zero value at the point $\mathbf{r}^{(1)}$. Eq. (19) then implies that

$$\tilde{F}(\mathbf{r}^{(1)}; 0, 0) = 0. \tag{30}$$

To determine the response of the system when the input field is partially coherent, we assume that the observation point $\mathbf{r}_1$ is in the immediate neighborhood of the point $\mathbf{r}^{(1)}$, and that the observation point $\mathbf{r}_2$ is in the immediate neighborhood of another, generally different, point $\mathbf{r}^{(2)}$. We may therefore perform a Taylor expansion of the functions $\tilde{F}(\mathbf{r}_i; 0, 0)$ and $\boldsymbol{\gamma}(\mathbf{r}_i)$ and keep the most significant (lowest-order) terms of each expansion, i.e.

$$\tilde{F}(\mathbf{r}_1; 0, 0) \approx \boldsymbol{\beta}^{(1)} \cdot (\mathbf{r}_1 - \mathbf{r}^{(1)}), \tag{31}$$

$$\tilde{F}(\mathbf{r}_2; 0, 0) \approx \alpha^{(2)}, \tag{32}$$

$$\boldsymbol{\gamma}(\mathbf{r}_i) \approx \boldsymbol{\gamma}^{(i)}, \tag{33}$$

where

$$\alpha^{(2)} = \tilde{F}(\mathbf{r}^{(2)}; 0, 0), \tag{34}$$

$$\beta_i^{(1)} = \frac{\partial}{\partial x_i} \tilde{F}(\mathbf{r}; 0, 0) \Big|_{\mathbf{r}=\mathbf{r}^{(1)}}, \tag{35}$$

$$\boldsymbol{\gamma}^{(j)} = \boldsymbol{\gamma}(\mathbf{r}^{(j)}). \tag{36}$$

The quantities $\alpha^{(2)}$, $\boldsymbol{\beta}^{(1)}$ and $\gamma^{(i)}$ are properties of the optical system, not of the incident field. α represents the system response at point $\mathbf{r}_2$, $\boldsymbol{\beta}$ characterizes the response of the system at point $\mathbf{r}_1$, and γ represents the response of the system to the coherence properties of the field. On substituting from these equations into Eq. (29), and making use of Eq. (19), it follows that

$$W(\mathbf{r}_1, \mathbf{r}_2) = (2\pi)^4 \{\alpha^{(2)} [\boldsymbol{\beta}^{(1)} \cdot (\mathbf{r}_1 - \mathbf{r}^{(1)})]^* + \boldsymbol{\gamma}^{(1)*} \cdot \boldsymbol{\gamma}^{(2)} D_0/\Delta^2\}. \tag{37}$$

This equation represents the local form of the cross-spectral density of the light after passing through a linear optical system, for the case that the point $\mathbf{r}_1$ is in the immediate neighborhood of a phase singularity of $\tilde{F}(\mathbf{r}; 0, 0)$, i.e. a phase singularity of the corresponding coherent field. To get the most common local forms of the cross-spectral density, or 'generic' forms, we may replace the vector $\boldsymbol{\beta}^{(1)}$ by the vector appropriate for a generic optical vortex.

It is to be noted that Eq. (37) consists of two pieces, as does its more general form, Eq. (29): a 'coherent part' (the first, position-dependent term) and a 'partially coherent' part, dependent on Δ, the correlation length. There are clearly many degrees of freedom in this equation and hence many types of behaviors to analyze; we begin by holding $\mathbf{r}_2$ fixed and consider how the singularity behaves as the state of coherence is changed.

We may consider several 'generic' singularities of a coherent optical field, namely pure *screw dislocations* [13, Section 5.2.1], for which

$$U^{(\mathrm{coh})}(\mathbf{r}_1) \approx (2\pi)^2 [\boldsymbol{\beta}^{(1)} \cdot (\mathbf{r}_1 - \mathbf{r}^{(1)})] = \beta_0 [(x_1 - x^{(1)}) + \mathrm{i}(y_1 - y^{(1)})] \tag{38}$$

and pure *edge dislocations* [13, Section 5.2.2], for which

$$U^{(\mathrm{coh})}(\mathbf{r}_1) \approx (2\pi)^2 [\boldsymbol{\beta}^{(1)} \cdot (\mathbf{r}_1 - \mathbf{r}^{(1)})] = \beta_0 [a(x_1 - x^{(1)}) + \mathrm{i}(z_1 - z^{(1)})], \tag{39}$$

where a and β_0 are constants. Let us define

$$\zeta_0 \equiv \beta_0^* \alpha^{(2)} \tag{40}$$

and

$$\eta_0 \equiv D_0 \boldsymbol{\gamma}^{(1)*} \cdot \boldsymbol{\gamma}^{(2)}. \tag{41}$$

If the point $\mathbf{r}^{(1)}$ is the location of a screw dislocation, the cross-spectral density from Eq. (37) may therefore be written in its generic form as

$$W(\mathbf{r}_1, \mathbf{r}_2) = (2\pi)^4 \zeta_0 \{[x_1 - (x^{(1)} + \delta_x)] - \mathrm{i}[y_1 - (y^{(1)} + \delta_y)]\}, \tag{42}$$

with

$$\delta_x = -\mathrm{Re}\left\{\frac{\eta_0}{\zeta_0 \Delta^2}\right\}, \tag{43}$$

$$\delta_y = \mathrm{Im}\left\{\frac{\eta_0}{\zeta_0 \Delta^2}\right\}. \tag{44}$$

Eq. (42) has the mathematical form of a screw dislocation. It demonstrates that the cross-spectral density has a screw dislocation (coherence vortex) with respect to $\mathbf{r}_1$ if the corresponding coherent field has a screw dislocation (optical vortex) at $\mathbf{r}^{(1)}$. The location of the coherence vortex is shifted spatially by (δ_x, δ_y) with respect to the position of the optical vortex. This shift depends on the position of the second observation point, $\mathbf{r}_2$, through the parameter ζ_0. However, in the fully coherent limit, $\Delta \to \infty$, this shift vanishes identically, the cross-spectral density factorizes, and the phase singularity of the coherence function becomes a zero of the field amplitude at the point $\mathbf{r}^{(1)}$, i.e. an optical vortex.

Similarly, if the point $\mathbf{r}^{(1)}$ is the location of an edge dislocation, the cross-spectral density may be written in its generic form as

$$W(\mathbf{r}_1, \mathbf{r}_2) = (2\pi)^4 \zeta_0 \{a[x_1 - (x^{(1)} + \delta_x)] - \mathrm{i}[z_1 - (z^{(1)} + \delta_z)]\}, \tag{45}$$

with δ_x now given by the expression

$$\delta_x = -\mathrm{Re}\left\{\frac{\eta_0}{a\zeta_0 \Delta^2}\right\} \tag{46}$$

and

$$\delta_z = \mathrm{Im}\left\{\frac{\eta_0}{\zeta_0 \Delta^2}\right\}. \tag{47}$$

We thus have the following observation regarding coherence vortices and optical vortices: a decrease in spatial coherence does not eliminate an optical vortex with respect to position $\mathbf{r}_1$, but only shifts its position by a distance $\Delta\mathbf{r} \equiv \hat{x}\delta_x + \hat{y}\delta_y$ (for a screw dislocation), and it becomes a coherence vortex. The position of this vortex is generally dependent on $\mathbf{r}_2$, and is only independent of it in the coherent limit.

We may conclude from this result that there is an intimate connection between the optical vortices and coherence vortices which appear in time-invariant linear optical systems. In fact, it is therefore reasonable to consider an optical vortex in such systems as a special case of the more general class of coherence vortices.

4. Examples

We consider here two examples which have been discussed in other articles and show how the generic form of a coherence vortex appears in each example.

4.1. Beam wander of a Laguerre–Gauss beam

In a paper which discusses ‘hidden’ singularities of partially coherent fields [4], an example was given of a Laguerre–Gauss beam whose center axis, containing the vortex core, is a random function of position. The probability distribution function for the center axis was taken to be a Gaussian distribution,

$$f(\mathbf{r}_0) = \frac{1}{\sqrt{\pi}\delta} \mathrm{e}^{-r_0^2/\delta^2}. \tag{48}$$

In the limit $\delta \to 0$, the beam axis does not wander at all and the beam is spatially coherent. For $\delta \neq 0$, the cross-spectral density is given by the expression

$$\begin{aligned} W(\mathbf{r}_1, \mathbf{r}_2, \omega) = {} & \frac{2\sqrt{\pi}|U_0(\omega)|^2}{w_0^6 A^3 \delta} \mathrm{e}^{-(\mathbf{r}_1-\mathbf{r}_2)^2/w_0^4 A} \mathrm{e}^{-(r_1^2+r_2^2)/\delta^2 w_0^2 A} \\ & \times \{[\gamma^2(x_1 + \mathrm{i}y_1) + (x_1 - x_2) + \mathrm{i}(y_1 - y_2)] \\ & \times [\gamma^2(x_2 - \mathrm{i}y_2) - (x_1 - x_2) + \mathrm{i}(y_1 - y_2)] + w_0^4 A\}, \end{aligned} \tag{49}$$

where $\gamma \equiv w_0/\delta$, $\mathbf{r} = (x, y)$, w_0 is the width of the beam waist and

$$A \equiv \left(\frac{2}{w_0^2} + \frac{1}{\delta^2}\right). \tag{50}$$

It is to be noted from the first exponential in Eq. (49) above that the correlation length Δ may be defined by the expression

$$\Delta^2 = w_0^4 A, \tag{51}$$

which means that δ is roughly an *inverse* correlation length, i.e.

$$\Delta^2 = B^2/\delta^2, \tag{52}$$

where $B \approx w_0^2$. If we consider points $\mathbf{r}_1$ and $\mathbf{r}_2$ very close to the origin, and assume that the field is very nearly fully coherent ($\delta \ll w_0$), $Aw_0^4 \approx B^2/\Delta^2$ and Eq. (49) reduces to the very simple form

$$W(\mathbf{r}_1, \mathbf{r}_2) \propto \gamma^4\{(x_1 + \mathrm{i}y_1)(x_2 - \mathrm{i}y_2) + B^2/\Delta^2\}, \tag{53}$$

which is in the generic form of a screw-type coherence vortex as given by Eq. (42), with $x^{(1)} = y^{(1)} = 0$, $\zeta_0 = \gamma^4(x_2 - \mathrm{i}y_2)/(2\pi)^4$ and $\eta_0 = \gamma^4 B^2/(2\pi)^4$. The transition of an optical vortex to a coherence vortex is illustrated in Fig. 3. The lines of constant phase of the cross-spectral density are plotted as a function of $\mathbf{r}_1$, with $\mathbf{r}_2$ fixed. The cross-spectral density is calculated with the formula (49). It can be seen that for small δ (high coherence), the cross-spectral density only has a phase singularity at the origin, and it coincides with the zero of intensity of the equivalent coherent field. However, as the coherence is decreased (δ is increased), the position of the vortex shifts, as predicted in Eq. (42). As the coherence is decreased significantly, a second coherence vortex appears, evidently ‘moving’ in from infinity. This vortex is does not appear in the generic form of the cross-spectral density, which is only a depiction of the behavior local to a particular phase singularity.

4.2. Coherence vortices in high Fresnel-number focusing systems

Coherence vortices and optical vortices in the focal region of high Fresnel-number focusing systems have been

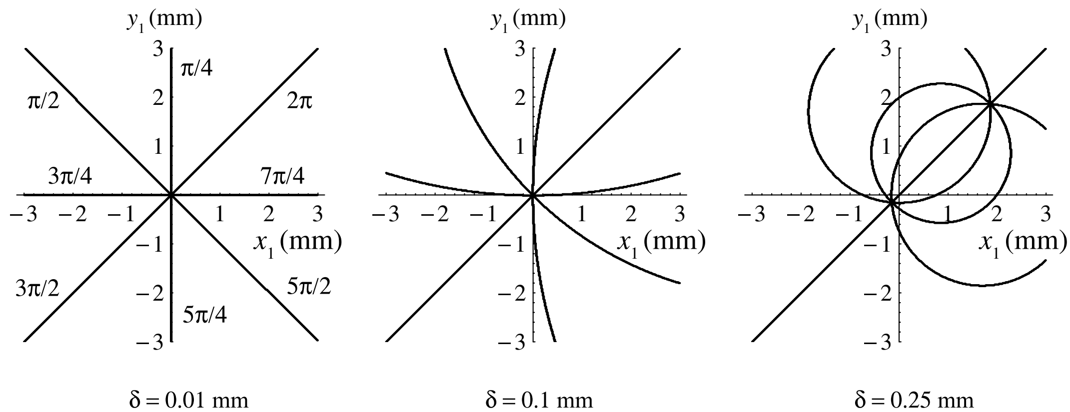

Fig. 3. Illustration of the transition of an optical vortex of the coherent field into a coherence vortex of the partially coherent field. For this example, $x_2 = 0.1$ mm, $y_2 = 0.1$ mm, $w_0 = 1.0$ mm. An increase in δ corresponds to a decrease in spatial coherence.

analyzed in [7], for the case that the field in the aperture is taken to be a Gaussian Schell-model field [12, Section 5.3.2]. The cross-spectral density of the field in the neighborhood of the geometrical focus is given by

$$W(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{(\lambda f)^2} \int\int_{\mathscr{W}} \mathrm{e}^{-(\boldsymbol{\rho}'' - \boldsymbol{\rho}')^2/2\sigma_g^2} \mathrm{e}^{\mathrm{i}k(\mathbf{q}' \cdot \mathbf{r}_1 - \mathbf{q}'' \cdot \mathbf{r}_2)} \, \mathrm{d}^2 r' \, \mathrm{d}^2 r'', \tag{54}$$

where f is the focal length of the system, $\lambda = 2\pi/k$, and σ_g is the correlation length of the input field. Also, $\mathbf{q}'$ is a unit vector in the direction $\mathbf{r}'$, with a similar definition for $\mathbf{q}''$, and the integral is over the wavefront $\mathscr{W}$ in the aperture (see Fig. 4). It is well-known [11, Section 8.8.4] that in the coherent limit there are phase singularities in the geometrical focal plane at radii from the geometrical focus such that

$$J_1(v) = 0, \tag{55}$$

where J_1 is the Bessel function of the first kind and order 1, $v = 2\pi a\sqrt{x^2 + y^2}/(\lambda f)$ is one of the so-called Lommel variables, and a is the radius of the aperture; these rings of zeros are typically referred to as the Airy rings. We consider the behavior of the cross-spectral density when the point $\mathbf{r}_1$ is in the immediate neighborhood of the first zero of $J_1(v)$, i.e. $v \approx 3.83$, and the spatial coherence of the input field is extremely high (i.e. $\sigma_g > a$). In this limit, we may approximate the spectral degree of coherence of the input field by a low-order Taylor series expansion, namely

$$\mathrm{e}^{-(\boldsymbol{\rho}'' - \boldsymbol{\rho}')^2/2\sigma_g^2} \approx 1 - \frac{(\boldsymbol{\rho}'' - \boldsymbol{\rho}')^2}{2\sigma_g^2}. \tag{56}$$

On substitution from Eq. (56) into Eq. (54), we have

$$W(\mathbf{r}_1, \mathbf{r}_2) \approx \frac{1}{(\lambda f)^2} \int\int_{\mathscr{W}} \mathrm{e}^{\mathrm{i}k(\mathbf{q}' \cdot \mathbf{r}_1 - \mathbf{q}'' \cdot \mathbf{r}_2)} \, \mathrm{d}^2 r' \, \mathrm{d}^2 r'' - \frac{1}{(\lambda f)^2} \frac{1}{2\sigma_g^2} \int\int_{\mathscr{W}} (\boldsymbol{\rho}'' - \boldsymbol{\rho}')^2 \mathrm{e}^{\mathrm{i}k(\mathbf{q}' \cdot \mathbf{r}_1 - \mathbf{q}'' \cdot \mathbf{r}_2)} \, \mathrm{d}^2 r' \, \mathrm{d}^2 r''. \tag{57}$$

At this point we have almost obtained the generic form of the cross-spectral density, for the first term of Eq. (57) can be factorized into the product of two coherent focused fields, i.e.

$$U^{(\mathrm{coh})}(\mathbf{r}) = \frac{1}{\lambda f} \int_{\mathscr{W}} \mathrm{e}^{\mathrm{i}k\mathbf{q}' \cdot \mathbf{r}} \, \mathrm{d}^2 r', \tag{58}$$

and the second term is already proportional to the inverse square of the correlation length, σ_g^{-2}. Assuming that $\mathbf{r}_2$ is fixed at point $\mathbf{r}^{(2)}$ and that $\mathbf{r}_1$ is in the immediate neighborhood of a point on the first Airy ring $\mathbf{r}^{(1)}$ (say along the $y_1 = 0$ line), we may keep only the lowest-order Taylor series terms of $W(\mathbf{r}_1, \mathbf{r}_2)$. We may therefore write

$$W(\mathbf{r}_1, \mathbf{r}_2) \approx U^{(\mathrm{coh})*}(\mathbf{r}_1) U^{(\mathrm{coh})}(\mathbf{r}_2) + \frac{\eta_0}{\sigma_g^2}, \tag{59}$$

where

$$\eta_0 \equiv -\frac{1}{2(\lambda f)^2} \int\int_{\mathscr{W}} (\boldsymbol{\rho}'' - \boldsymbol{\rho}')^2 \mathrm{e}^{\mathrm{i}k(\mathbf{q}' \cdot \mathbf{r}^{(1)} - \mathbf{q}'' \cdot \mathbf{r}^{(2)})} \, \mathrm{d}^2 r' \, \mathrm{d}^2 r''. \tag{60}$$

This is the generic form of the cross-spectral density, illustrated in Eqs. (29) and (37). It is to be noted that the right-hand term of Eq. (57) depends on the two positions of observation $\mathbf{r}_1$ and $\mathbf{r}_2$, whereas η_0 in Eq. (60) is independent of these positions. With the point $\mathbf{r}_1$ in the neighborhood of

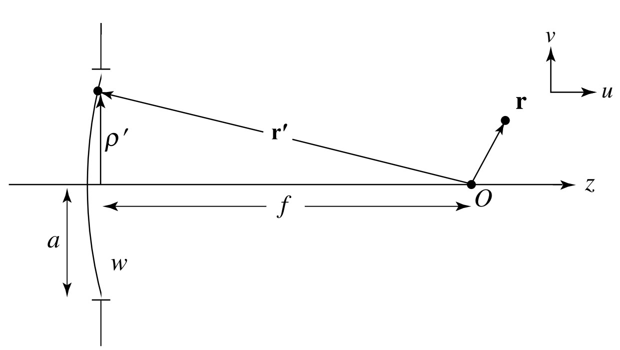

Fig. 4. Illustration of the notation used to describe high-Fresnel number focusing.

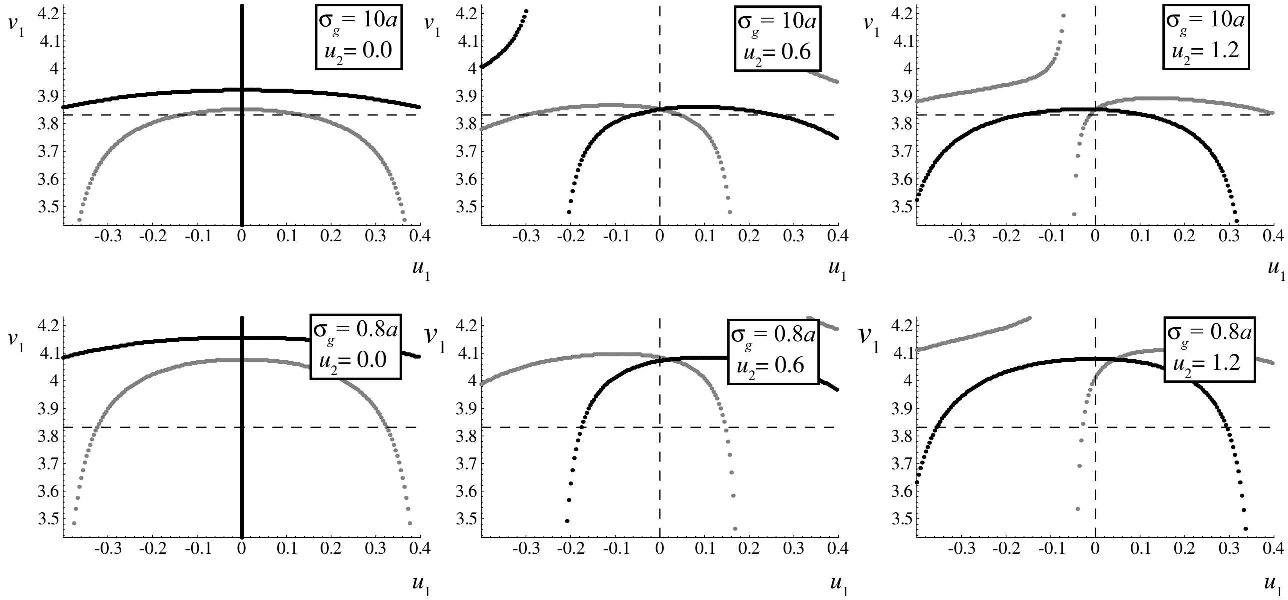

Fig. 5. The behavior of the coherence vortex associated with the first Airy ring of focusing. The grey curves indicate zeros of the real part of the cross-spectral density, while the black curves indicate zeros of the imaginary part of the cross-spectral density. Their intersection represents a zero of the cross-spectral density. For the upper row, $\sigma_g = 10a$, while for the lower row, $\sigma_g = 0.8a$. The columns represent the cases $u_2 = 0.0$, $u_2 = 0.6$, and $u_2 = 1.2$. The dashed line indicates the position of the first Airy ring at $u_1 = 0.0$, $v_1 = 3.83$.

an edge dislocation, we may further expand $U^{(\mathrm{coh})}$ to find the local form of the cross-spectral density.

The behavior of the coherence vortex near the first Airy ring is illustrated in Fig. 5, for two different values of the correlation length σ_g. The cross-spectral density was calculated numerically from Eq. (54). The figure is plotted in the dimensionless Lommel variables $u_1 = 2\pi a^2 z_1/(\lambda f^2)$ and $v_1 = 2\pi a\sqrt{x_1^2 + y_1^2}/(\lambda f)$. It can be seen, as expected, that for a high degree of coherence, the location of the singularity is very near the position of the first Airy ring (i.e. $u_2 = 0$, $v_2 = 3.86$) and only weakly dependent on the position of the point $\mathbf{r}_2$. For a lower degree of coherence, the singularity is displaced from the position of the first Airy ring and its location depends on the position of the point $\mathbf{r}_2$, as expected from Eqs. (45)–(47).

5. Conclusions

It is not clear from the preceding analysis how to physically explain the origin of the relation between the two types of vortices. We may, however, give two informal arguments concerning why this relation is physically reasonable. First, it is well known that a topological charge [1] may be associated with any optical vortex, and this charge is a conserved quantity through any continuous change in the system parameters. The conversion of an optical vortex to a coherence vortex may be interpreted as a generalized form of topological charge conservation, in which the topological charge 'moves' from the field to the coherence function as the field coherence is decreased. Furthermore, it is known [14] that optical vortices are often – though not always – associated with orbital angular momentum of an optical field. Since it is reasonable to expect that this orbital angular momentum is conserved even as the field coherence is decreased, it is also reasonable to expect that the vortices associated with it must also remain.

It is to be noted that although our analysis has demonstrated that optical vortices always evolve into coherence vortices as the coherence of the field decreases, the converse is not necessarily true – that is, a phase singularity or vortex of a coherence function cannot always be connected to some vortex in a corresponding coherent field. For instance, the correlation functions of black body radiation are known to possess an infinite number of phase singularities (related to the zeros of the spherical Bessel functions [15]). There is no way to define a system of 'coherent black body radiation' – black body radiation is always partially coherent – and so the analysis of this paper does not apply.

Further investigation will hopefully clarify the physical origin of the relation between coherence vortices and optical vortices.

In conclusion, we have analyzed the response of a time-invariant linear optical system for the case that the state of coherence of the input field gradually changes from fully coherent to partially coherent. It was shown that under this change, optical vortices evolve into coherence vortices, and the generic features of this process were described. It was illustrated by two recently discussed examples.

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