Rotational frequency shifts for electromagnetic fields of arbitrary states of coherence and polarization

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The rotational frequency shift is studied for fields of arbitrary states of coherence and polarization. It is shown that the power spectrum of the field in the rotating frame is influenced both by the degree of polarization and the degree of coherence. Examples for some model field classes are given. © 2006 Optical Society of America

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In contrast to the ordinary Doppler shift, in which light emitted or absorbed by translating bodies is frequency shifted with respect to the source, it is known that there exists a rotational Doppler shift as well, in which light emitted or absorbed by rotating bodies can be frequency shifted.[1,2] Such shifts have been studied in both classical[3] and quantum mechanical[4] optical systems. Rotational Doppler shifts can be considered a special case of more general rotational frequency shifts, in which both the central frequency of the spectrum as well as the line shape are modified in the rotating frame. These rotational frequency shifts have two possible contributions, one due to polarization effects of the field and one due to azimuthal asymmetry of the field. Recently, researchers demonstrated that the spatial coherence of circularly polarized, partially coherent light plays a large role in the nature and magnitude of frequency shifts.[5] This observation is significant both because partially coherent sources of light are common, and the state of coherence provides an extra tunable degree of freedom in an optical system. It is expected that the partial polarization of the field will also play a nontrivial role in any observed shift; however, no study of partial polarization has yet been undertaken. In this Letter we present a unified theory of partial coherence and partial polarization for rotational frequency shifts. It is shown that the statistical properties of light have a unique effect on the field in the rotating reference frame.

We begin by considering the general statistical properties of a random electromagnetic beam in a nonrotating reference frame. In such studies, it is common to work with the coherence matrix (Ref. 6, Section 6.5), which measures the correlations between all linearly polarized components of the field. It is more convenient for our purposes, however, to define a coherence matrix in a circular polarization basis,

$$\mathbf{\Gamma}(\mathbf{r}_1,\mathbf{r}_2,\tau) = \begin{bmatrix} \langle E_+^*(\mathbf{r}_1,t_1)E_+(\mathbf{r}_2,t_2)\rangle & \langle E_+^*(\mathbf{r}_1,t_1)E_-(\mathbf{r}_2,t_2)\rangle \\ \langle E_-^*(\mathbf{r}_1,t_1)E_+(\mathbf{r}_2,t_2)\rangle & \langle E_-^*(\mathbf{r}_1,t_1)E_-(\mathbf{r}_2,t_2)\rangle \end{bmatrix}, \quad (1)$$

where $E_\pm(\mathbf{r},t)$ is the right-handed or left-handed circularly polarized component of the field, $\tau=t_2-t_1$, and the angular brackets denote ensemble averaging. It is assumed that the field is statistically stationary in the nonrotating frame, at least in the wide sense. We will denote the components of the coherence matrix by Γ_{++} and so on.

We now consider transformation to a reference frame rotating at frequency Ω. The rotation matrix in the circular polarization basis takes on the simple form

$$R_c(t) = \begin{bmatrix} \exp[-\mathrm{i}\Omega t] & 0 \\ 0 & \exp[\mathrm{i}\Omega t] \end{bmatrix}. \quad (2)$$

In the rotating reference frame, the coherence matrix may therefore be written by a similarity transformation as

$$\mathbf{\Gamma}^{(r)}(\mathbf{r}_1,\mathbf{r}_2,\tau,T) = \begin{bmatrix} \tilde{\Gamma}_{++}\exp[\mathrm{i}\Omega\tau] & \tilde{\Gamma}_{+-}\exp[-2\mathrm{i}\Omega T] \\ \tilde{\Gamma}_{-+}\exp[2\mathrm{i}\Omega T] & \tilde{\Gamma}_{--}\exp[-\mathrm{i}\Omega\tau] \end{bmatrix}, \quad (3)$$

where $T=(t_1+t_2)/2$,

$$\tilde{\mathbf{\Gamma}}_{ij} = \Gamma_{ij}[R_\Omega(t_1)\mathbf{r}_1, R_\Omega(t_2)\mathbf{r}_2,\tau], \quad (4)$$

and $R_\Omega(t)$ is the rotation matrix for the position vector

$$R_\Omega(t) = \begin{bmatrix} \cos\Omega t & \sin\Omega t \\ -\sin\Omega t & \cos\Omega t \end{bmatrix}. \quad (5)$$

Equation (3) is the most general form of the coherence matrix in the rotating frame and illustrates a number of significant properties. First, it is to be noted that there exist two possible contributions to a frequency shift of the field: a polarization part (characterized by the exponential factors) and a spatial part (characterized by the rotation matrices R_Ω). For a general electromagnetic field, rotational frequency shifts will consist of a combination of these factors. Second, as has been noted previously,[5] the field in the rotating frame is not in general stationary, as demonstrated by the presence of the time variable T in the matrix. This can happen if the intensity profile of the field or its polarization properties are not rotationally symmetric: for instance, an elliptical beam spot will be rotating in the rotating frame.

The instantaneous spectral properties of the field can be characterized by the cross-spectral density matrix, defined by (Ref. 6, Section 4.3.2)

$$\mathbf{W}^{(r)}(\mathbf{r}_1,\mathbf{r}_2,\omega,T) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{\Gamma}^{(r)}(\mathbf{r}_1,\mathbf{r}_2,\tau,T)\exp[\mathrm{i}\omega\tau]\mathrm{d}\tau. \quad (6)$$

To apply this equation, we look at some specific examples.

First, we consider Bessel-correlated fields of arbitrary uniform state of polarization and Gaussian intensity. Such a field has a coherence matrix in the nonrotating frame given by

$$\mathbf{\Gamma}(\mathbf{r}_1,\mathbf{r}_2,\tau) = I_0\mathbf{P}_0 \exp[-r_1^2/2\sigma_I^2]\exp[-r_2^2/2\sigma_I^2] \times J_0[|\mathbf{r}_2-\mathbf{r}_1|/\sigma_\mu]g(\tau), \quad (7)$$

where σ_I is the waist size of the beam, σ_μ is the correlation length, $g(\tau)$ is the complex degree of coherence, and $\mathbf{P}_0$ is the matrix that describes the polarization state of the field. We consider a field that is in general partially polarized, for which $\mathbf{P}_0$ may be written as (Ref. 6, Section 6.3.3)

$$\mathbf{P}_0 = (1-P)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + 2P\begin{bmatrix}|a|^2 & a^*b\\ b^*a & |b|^2\end{bmatrix}, \quad (8)$$

where $0\leqslant P\leqslant 1$ is the degree of polarization, $P=0$ represents an unpolarized field, and $P=1$ represents a polarized field with complex polarization vector (a,b) in the circular basis. The calculation for the frequency shift can be made more tractable by using a well-known expansion[7] for the Bessel function,

$$J_0(|\mathbf{r}_2-\mathbf{r}_1|/\sigma_\mu) = \sum_{-\infty}^{\infty} J_n(r_1/\sigma_\mu)J_n(r_2/\sigma_\mu)\exp[\mathrm{i}n(\phi_2-\phi_1)], \quad (9)$$

where $\mathbf{r}_1\equiv(r_1,\phi_1)$ in polar coordinates. On converting our field into the rotating coordinate system (by the transformation $\phi_1\to\phi_1-\Omega t_1$) and transforming to the frequency domain, it is found that

$$\mathbf{W}^{(r)}(\mathbf{r}_1,\mathbf{r}_2,\omega) = I_0\sum_{n=-\infty}^{\infty}\mathbf{P}_n(\omega,T)\exp[\mathrm{i}n(\phi_2-\phi_1)] \times\exp[-r_1^2/2\sigma_I^2]\exp[-r_2^2/2\sigma_I^2] \times J_n(r_1/\sigma_\mu)J_n(r_2/\sigma_\mu), \quad (10)$$

where $\tilde{g}(\omega)$ is the temporal Fourier transform of $g(\tau)$ and hence the power spectrum of the field in the nonrotating reference frame, and

$$\mathbf{P}_n(\omega,T) = \begin{bmatrix}[(1-P)+2P|a|^2]\tilde{g}(\omega+\Omega-n\Omega) & 2Pa^*b\tilde{g}(\omega-n\Omega)\exp[-2\mathrm{i}\Omega T]\\ 2Pb^*a\exp[2\mathrm{i}\Omega T]\tilde{g}(\omega-n\Omega) & [(1-P)+2P|b|^2]\tilde{g}(\omega-\Omega-n\Omega)\end{bmatrix} \quad (11)$$

is a matrix representing the polarization and spectral properties of the field. If we restrict ourselves to looking at the spectral density of the field, defined as

$$S^{(r)}(\mathbf{r},\omega) = W_{++}^{(r)}(\mathbf{r},\mathbf{r},\omega) + W_{--}^{(r)}(\mathbf{r},\mathbf{r},\omega), \quad (12)$$

we readily find that

$$S^{(r)}(\mathbf{r},\omega) = I_0\sum_{n=-\infty}^{\infty}[[(1-P)+2P|a|^2]\tilde{g}(\omega+\Omega-n\Omega) + [(1-P)+2P|b|^2]\tilde{g}(\omega-\Omega-n\Omega)] \times\exp[-r^2/\sigma_I^2][J_n(r/\sigma_\mu)]^2. \quad (13)$$

It can be seen that each term of the series expansion has two contributions to its frequency shift: a spatial part (indicated by the $n\Omega$ shift) and a polarization part (indicated by the single Ω shift). The combination of effects introduces a change in the spectral line shape, as well as the central frequency. The results are illustrated in Fig. 1, for a Lorentzian power spectrum. Figure 1(a) demonstrates the dependence of the spectrum of the field on the frequency of rotation in the rotating frame for circularly polarized light. It can be seen that there is a uniform frequency shift due to the state of polarization, as well as a broadening and distortion of the spectrum that depends on the frequency of rotation; these results are similar to those presented in Ref. 5. In Fig. 1(b), the dependence of the spectrum on the degree of polarization is illustrated for $a=0$, $b=1$ (the polarized part of the field is left-circularly polarized). It can be seen that the degree of polarization plays a significant role in the symmetry properties of the spectrum.

It is to be noted that light that is partially linearly polarized (for instance, $a=b=1/\sqrt{2}$) will have a spectrum independent of the degree of polarization.

A careful examination of Eq. (13) shows that the effect of partial polarization and partial coherence in the rotating frame is to split the spectrum into multiple, Doppler-shifted copies of the original spectrum. This can be seen in Fig. 1 though the bandwidth of the field is large enough that the shifted spectral lines overlap appreciably. A change in the degree of polarization can be used to change the relative weights of these shifted lines, as the next example demonstrates.

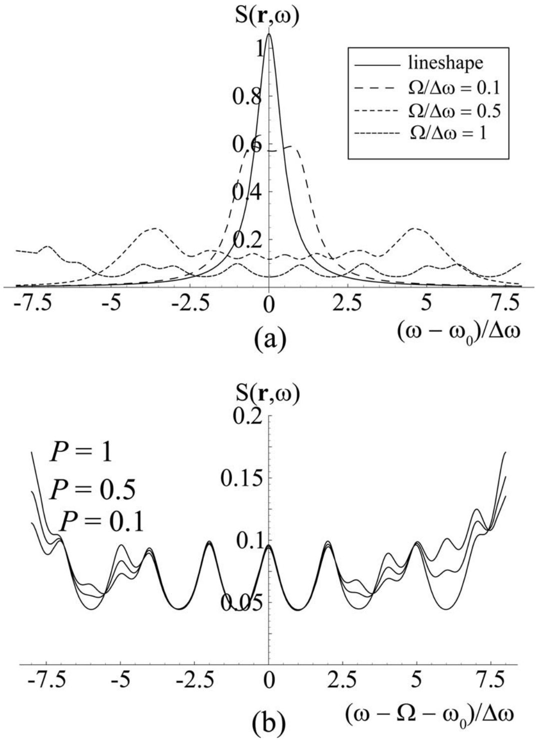

Fig. 1. Changes in the (normalized) spectrum of a Bessel-correlated field for (a) a circularly polarized field at different rates of rotation Ω normalized by linewidth $\Delta\omega$ and (b) a partially polarized field with different degrees of polarization P, with $\Omega/\Delta\omega=1$. In both cases, $r/\sigma_\mu=10$, where r is the radial distance from the center of rotation and σ_μ is the correlation length. The power spectrum in the nonrotating frame is Lorentzian.

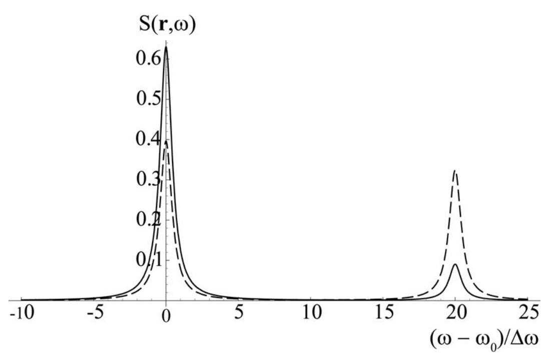

Fig. 2. Changes in the (normalized) spectrum of a partially polarized and partially coherent field for $P=0.1$ (dashed curve) and $P=0.75$ (solid curve), with $\Omega/\Delta\omega=10$ and $r\sigma_\mu=0.25$.

We consider a field with a coherence matrix given by

$$\mathbf{\Gamma}(\mathbf{r}_1,\mathbf{r}_2,\tau)=I_0\mathbf{P}_0J_1(r_1/\sigma_\mu)J_1(r_2/\sigma_\mu)\times\exp[\mathrm{i}(\phi_2-\phi_1)]g(\tau). \quad (14)$$

This field is partially polarized and spatially coherent. It follows in a manner similar to the previous example that

$$S^{(r)}(\mathbf{r},\omega)=I_0[J_1(r/\sigma_\mu)]^2\{[(1-P)+2P|a|^2]\tilde{g}(\omega)+[(1-P)+2P|b|^2]\tilde{g}(\omega-2\Omega)\}. \quad (15)$$

If we consider light that is, in a polarized state, right-circularly polarized ($a=1$, $b=0$), we have

$$S^{(r)}(\mathbf{r},\omega)=I_0[J_1(r/\sigma_\mu)]^2\times[(1+P)\tilde{g}(\omega)+(1-P)\tilde{g}(\omega-2\Omega)]. \quad (16)$$

In this case the degree of polarization controls the weight of two spectral lines, one of which is not frequency shifted and one that is frequency shifted by 2Ω. Figure 2 illustrates the power spectrum in the rotated frame for two values of the degree of polarization. For $P=1$ there exists only the single, nonshifted spectral line, while for $P=0$ there exist two equally weighted lines, one shifted by 2Ω and one unshifted. The unshifted line can be shown to correspond to a radially polarized field; such fields are known to have a number of interesting properties, particularly on focusing.[8]

It is important to note that rotational frequency shifts appear in fields even when the field is on average rotationally isotropic. In both examples given, the degree of coherence is rotationally invariant, and the polarization state is invariant when the field is unpolarized. In both examples rotational frequency shifts appear nonetheless.

Spectral changes related to the statistical properties of light are not unprecedented. So-called correlation-induced spectral changes,[9] in which the initial state of coherence of a wave field influences its spectral behavior on propagation, are well established.

The introduction of partial coherence and partial polarization to the study of rotational frequency shifts introduces new complexity to the problem and new degrees of freedom to exploit in applying the effect. Such frequency shifts may prove useful in remote sensing, or in developing new forms of optical gyroscopes. The study of fields with special coherence and polarization properties will provide new avenues of study.

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