Angular spectrum representation for the propagation of arbitrary coherent and partially coherent beams through atmospheric turbulence

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An angular spectrum representation is applied for a description of statistical properties of arbitrary beamlike fields propagating through atmospheric turbulence. The Rytov theory is used for the characterization of the perturbation of the field by the atmosphere. In particular, we derive expressions for the cross-spectral density of a coherent and a partially coherent beam of arbitrary type in the case when the power spectrum of atmospheric fluctuations is described by the von Karman model. We illustrate the method by applying it to the propagation of several model beams through the atmosphere. © 2007 Optical Society of America

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1. INTRODUCTION

There is much interest in using laser light for point-to-point communications and for laser radar systems, due to both the secure nature of highly directional lasers and the high bandwidth associated with optical frequencies. However, atmospheric turbulence can significantly degrade an optical signal over even relatively short distances, and much work has been done to study the effects of turbulence on the spreading, coherence, and scintillation characteristics of light.

In recent years, there has been an increasing body of research, which suggests that partially coherent beams may be less affected by turbulence than their fully coherent counterparts and hence might be better suited to atmospheric applications. Earlier work focused on the average beam width[1–5] and coherence properties of such beams, while more recent studies[6–8] have shown that under proper circumstances the scintillation properties could be superior for partially coherent beams.

The studies mentioned above focus primarily either on coherent Gaussian beams or on Gaussian Schell-model beams,[9] a broad but special class of optical fields with a Gaussian intensity profile and a Gaussian correlation function. There is significant evidence, however, that other classes of beams might provide even better improvement. Notable among these are nondiffracting beams,[10] which have been shown theoretically to be resistant to amplitude and phase fluctuations[11] and therefore possibly to turbulence,[12,13] and also vortex beams,[14,15] whose inherent orbital angular momentum may provide additional resistance.[16]

Due to the complex nature of atmospheric turbulence, calculations involving beams of arbitrary shape and coherence are difficult and must be started over for every new field. A formalism for studying general beam classes is therefore desirable but does not currently exist. In physical optics, however, the angular spectrum representation[9] of wave fields has been extremely successful in the study of both coherent and partially coherent fields of arbitrary shape. This representation involves the decomposition of a general field into a sum of plane waves of different directions, which can then be individually propagated through the optical system of interest. The field at any point throughout the optical system can be calculated by recombining the contributions of the individual plane waves. Because the propagation characteristics of plane waves in turbulence are well known,[17] such an approach seems helpful for atmospheric propagation problems as well.

In this paper, we introduce an angular spectrum representation for partially coherent fields of arbitrary intensity profile and spatial coherence propagating in atmospheric turbulence. In particular, we derive an expression for the cross-spectral density function of the field (second-order coherence properties). Also, the method is outlined for the calculation of the intensity correlations (fourth-order coherence properties) in this representation. We illustrate the method by applying it to the propagation of several model beams. We first demonstrate that the techniques lead to the results, which are in good agreement with ones based on the classic approach, using (coherent) Gaussian beams and Gaussian Schell-model beams. We then show how the method works for (coherent) Bessel beams.

2. ANGULAR SPECTRUM REPRESENTATION OF OPTICAL FIELDS PROPAGATING THROUGH A RANDOM MEDIUM

A. Coherent Fields

We start by considering a monochromatic field $U(\mathbf{r},\omega)$ at frequency ω at a point with position vector $\mathbf{r}=(\boldsymbol{\rho},z)$, $\boldsymbol{\rho}=(x,y)$ propagating in vacuum from the plane $z=0$ into a positive half-space $z>0$ (see Fig. 1). The space-dependent part of this field can be expressed in terms of its angular spectrum of plane waves (see Ref. 9, Subsection 3.2.2), i.e.,

$$U(\mathbf{r},\omega)=\int\int a(\mathbf{u},\omega)P_{\mathbf{u}}(\mathbf{r},\omega)\mathrm{d}^2u_{\perp}, \tag{2.1}$$

where

$$P_{\mathbf{u}}(\mathbf{r},\omega)=\exp[ik(\mathbf{u}\cdot\mathbf{r})] \tag{2.2}$$

is the plane wave propagating in the direction specified by unit vector $\mathbf{u}=(u_x,u_y,u_z)$, $k=\omega/c$ is the wavenumber, c is the speed of light in vacuum, $\mathbf{u}_{\perp}=(u_x,u_y,0)$, and

$$u_z=+\sqrt{1-u_{\perp}^2}. \tag{2.3}$$

We take the integration in Eq. (2.1) over the region $|u_{\perp}|\leq 1$, which is equivalent to assuming that the evanescent waves of the angular spectrum can be neglected. In Eq. (2.1), $a(\mathbf{u},\omega)$ is the amplitude of a plane wave in the angular spectrum, which can be determined from the field $U_0(\mathbf{r}',\omega)$ in the source plane by the formula

$$a(\mathbf{u},\omega)=\frac{1}{(2\pi)^2}\int\int U_0(\mathbf{r}',\omega)P_{\mathbf{u}}^*(\mathbf{r}',\omega)\mathrm{d}^2r', \tag{2.4}$$

where $\mathbf{r}'=(\boldsymbol{\rho}',0)$, $\boldsymbol{\rho}'=(x',y')$ is a point in the source plane, and the asterisk denotes complex conjugation.

If the field propagates in a random medium, which fills the half-space $z>0$, we may write a formula similar to Eq. (2.1),

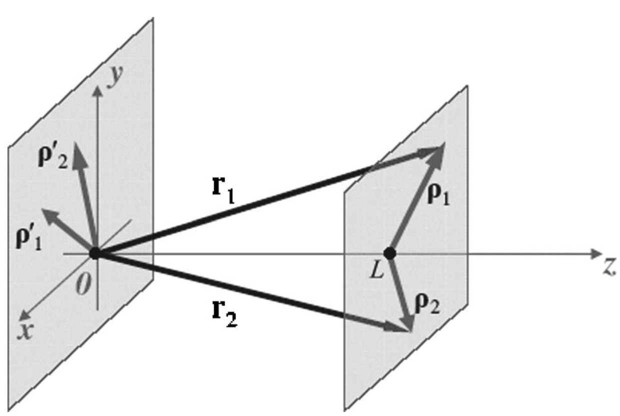

Fig. 1. Illustrating the notation.

$$U(\mathbf{r},\omega)=\int\int a(\mathbf{u},\omega)P_{\mathbf{u}}^T(\mathbf{r},\omega)\mathrm{d}^2u_{\perp}, \tag{2.5}$$

where $P_{\mathbf{u}}^T(\mathbf{r},\omega)$ represents a plane wave distorted by the random medium with direction $\mathbf{u}$ and evaluated at a point $\mathbf{r}$.

The statistical moments of any order of the field at points with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ can then be evaluated from Eq. (2.5). For example, the second moment of the field at a pair of points, also called the cross-spectral density function, is defined as

$$W(\mathbf{r}_1,\mathbf{r}_2,\omega)=\langle U^*(\mathbf{r}_1,\omega)U(\mathbf{r}_2,\omega)\rangle, \tag{2.6}$$

where angular brackets denote the ensemble average in the sense of the coherence theory in the space-frequency domain.[8]

When the field propagates in the medium, the cross-spectral density function then takes the form

$$\begin{aligned}W(\mathbf{r}_1,\mathbf{r}_2,\omega)&=\langle U^*(\mathbf{r}_1,\omega)U(\mathbf{r}_2,\omega)\rangle\\&=\int\int\int\int a^*(\mathbf{u}_1,\omega)a(\mathbf{u}_2,\omega)\\&\quad\times\langle P_{\mathbf{u}_1}^{T*}(\mathbf{r}_1,\omega)P_{\mathbf{u}_2}^{T}(\mathbf{r}_2,\omega)\rangle_T\mathrm{d}^2u_{1\perp}\mathrm{d}^2u_{2\perp}.\end{aligned} \tag{2.7}$$

Here, subscript T denotes the average taken over the realizations of the fluctuating medium.

The beam intensity $I(\mathbf{r})$ at a point with position vector $\mathbf{r}$ can be calculated from the last expression by taking $\mathbf{r}_1\equiv\mathbf{r}_2\equiv\mathbf{r}$, i.e.,

$$I(\mathbf{r})\equiv W(\mathbf{r},\mathbf{r},\omega)=\langle U^*(\mathbf{r},\omega)U(\mathbf{r},\omega)\rangle. \tag{2.8}$$

B. Partially Coherent Fields

Fluctuations in a partially coherent field at a pair of points $\mathbf{r}_1=(\boldsymbol{\rho}_1,z)$ and $\mathbf{r}_2=(\boldsymbol{\rho}_2,z)$ may be described in terms of the cross-spectral density function,

$$W(\mathbf{r}_1,\mathbf{r}_2,\omega)=\langle U^*(\mathbf{r}_1,\omega)U(\mathbf{r}_2,\omega)\rangle, \tag{2.9}$$

points $\mathbf{r}_1$ and $\mathbf{r}_2$ belonging to the positive half-space $z>0$. In vacuum, the cross-spectral density function can be represented in terms of the angular spectrum of the plane waves of the form[9]

$$\begin{aligned}W(\mathbf{r}_1,\mathbf{r}_2,\omega)&=\int\int\int\int A(\mathbf{u}_1,\mathbf{u}_2,\omega)P_{\mathbf{u}_1}^*(\mathbf{r}_1,\omega)P_{\mathbf{u}_2}(\mathbf{r}_2,\omega)\\&\quad\times\mathrm{d}^2u_{1\perp}\mathrm{d}^2u_{2\perp},\end{aligned} \tag{2.10}$$

where

$$A(\mathbf{u}_1,\mathbf{u}_2,\omega)=\langle a^*(\mathbf{u}_1,\omega)a(\mathbf{u}_2,\omega)\rangle \tag{2.11}$$

is the angular correlation function given by the formula

$$\begin{aligned}A(\mathbf{u}_1,\mathbf{u}_2,\omega)&=\frac{1}{(2\pi)^4}\int\int\int\int W^{(0)}(\mathbf{r}_1',\mathbf{r}_2',\omega)\\&\quad\times P_{\mathbf{u}_1}^*(\mathbf{r}_1',\omega)P_{\mathbf{u}_2}(\mathbf{r}_2',\omega)\mathrm{d}^2r_1'\mathrm{d}^2r_2'.\end{aligned} \tag{2.12}$$

When a partially coherent field described by Eq. (2.10) propagates in a random medium, its mutual coherence function is given by the expression

$$W(\mathbf{r}_1,\mathbf{r}_2,\omega)=\int\int\int\int A(\mathbf{u}_1,\mathbf{u}_2,\omega)\times\langle P_{\mathbf{u}_1}^{T*}(\mathbf{r}_1,\omega)P_{\mathbf{u}_2}^{T}(\mathbf{r}_2,\omega)\rangle \mathrm{d}^2u_{1\perp}\mathrm{d}^2u_{2\perp}. \tag{2.13}$$

It is to be noted that this formula incorporates two ensemble averages: the average over the fluctuations of the field in the source plane and the average over the fluctuations of the random medium. It is assumed here that these two averages are statistically independent. Also, it can be seen from Eq. (2.13) that the statistical properties of the field are contained entirely in the function $A(\mathbf{u}_1,\mathbf{u}_2,\omega)$, which is completely independent of the statistical properties of the medium. Hence, once the statistical properties have been calculated for a given medium, the propagation characteristics of a beam of any type may be evaluated in a straightforward manner by carrying out the integral above.

3. SECOND-ORDER RYTOV THEORY FOR TILTED PLANE WAVES

Consider a wide-sense statistically stationary plane-wave field $P_{\mathbf{u}}^T(\mathbf{r},\omega)$ propagating through a weakly scattering random medium. At any point with position vector $\mathbf{r}$ at frequency ω, the field can be represented by a so-called Rytov series (cf. Ref. 17, p. 102),

$$P_{\mathbf{u}}^T(\mathbf{r},\omega)=P_{\mathbf{u}}(\mathbf{r},\omega)\exp[\psi_{\mathbf{u}}^{(1)}(\mathbf{r},\omega)+\psi_{\mathbf{u}}^{(2)}(\mathbf{r},\omega)+\cdots], \tag{3.1}$$

where $P_{\mathbf{u}}(\mathbf{r},\omega)$ is the plane wave in the absence of the medium and $\psi_{\mathbf{u}}^{(1)}(\mathbf{r},\omega)$, $\psi_{\mathbf{u}}^{(2)}(\mathbf{r},\omega)$ are the complex phase perturbations of the first order and of the second order, respectively.

Keeping terms to only second order in the Rytov approximation, we may write the cross-spectral density function $W_{\mathbf{u}_1,\mathbf{u}_2}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ for two plane waves propagating in turbulence as

$$W_{\mathbf{u}_1,\mathbf{u}_2}(\mathbf{r}_1,\mathbf{r}_2,\omega)=P_{\mathbf{u}_1}^*(\mathbf{r}_1,\omega)P_{\mathbf{u}_2}(\mathbf{r}_2,\omega)\langle\exp[\psi_{\mathbf{u}_2}^{(1)*}(\mathbf{r}_1,\omega)+\psi_{\mathbf{u}_1}^{(2)*}(\mathbf{r}_1,\omega)+\psi_{\mathbf{u}_2}^{(1)}(\mathbf{r}_2,\omega)+\psi_{\mathbf{u}_2}^{(2)}(\mathbf{r}_2,\omega)]\rangle. \tag{3.2}$$

This expression can be simplified by using the method of cumulants; keeping terms to second order, we may approximate the average of an exponential function as

$$\langle\exp[\psi]\rangle\approx\exp\left[\langle\psi\rangle+\frac{1}{2}(\langle\psi\rangle^2-\langle\psi\rangle^2)\right]. \tag{3.3}$$

We show in Appendix A that by applying the Rytov approximation together with the method of cumulants, the cross-spectral density may be written in the form

$$W_{\mathbf{u}_1,\mathbf{u}_2}(\mathbf{r}_1,\mathbf{r}_2,\omega)=P_{\mathbf{u}_1}^*(\mathbf{r}_1,\omega)P_{\mathbf{u}_2}(\mathbf{r}_2,\omega)\exp[2E_{\mathbf{u}_1,\mathbf{u}_2}^{(1)}(\mathbf{r}_1,\mathbf{r}_2,\omega)+E_{\mathbf{u}_1,\mathbf{u}_2}^{(2)}(\mathbf{r}_1,\mathbf{r}_2,\omega)], \tag{3.4}$$

where

$$E_{\mathbf{u}_1,\mathbf{u}_2}^{(1)}(\mathbf{r}_1,\mathbf{r}_2,\omega)=-\pi k^2\int_0^L \mathrm{d}z\int\int \mathrm{d}^2\kappa\Phi_n(z,\kappa), \tag{3.5}$$

$$E_{\mathbf{u}_1,\mathbf{u}_2}^{(2)}(\mathbf{r}_1,\mathbf{r}_2,\omega)=2\pi k^2\int_0^L \mathrm{d}z\int\int \mathrm{d}^2\kappa\Phi_n(z,\boldsymbol{\kappa})\times\exp[-i(L-z)(\mathbf{u}_1-\mathbf{u}_2)\cdot\boldsymbol{\kappa}]\times\exp[-i(\mathbf{r}_2-\mathbf{r}_1)\cdot\boldsymbol{\kappa}], \tag{3.6}$$

where $\Phi_n(z,\boldsymbol{\kappa})$ is the power spectrum of atmospheric fluctuations, $\boldsymbol{\kappa}=(\kappa_x,\kappa_y,\kappa_z)$ is the spatial frequency vector. Similar expressions have been derived previously for the special cases of a single, normally incident plane wave, a spherical wave, and a Gaussian beam (Ref. 17, Subsection 5.5.3).

In the case when the atmosphere is isotropic, Eqs. (3.5) and (3.6) reduce to the expressions

$$E_{\mathbf{u}_1,\mathbf{u}_2}^{(1)}(\mathbf{r}_2-\mathbf{r}_1,\omega)=-2\pi^2k^2\int_0^L \mathrm{d}z\int_0^\infty \kappa \mathrm{d}\kappa\Phi_n(z,\kappa), \tag{3.7}$$

$$E_{\mathbf{u}_1,\mathbf{u}_2}^{(2)}(\mathbf{r}_2-\mathbf{r}_1,\omega)=4\pi^2k^2\int_0^L \mathrm{d}z\int_0^\infty \kappa \mathrm{d}\kappa\Phi_n(z,\kappa)\times J_0[\kappa|(\mathbf{r}_{2\perp}-\mathbf{r}_{1\perp})-(L-z)(\mathbf{u}_{2\perp}-\mathbf{u}_{1\perp})|], \tag{3.8}$$

where J_0 is the Bessel function of the first kind of zero order and $\kappa=|\boldsymbol{\kappa}|$.

If the power spectrum of atmospheric fluctuations does not depend on z, e.g., in the case when the beam propagates along the horizontal path, Eqs. (3.7) and (3.8) can be further reduced. The moment $E_{\mathbf{u}_1,\mathbf{u}_2}^{(1)}(\mathbf{r}_2-\mathbf{r}_1,\omega)$ then takes the form

$$E_{\mathbf{u}_1,\mathbf{u}_2}^{(1)}(\mathbf{r}_2-\mathbf{r}_1,\omega)=-2\pi^2k^2L\int_0^\infty \kappa \mathrm{d}\kappa\Phi_n(\kappa). \tag{3.9}$$

The expression for the moment $E_{\mathbf{u}_1,\mathbf{u}_2}^{(2)}(\mathbf{r}_2-\mathbf{r}_1,\omega)$ may be made more analytically tractable by expanding the Bessel function in a familiar series form,

$$J_0[\kappa|(\mathbf{r}_{2\perp}-\mathbf{r}_{1\perp})-z(\mathbf{u}_{2\perp}-\mathbf{u}_{1\perp})|]=\sum_m J_m[\kappa|\mathbf{r}_{2\perp}-\mathbf{r}_{1\perp}|]J_m[\kappa z|\mathbf{u}_{2\perp}-\mathbf{u}_{1\perp}|]e^{im(\phi_r-\phi_u)}, \tag{3.10}$$

where J_m is the Bessel function of the first kind of order m, ϕ_r, and ϕ_u are the angles that the vectors $\mathbf{r}_{2\perp}-\mathbf{r}_{1\perp}$ and $\mathbf{u}_{2\perp}-\mathbf{u}_{1\perp}$ make with the x axis, respectively. We then use the closed form [Ref. 18, Eqs. (11.1.3) and (11.1.4)] for the integral over the propagation path z,

$$\int_0^L \mathrm{d}z J_m[\kappa z|\mathbf{u}_2-\mathbf{u}_1|] = \frac{1}{\kappa|\mathbf{u}_2-\mathbf{u}_1|}\left[\int_0^{\kappa L|\mathbf{u}_2-\mathbf{u}_1|} J_0(t)\mathrm{d}t - 2\sum_{k=0}^{m-1} J_{2k+1}(\kappa L|\mathbf{u}_2-\mathbf{u}_1|)\right] \tag{3.11}$$

if m is even and

$$\int_0^L \mathrm{d}z J_m[\kappa z|\mathbf{u}_2-\mathbf{u}_1|] = \frac{1}{\kappa|\mathbf{u}_2,\mathbf{u}_1|}\left[1 - J_0(\kappa L|\mathbf{u}_2-\mathbf{u}_1|) - 2\sum_{k=0}^{m} J_{2k}(\kappa L|\mathbf{u}_2-\mathbf{u}_1|)\right] \tag{3.12}$$

if m is odd. The integral for $E^{(2)}_{\mathbf{u}_1,\mathbf{u}_2}(\mathbf{r}_2-\mathbf{r}_1,\omega)$ can therefore be written as a sum of a series of one-dimensional integrals in κ.

With these results, the propagation of a general beamlike field may be calculated by substituting from Eq. (3.4) into Eq. (2.13), and one finds that

$$W(\mathbf{r}_1,\mathbf{r}_2,\omega) = \int\int\int\int A(\mathbf{u}_1,\mathbf{u}_2,\omega)P^*_{\mathbf{u}_1}(\mathbf{r}_1,\omega)P_{\mathbf{u}_2}(\mathbf{r}_2,\omega) \times \exp[2E^{(1)}_{\mathbf{u}_1,\mathbf{u}_2}(\mathbf{r}_1,\mathbf{r}_2,\omega) + E^{(2)}_{\mathbf{u}_1,\mathbf{u}_2}(\mathbf{r}_1,\mathbf{r}_2,\omega)] \times \mathrm{d}^2u_{\perp 1}\mathrm{d}^2u_{\perp 2}. \tag{3.13}$$

This formula is the main result of our paper. It is an expression, which can be used to calculate the second-order characteristics of a general partially coherent beam propagating in turbulence.

It is to be noted that the results described here can be extended to calculate the fourth-order characteristics of a partially coherent field propagating in turbulence, as would be needed in studies of intensity fluctuations. The needed turbulence characteristic is the average (over the turbulence) of the product of four plane-wave fields, and it can be shown in a manner similar to that used above that this quantity has the form

$$\langle P^{*T}_{\mathbf{u}_1}(\mathbf{r}_1)P^{T}_{\mathbf{u}_2}(\mathbf{r}_1)P^{*T}_{\mathbf{u}_3}(\mathbf{r}_2)P^{T}_{\mathbf{u}_4}(\mathbf{r}_2)\rangle = P^*_{\mathbf{u}_1}(\mathbf{r}_1)P_{\mathbf{u}_2}(\mathbf{r}_2)P^*_{\mathbf{u}_3}(\mathbf{r}_3)P_{\mathbf{u}_4}(\mathbf{r}_4) \times \exp\Big[4E^{(1)} + E^{(2)}_{1,2}(1,1) + E^{(2)}_{1,4}(1,2) + E^{(2)}_{3,2}(2,1) + E^{(2)}_{3,4}(2,2) + E^{(3)}_{1,3}(1,2) + E^{(3)*}_{2,4}(1,2)\Big], \tag{3.14}$$

where

$$E^{(3)}_{a,b}(1,2) = -2\pi k^2 \int_0^L \mathrm{d}z \int\int \mathrm{d}^2\kappa \Phi_n(z,\kappa) e^{-iz\boldsymbol{\kappa}\cdot(\mathbf{u}_a-\mathbf{u}_b)} \times e^{i\boldsymbol{\kappa}\cdot(\mathbf{r}_1-\mathbf{r}_2)} e^{-iz\kappa^2/k},$$

and $E^{(1)}$ and $E^{(2)}$ are defined as above. These calculations will, in general, be difficult, due to the eightfold angular spectrum integral required as well as known challenges in defining higher-order coherence functions in the space-frequency domain.[19] Fourth-order propagation characteristics will be considered in more detail in future work.

4. EXAMPLES

We illustrate the effectiveness of our theory by considering the propagation of several well-known classes of beams. In all cases, the turbulence is modeled by the von Karman spectrum,[17] given by the relation

$$\Phi_n(\kappa) = 0.33C_n^2 \frac{\exp[-\kappa^2/\kappa_m^2]}{(\kappa^2+\kappa_0^2)^{11/6}}, \tag{4.1}$$

where $C_n^2 = 10^{-14}\,\mathrm{m}^{-2/3}$, $\kappa_m = 5.92/l_m$, with inner scale $l_m = 1\,\mathrm{mm}$, and $\kappa_0 = 1/l_0$, with outer scale $l_0 = 10\,\mathrm{m}$.

A. Coherent Gaussian Beams

A fully coherent Gaussian beam in the source plane is given by the expression

$$U_0(\mathbf{r},\omega) = U_0 \exp\left(-\frac{r^2}{2\sigma^2}\right), \tag{4.2}$$

where U_0 is the amplitude and σ is the width of the beam; it is to be noted that both parameters may, in general, depend on frequency ω. On substituting from Eq. (4.2) into

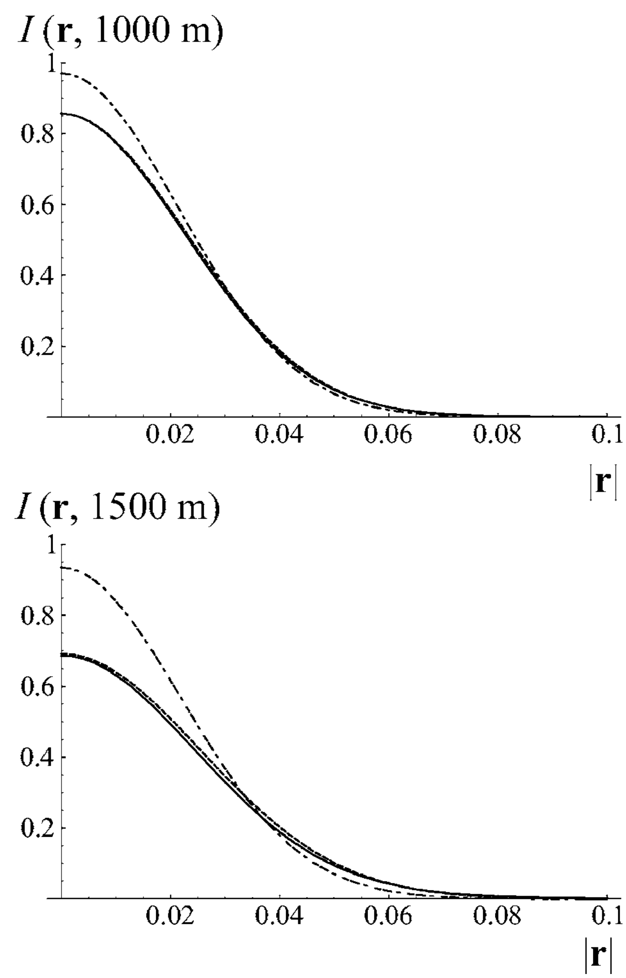

Fig. 2. Intensity profile of a typical coherent Gaussian beam, with $\sigma = 3$ cm, at propagation distances $L = 1000$ m and $L = 1500$ m. The solid curve is the angular spectrum result, while the dashed curve is based on a standard method. The dashed–dotted curve shows the profile of the beam in the absence of turbulence.

Eq. (2.4), the angular spectrum is readily found to be given by the formula

$$a(\mathbf{u},\omega) = \frac{\sigma^2}{2\pi} U_0 \exp\left(-\frac{k^2 u^2 \sigma^2}{2}\right). \tag{4.3}$$

This formula may be substituted into Eq. (2.7) and numerically evaluated to determine the cross-spectral density of the field at any propagation distance L through the atmosphere. Figure 2 shows the radial intensity profile of the Gaussian beam at distances $L=1000$ m and $L=1500$ m from the source plane, with $\sigma=2$ cm. The solid curve is based on our angular spectrum calculation, while the dashed curve is based on a standard approximate method of calculating Gaussian beam propagation.[16] Excellent agreement can be seen between the curves. The dashed–dotted curve shows the profile of the beam in the absence of turbulence.

B. Gaussian Schell-Model Beams

The cross-spectral density function of a Gaussian Schell-model beam (Ref. 9, Subsection 5.6.4) in the source plane is given by

$$W_0(\mathbf{r}_1,\mathbf{r}_2,\omega) = I_0 \exp\left[-\frac{r_1^2 + r_2^2}{2\sigma_I^2}\right] \exp\left[-\frac{(\mathbf{r}_2 + \mathbf{r}_1)^2}{2\sigma_\mu^2}\right], \tag{4.4}$$

where I_0 is the on-axis intensity, σ_I is the rms width of the intensity, and σ_μ is the correlation length of the field. On substituting from Eq. (4.4) into Eq. (2.12), one obtains for the angular correlation function the expression

$$A(\mathbf{u}_1,\mathbf{u}_2,\omega) = \frac{1}{8\pi^2} I_0 \sigma_I^2 \sigma^2 \exp\left[-\frac{k^2(\mathbf{u}_2 + \mathbf{u}_1)^2 \sigma_I^2}{4}\right] \times \exp\left[-\frac{k^2(\mathbf{u}_2^2 - \mathbf{u}_1^2)\sigma^2}{8}\right], \tag{4.5}$$

where

$$\sigma^2 = \frac{1}{2}\left(\frac{1}{4\sigma_I^2} + \frac{1}{2\sigma_\mu^2}\right)^{-1}. \tag{4.6}$$

We may substitute from Eq. (4.5) into Eq. (3.13) and numerically evaluate the resulting integral to determine the cross-spectral density of the field at any propagation distance through the atmosphere. Figure 3 shows the radial intensity profile of a Gaussian Schell-model beam at $L=500$ m and $L=1000$ m from the source plane, with $\sigma_I=2$ cm and $\sigma_\mu=2$ mm. The labeling of the curves is the same as in Fig. 2. It can be seen that there is excellent agreement again between the angular spectrum method and the standard method of calculation. Also, it is to be noted that there is less difference between the beam profile in free space and turbulence as compared to the coherent case, a rough indication that partially coherent beams are less affected by the turbulence.

C. Nondiffracting Beams

In recent years, so-called nondiffracting beams (so named because an ideal example will not spread on propagation; also referred to as Bessel beams) have been studied extensively and hold promise for a number of applications.[9] Among the intriguing properties of such beams is their ability to self-reconstruct after diffraction by a semitransparent obstacle.[11] This property has led a number of authors[12,13] to suggest that nondiffracting beams are less affected by turbulence than Gaussian beams and plane waves.

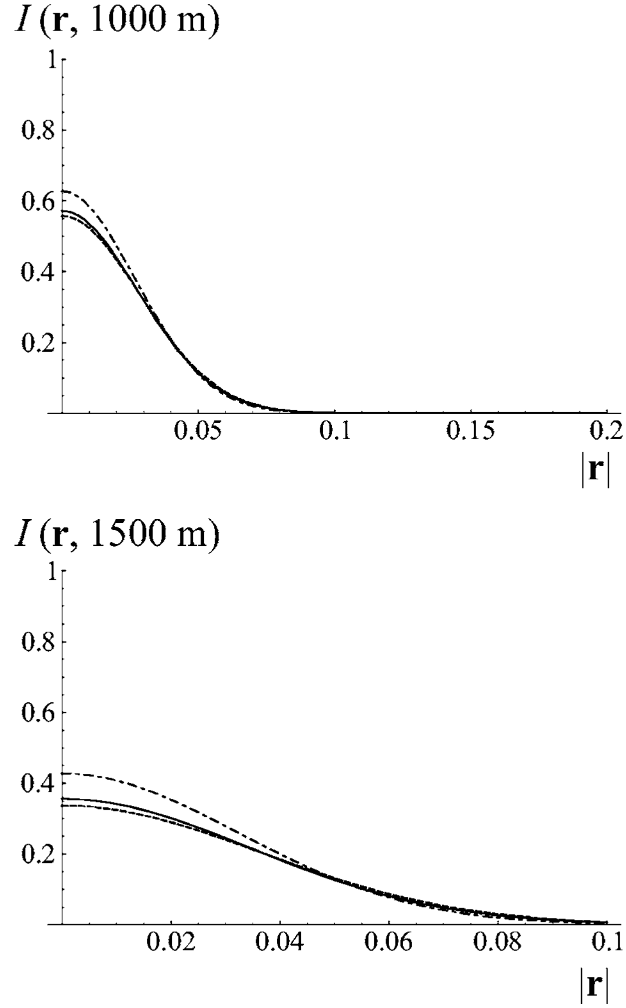

Fig. 3. Intensity profile of a typical Gaussian Schell-model beam, with $\sigma_I=3$ cm and $\sigma_\mu=1$ cm, at propagation distances $L=1000$ m and $L=1500$ m. The solid curve is the angular spectrum result, while the dashed curve is based on a standard method. The dashed–dotted curve shows the profile of the beam in the absence of turbulence.

We have applied the angular spectrum method to study the intensity profile of ideal nondiffracting beams on propagation. The field of such a beam in the source plane is given by

$$U_0(\mathbf{r},\omega) = A_0 J_0(k_\alpha |\mathbf{r}|), \tag{4.7}$$

where J_0 is the Bessel function of the first kind and order zero and $r_0=2.40/k_\alpha$ is the radius of the center lobe of the beam.

The angular spectrum of such a beam can be found using Eq. (2.4) to be given by the formula,

$$a(\mathbf{u},\omega) = A_0 \delta(k|\mathbf{u}| - k_\alpha), \tag{4.8}$$

where δ is a delta function.

Figure 4 shows the radial intensity profile of a beam with $r_0 = 1$ cm for various propagation distances. This beam is much more influenced by turbulence than the coherent Gaussian beam considered earlier; the on-axis intensity of the Gaussian beam is reduced to roughly 70% of its initial value by 1500 m, while the on-axis intensity of the Bessel beam is reduced to roughly 35% by 1500 m.

For a beam with $r_0 = 3$ cm, however, the propagation behavior is significantly improved. Figure 5 shows the radial intensity profile of the beam at various distances. The on-axis intensity of the beam is reduced to only 75% by 1500 m, a significant improvement over the Gaussian case.

This behavior is markedly different from that of a coherent Gaussian beam, for which a wider beam corresponds to a shorter degradation-free path. In contrast, for an ideal Bessel beam, a wider beam corresponds to a longer degradation-free path. Our results suggest that a Bessel beam can show resistance in turbulence, but that this resistance is highly dependent on the beam width.

The results shown here are for an ideal Bessel beam, which is unrealizable in practice due to its infinite energy—it is expected that the performance of a realistic Bessel beam will be limited also by its finite free-space spreading.

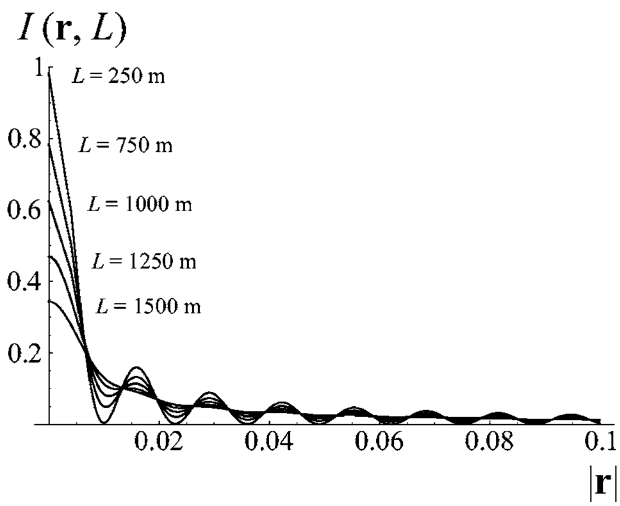

Fig. 4. Intensity profile of a nondiffracting beam, with $r_0 = 1$ cm, for different propagation distances L.

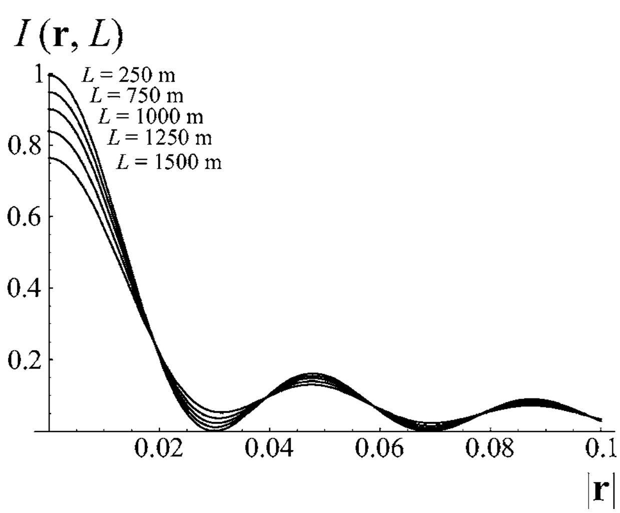

Fig. 5. Intensity profile of a nondiffracting beam, with $r_0 = 3$ cm, for different propagation distances L.

5. CONCLUDING REMARKS

We have developed a technique based on the angular spectrum representation of the fields and on the Rytov approximation for the calculation of the second-order statistical properties of arbitrary coherent and partially coherent beamlike fields. Our method is valid in the regime of weak atmospheric fluctuations; however, it can be, in principle, applied for any model of the atmospheric power spectrum. Moreover, for a chosen model of power spectrum and for the fixed propagation path, the basic quantities of the Rytov theory [i.e., the statistical moments $E^{(1)}$ and $E^{(2)}$ for a pair of plane waves] are calculated only once and can be tabulated. The cross-spectral density of the beam (and, hence, various beam characteristics of interest, e.g., the intensity, the beam spread, the degree of coherence) can then be evaluated for different classes of beams. We have suggested that the same method also works for the calculation of the higher-order moments of the beam, subject to the additional evaluation of the statistical moment $E^{(3)}$ (see Ref. 17, p. 106).

The results presented in this paper might result in the improvement of the quality of the imaging systems and communication systems operating over turbulent channels, as the suggested method provides a fast tool for selecting and/or adjusting a beam source optimal for the chosen atmospheric path.

APPENDIX A: DERIVATION OF $E^{(1)}$ AND $E^{(2)}$ FOR A PAIR OF TILTED PLANE WAVES

In this appendix, we derive the expressions for $E^{(1)}$ and $E^{(2)}$ given in the paper. We first note that the first-order Rytov approximation for a tilted plane wave has been derived previously.[20,21] This expression is given by the formula

$$\phi_{\mathbf{u}}^{(1)}(\mathbf{r},\omega) = -ik\int_0^L \mathrm{d}z \int\int \mathrm{d}^2\kappa g_n(z,\boldsymbol{\kappa})\exp[-i(L-z)\mathbf{u}\cdot\boldsymbol{\kappa}] \times\exp[i\boldsymbol{\kappa}\cdot\mathbf{r}]\exp\left[-\frac{i(L-z)}{2k}\kappa^2\right], \quad \text{(A1)}$$

where $k = 2\pi/\lambda$ is the wavenumber, λ is the wavelength, and $g_n(z,\boldsymbol{\kappa})$ is the two-dimensional Fourier transform of the fluctuating part $n_1(\mathbf{r},z)$ of the refractive index $n(\mathbf{r},z)$, i.e.,

$$g_n(z,\boldsymbol{\kappa}) = \frac{1}{4\pi^2}\int\int n_1(\mathbf{r},z)\exp[-i\boldsymbol{\kappa}\cdot\mathbf{r}]. \quad \text{(A2)}$$

The fluctuating part of the refractive index n_1 is assumed to have zero mean; this implies that

$$\langle\phi_{\mathbf{u}}^{(1)}(\mathbf{r},\omega)\rangle = 0. \quad \text{(A3)}$$

We now apply the method of cumulants to the phase factor in Eq. (3.2), and it is straightforward to show that

$$\langle \exp[\psi_{\mathbf{u}_1}^{(1)*}(\mathbf{r}_1,\omega) + \psi_{\mathbf{u}_1}^{(2)*}(\mathbf{r}_1,\omega) + \psi_{\mathbf{u}_2}^{(1)}(\mathbf{r}_2,\omega) + \psi_{\mathbf{u}_2}^{(2)}(\mathbf{r}_2,\omega)]\rangle = \exp\left[\langle \psi_{\mathbf{u}_1}^{(2)*}(\mathbf{r}_1,\omega) + \psi_{\mathbf{u}_2}^{(2)}(\mathbf{r}_2,\omega)\rangle + \frac{1}{2}\langle (\psi_{\mathbf{u}_1}^{(1)*}(\mathbf{r}_1,\omega) + \psi_{\mathbf{u}_2}^{(1)}(\mathbf{r}_2,\omega))^2\rangle \right], \tag{A4}$$

where only terms of second order in the refractive index have been retained. The terms of this formula may be rearranged into the form

$$\langle \exp[\psi_{\mathbf{u}_1}^{(1)*}(\mathbf{r}_1,\omega) + \psi_{\mathbf{u}_1}^{(2)*}(\mathbf{r}_1,\omega) + \psi_{\mathbf{u}_2}^{(1)}(\mathbf{r}_2,\omega) + \psi_{\mathbf{u}_2}^{(2)}(\mathbf{r}_2,\omega)]\rangle = \exp\left[\left\langle \psi_{\mathbf{u}_1}^{(2)*}(\mathbf{r}_1,\omega) + \frac{1}{2}(\psi_{\mathbf{u}_1}^{(1)*}(\mathbf{r}_1,\omega))^2 + \psi_{\mathbf{u}_2}^{(2)}(\mathbf{r}_2,\omega) + \frac{1}{2}(\psi_{\mathbf{u}_2}^{(1)}(\mathbf{r}_2,\omega))^2 \right\rangle \right] \exp[\langle \psi_{\mathbf{u}_1}^{(1)*}(\mathbf{r}_1,\omega)\psi_{\mathbf{u}_2}^{(1)}(\mathbf{r}_2,\omega)\rangle]. \tag{A5}$$

The average of the right-most term may be determined in a straightforward way, using Eq. (A1) and the definition of the refractive index power spectrum,

$$2\pi\Phi_n(z,\boldsymbol{\kappa})\delta(z-z')\delta^{(2)}(\boldsymbol{\kappa}-\boldsymbol{\kappa}') = \langle g_n^*(z',\boldsymbol{\kappa}')g_n(z,\boldsymbol{\kappa})\rangle, \tag{A6}$$

and takes on the form

$$\langle \psi_{\mathbf{u}_1}^{(1)*}(\mathbf{r}_1,\omega)\psi_{\mathbf{u}_2}^{(1)}(\mathbf{r}_2,\omega)\rangle = 4\pi k^2 \int_0^L \mathrm{d}z \int\int \mathrm{d}^2\kappa \Phi_n(z,\boldsymbol{\kappa}) \times \exp[iz(\mathbf{u}_1-\mathbf{u}_2)\cdot\boldsymbol{\kappa} + i\boldsymbol{\kappa}\cdot(\mathbf{r}_2-\mathbf{r}_1)]. \tag{A7}$$

This expression is exactly $E^{(2)}$, as given in Eq. (3.5). To evaluate the left-most term of Eq. (A5), we need to determine the second-order Rytov approximation for a tilted plane wave. It can be shown that the first- and second-order Rytov approximations, $\psi^{(1)}$ and $\psi^{(2)}$, are related to the first- and second-order Born approximations $\phi^{(1)}$ and $\phi^{(2)}$ by the formulas

$$\psi_{\mathbf{u}}^{(1)}(\mathbf{r},\omega) \equiv \phi_{\mathbf{u}}^{(1)}(\mathbf{r},\omega), \tag{A8}$$

$$\psi_{\mathbf{u}}^{(2)}(\mathbf{r},\omega)\phi_{\mathbf{u}}^{(2)}(\mathbf{r},\omega) - \frac{1}{2}\phi_{\mathbf{u}}^{(1)2}(\mathbf{r},\omega). \tag{A9}$$

This latter equation can then be written as

$$\phi_{\mathbf{u}}^{(2)}(\mathbf{r},\omega) = \psi_{\mathbf{u}}^{(2)}(\mathbf{r},\omega) + \frac{1}{2}\psi_{\mathbf{u}}^{(1)2}(\mathbf{r},\omega), \tag{A10}$$

which is exactly the left-most term of Eq. (A5). The second-order Born approximation can be shown to have the expression

$$\phi_{\mathbf{u}}^{(2)}(\mathbf{r},\omega) = -\frac{ik^3}{2\pi}\int_0^L \mathrm{d}z \int_0^z \mathrm{d}z' \int\int \mathrm{d}^2 s \int\int \mathrm{d}^2\kappa \int\int \mathrm{d}^2\kappa' \times \exp[ik(L-z)]\exp\left[\frac{ik|\mathbf{s}-\mathbf{r}|}{2(L-z)}\right]\exp[iu_z(z-L)] \times \exp[ik\mathbf{u}\cdot(\mathbf{s}-\mathbf{r})] \times \exp[i(\boldsymbol{\kappa}+\boldsymbol{\kappa}')\cdot\mathbf{s}]g_n(z,\boldsymbol{\kappa})g_n(z',\boldsymbol{\kappa}') \times \exp\left[-i(z-z')\mathbf{u}\cdot\boldsymbol{\kappa}'\right]\exp\left[-\frac{-i(z-z')}{2k}\kappa'^2\right]. \tag{A11}$$

The ensemble average of this quantity can be evaluated using the complementary formula for the refractive index power spectrum,

$$2\pi\Phi_n(z,\boldsymbol{\kappa})\delta(z-z')\delta^{(2)}(\boldsymbol{\kappa}+\boldsymbol{\kappa}') = \langle g_n(z',\boldsymbol{\kappa}')g_n(z,\boldsymbol{\kappa})\rangle, \tag{A12}$$

and by evaluating the integral of the spatial variable **s**. We are left with the expression

$$\phi_{\mathbf{u}}^{(2)}(\mathbf{r},\omega) = -4k^2\int_0^L \mathrm{d}z \int_0^\infty \kappa \mathrm{d}\kappa \Phi_n(z,k).$$

We define this expression as $2E^{(1)}$ and we have completed the derivation of Eq. (3.4).

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