Angular spectrum representation for propagation of random electromagnetic beams in a turbulent atmosphere

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The combination of an angular spectrum representation (in the space–frequency domain) and the second-order Rytov perturbation theory is applied for description of the second-order statistical properties of arbitrary (coherent and partially coherent) stochastic electromagnetic beamlike fields that propagate in a turbulent atmosphere. In particular, we derive the expressions for the elements of the cross-spectral density matrix of the beam, from which its spectral, coherence, and polarization properties can be found. We illustrate the method by applying it to the propagation of several electromagnetic model beams through the atmosphere. © 2007 Optical Society of America

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1. INTRODUCTION

In a recent publication [1] a new theory was developed for the calculation of the second-order statistical properties of random *scalar* beams, generated by planar secondary sources with arbitrary spectral density and arbitrary spectral degree of coherence, which propagate in weak atmospheric turbulence. That theory relied on angular spectrum representation [2] for description of sources that generate the beams and on Rytov's perturbation theory [3] for characterization of fluctuations of the refractive index in the atmosphere. The results based on that theory were found to be in good agreement with those calculated from the standard methods known for several classes of beams, e.g., Gaussian beams and Gaussian Schell-model beams [2]. Moreover, the behavior of statistical properties of the other previously untreated types of beams, e.g., Bessel beams, has been obtained.

Recently, a good deal of research was carried out to describe the behavior of stochastic *electromagnetic* beams in free space [4–6] and in random media, such as the atmosphere [7,8] or human tissue [9,10]. For electromagnetic fields, in addition to their spectral and coherence properties, the polarization properties are also of interest. If the electromagnetic beam is monochromatic, then it may be described in terms of the two-dimensional Jones vector and its state of polarization at any point is represented by the polarization ellipse [2]. If the electromagnetic beam has a random nature, then it may be characterized by the 2×2 cross-spectral density matrix [11] and its polarization properties consist of the degree of polarization and the state of polarization of its completely polarized part (i.e., its polarization ellipse) [2]. Especially important results have been recently found relating to the possible changes in all polarization properties of electromagnetic random beams on propagation, even in free space [4,6]. However, the only model beams that have been used for such calculations are the so-called electromagnetic Gaussian Schell-model beams [5].

In the present paper we extend our work in [1] to the electromagnetic domain; i.e., we develop the technique for the description of the (second-order) statistical properties of arbitrary electromagnetic beams that propagate in weak atmospheric turbulence. We first consider monochromatic electromagnetic beams and then random electromagnetic beams.

We also demonstrate the usefulness of the new approach by calculating the polarization properties of several model beam classes on propagation through atmospheric turbulence. In particular, we compare the results based on the new technique with those obtained by the standard technique [3] relating to propagation of the degree of polarization of electromagnetic Gaussian Schell-model beams in the atmosphere [7].

2. ANGULAR SPECTRUM REPRESENTATION OF ELECTROMAGNETIC BEAMLIKE FIELDS PROPAGATING IN WEAK ATMOSPHERIC TURBULENCE

A. Monochromatic Fields

We start by considering a monochromatic electric vector field $\mathbf{E}(\mathbf{r},\omega)=[E_x(\mathbf{r},\omega),E_y(\mathbf{r},\omega)]$ at frequency ω at a point with position vector $\mathbf{r}=(\boldsymbol{\rho},z),\boldsymbol{\rho}=(x,y)$ propagating in vacuum from the plane $z=0$ into the positive half-space $z>0$ (see Fig. 1). Vector $\mathbf{E}(\mathbf{r},\omega)$ is conventionally known as a Jones vector, with $E_x(\mathbf{r},\omega)$ and $E_y(\mathbf{r},\omega)$ being its two mutually orthogonal components. The space-dependent part of each of the two components of the Jones vector can

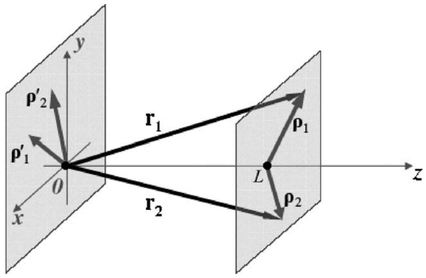

Fig. 1. Illustraton of the notation relating to the propagation of beams.

be expressed in terms of its angular spectrum of plane waves [2]. The vector $\mathbf{a}(\mathbf{u},\omega)=[a_x(\mathbf{u},\omega),a_y(\mathbf{u},\omega)]$ is the electromagnetic generalization of the scalar angular spectrum function, and we have

$$E_i(\mathbf{r},\omega)=\int\int a_i(\mathbf{u},\omega)P_{\mathbf{u}}(\mathbf{r},\omega)\mathrm{d}^2u_{\perp}, \qquad (i=x,y), \tag{2.1}$$

where

$$P_{\mathbf{u}}(\mathbf{r},\omega)=\exp[ik(\mathbf{u}\cdot\mathbf{r})] \tag{2.2}$$

is the plane wave propagating in the direction specified by unit vector $\mathbf{u}=(u_x,u_y,u_z)$, $k=\omega/c$ is the wavenumber, c is the speed of light in vacuum, $\mathbf{u}_{\perp}=(u_x,u_y,0)$, and

$$u_z=+\sqrt{1-u_{\perp}^2}. \tag{2.3}$$

We take the integration in Eq. (2.1) over the region $|u_{\perp}|\leqslant 1$, which is equivalent to assuming that the evanescent waves of the angular spectrum are neglected. The amplitudes of plane waves $a_i(\mathbf{u},\omega)$, $(i=x,y)$ in the angular spectrum [see Eq. (2.1)] can be determined from the field $E_i^{(0)}(\mathbf{r}',\omega)$ in the source plane by the formula

$$a_i(\mathbf{u},\omega)=\frac{1}{(2\pi)^2}\int\int E_i^{(0)}(\mathbf{r}',\omega)P_{\mathbf{u}}^{*}(\mathbf{r}',\omega)\mathrm{d}^2r'_{\perp}, \qquad (i=x,y), \tag{2.4}$$

where $\mathbf{r}'=(\boldsymbol{\rho}',0)$, $\boldsymbol{\rho}'=(x',y')$ is a point in the source plane and the asterisk denotes complex conjugation.

In order to describe one realization of the field that propagates in a random medium filling the half-space $z>0$, we may write a formula similar to Eq. (2.1),

$$E_i(\mathbf{r},\omega)=\int\int a_i(\mathbf{u},\omega)P_{\mathbf{u}}^{T}(\mathbf{r},\omega)\mathrm{d}^2u_{\perp}, \qquad (i=x,y), \tag{2.5}$$

where $P_{\mathbf{u}}^{T}(\mathbf{r},\omega)$ represents a plane wave propagating through the random medium along direction $\mathbf{u}$. It is to be noted that $P^{T}(\mathbf{r},\omega)$ itself is a random function and that its exact behavior not known, but its average properties can be determined, as we shall see.

The statistical moments of any order of the field randomized by the medium in which it propagates can be evaluated from Eq. (2.5). For example, the second moment of the field at a pair of points with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, called the cross-spectral density matrix, is defined as [11]

$$W_{ij}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\langle E_i^{*}(\mathbf{r}_1,\omega)E_j(\mathbf{r}_2,\omega)\rangle_T, \qquad (i,j=x,y), \tag{2.6}$$

where angle brackets denote the ensemble average over the realizations of the medium. For convenience, this matrix will occasionally be referred to by its dyadic form, $\overleftrightarrow{W}(\mathbf{r}_1,\mathbf{r}_2;\omega)$.

On substituting from Eq. (2.6) into (2.5) we find that

$$W_{ij}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\int\int\int\int a_i^{*}(\mathbf{u}_1,\omega)a_j(\mathbf{u}_2,\omega)\times\langle P_{\mathbf{u}_1}^{T*}(\mathbf{r}_1,\omega)P_{\mathbf{u}_2}^{T}(\mathbf{r}_2,\omega)\rangle_T\mathrm{d}^2u_{1\perp}\mathrm{d}^2u_{2\perp}, \qquad (i,j=x,y). \tag{2.7}$$

B. Random Fields

Fluctuations in a partially coherent electromagnetic field at a pair of points $\mathbf{r}_1=(\boldsymbol{\rho}_1,z)$ and $\mathbf{r}_2=(\boldsymbol{\rho}_2,z)$ located in the half-space $z>0$ may be described in terms of the 2×2 cross-spectral density matrix with elements [11]

$$W_{ij}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\langle E_i^{*}(\mathbf{r}_1,\omega)E_j(\mathbf{r}_2,\omega)\rangle, \qquad (i,j=x,y), \tag{2.8}$$

where the ensemble average is taken over the realizations of the random field. In vacuum the elements of the cross-spectral density matrix can be represented in terms of an angular spectrum of plane waves of the form [12]

$$W_{ij}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\int\int\int\int A_{ij}(\mathbf{u}_1,\mathbf{u}_2,\omega)P_{\mathbf{u}_1}^{*}(\mathbf{r}_1,\omega)\times P_{\mathbf{u}_2}(\mathbf{r}_2,\omega)\mathrm{d}^2u_{1\perp}\mathrm{d}^2u_{2\perp}, \qquad (i,j=x,y), \tag{2.9}$$

where

$$A_{ij}(\mathbf{u}_1,\mathbf{u}_2,\omega)=\langle a_i^{*}(\mathbf{u}_1,\omega)a_j(\mathbf{u}_2,\omega)\rangle \tag{2.10}$$

are the elements of the 2×2 angular correlation matrix given by the formula [12]

$$A_{ij}(\mathbf{u}_1,\mathbf{u}_2,\omega)=\frac{1}{(2\pi)^4}\int\int\int\int W_{ij}^{(0)}(\mathbf{r}'_1,\mathbf{r}'_2,\omega)P_{\mathbf{u}_1}^{*}(\mathbf{r}'_1,\omega)\times P_{\mathbf{u}_2}(\mathbf{r}'_2,\omega)\mathrm{d}^2r'_{1\perp}\mathrm{d}^2r'_{2\perp}, \qquad (i,j=x,y). \tag{2.11}$$

Angular brackets in Eq. (2.10) denote averaging over the ensemble of realizations of the field. This matrix is the electromagnetic generalization of the well-known angular correlation function of the scalar coherence theory [2].

When a partially coherent field described by Eq. (2.9) propagates in a random medium, its mutual coherence function is given by the expression

$$W_{ij}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\int\int\int\int A_{ij}(\mathbf{u}_1,\mathbf{u}_2,\omega)\langle P^{T*}_{\mathbf{u}_1}(\mathbf{r}_1,\omega)\times P^{T}_{\mathbf{u}_2}(\mathbf{r}_2,\omega)\rangle_T \mathrm{d}^2u_{1\perp}\mathrm{d}^2u_{2\perp}, \qquad (i,j=x,y). \tag{2.12}$$

This formula incorporates two ensemble averages: the average over the fluctuations of the field in the source plane and the average over the fluctuations of the random medium. It is assumed here that these two averages are statistically independent. The statistical properties of the field are contained entirely in the four angular correlation functions $A_{ij}(\mathbf{u}_1,\mathbf{u}_2,\omega)$, $(i,j=x,y)$, which are completely independent of the statistical properties of the medium. Hence, once the statistical properties have been calculated for a given medium, the propagation characteristics of a beam of any type may be evaluated in a straightforward manner by carrying out four integrals, one for each of the components of the cross-spectral density matrix.

The expression for the pair-correlation function of two tilted plane waves that propagate in the atmosphere was derived in [1] on the basis of Rytov's perturbation theory. When the atmosphere is homogeneous and isotropic and its power spectrum does not depend on propagation path, the cross-spectral density function of the two plane waves propagating in turbulence can be expressed as

$$\langle P^{T*}_{\mathbf{u}_1}(\mathbf{r}_1,\omega)P^{T}_{\mathbf{u}_2}(\mathbf{r}_2,\omega)\rangle=P^{*}_{\mathbf{u}_1}(\mathbf{r}_1,\omega)P_{\mathbf{u}_2}(\mathbf{r}_2,\omega)\times\exp[2H^{(1)}_{\mathbf{u}_1,\mathbf{u}_2}(\omega)+H^{(2)}_{\mathbf{u}_1,\mathbf{u}_2}(\omega)], \tag{2.13}$$

where the statistical moments $H^{(1)}$ and $H^{(2)}$ of the pair of tilted plane waves are given by the formulas

$$H^{(1)}_{\mathbf{u}_1,\mathbf{u}_2}(\omega)=-2\pi^2k^2L\int_0^L \kappa\Phi_n(\kappa,z)\mathrm{d}\kappa, \tag{2.14}$$

$$H^{(2)}_{\mathbf{u}_1,\mathbf{u}_2}(\omega)=4\pi^2k^2\int_0^L \mathrm{d}z\int_0^\infty \kappa\mathrm{d}\kappa\Phi_n(\boldsymbol{\kappa},z)\times J_0[\kappa|(\mathbf{r}_{2\perp}-\mathbf{r}_{1\perp})-(L-z)(\mathbf{u}_{2\perp}-\mathbf{u}_{1\perp})|]. \tag{2.15}$$

Here $\Phi_n(\kappa,z)$ is the power spectrum of atmospheric fluctuations, κ is the spatial frequency, and J_0 is the Bessel function of the first kind of order zero. The integral in Eq. (2.15) can be made computationally tractable using properties of Bessel functions, as discussed in [1].

On substituting from Eqs. (2.13)–(2.15) into Eqs. (2.7) and (2.12), we obtain, respectively, the expressions for the elements of the cross-spectral density matrix of the coherent (monochromatic) and partially coherent electromagnetic beams propagating in the atmosphere.

3. STATISTICAL PROPERTIES OF ELECTROMAGNETIC FIELDS: SPECTRAL DENSITY, DEGREE OF COHERENCE, AND DEGREE AND STATE OF POLARIZATION

A. Monochromatic Fields

In the case when the electromagnetic beam is generated by a monochromatic source, then the only properties of interest in the source plane are the spectral density and the spectral polarization ellipse. Assume that the electric vector field in the source plane is given by the expression

$$\mathbf{E}^{(0)}(\mathbf{r}';\omega)=[E^{(0)}_x(\mathbf{r}';\omega),E^{(0)}_y(\mathbf{r}';\omega)]. \tag{3.1}$$

The spectral density of the beam at a point $\mathbf{r}'$ is defined by the formula

$$S^{(0)}(\mathbf{r}';\omega)=E^{(0)*}_x(\mathbf{r}';\omega)E^{(0)}_x(\mathbf{r}';\omega)+E^{(0)*}_y(\mathbf{r}';\omega)E^{(0)}_y(\mathbf{r}';\omega). \tag{3.2}$$

The state of polarization of the beam at this point is defined by the parameters of polarization ellipse given by the equation

$$\left(\frac{E^{(0)}_x(\mathbf{r}',\omega)}{a_1}\right)^2+\left(\frac{E^{(0)}_y(\mathbf{r}',\omega)}{a_2}\right)^2-2\frac{E^{(0)}_x(\mathbf{r}',\omega)}{a_1}\frac{E^{(0)}_y(\mathbf{r}',\omega)}{a_2}\cos\delta=\sin^2\delta, \tag{3.3}$$

where a_1, a_2 are the amplitudes of the electric field components and $\delta=\delta_2-\delta_1$ is their phase difference. Since the beam is deterministic at the source, the degree of polarization and the degree of coherence are unity in this case.

After propagation through atmospheric turbulence, the monochromatic electromagnetic beam becomes stochastic. Then its spectral density at any point $\mathbf{r}$ within the beam can be found from the expression [11]

$$S(\mathbf{r};\omega)=\mathrm{Tr}[\overleftrightarrow{W}(\mathbf{r},\mathbf{r};\omega)], \tag{3.4}$$

where Tr is the trace of the matrix.

The degree of polarization $\mathcal{P}(\mathbf{r};\omega)$ of the beam is defined by the expression [11]

$$\mathcal{P}(\mathbf{r};\omega)=\sqrt{1-\frac{4\,\mathrm{Det}[\overleftrightarrow{W}(\mathbf{r},\mathbf{r};\omega)]}{\mathrm{Tr}^2[\overleftrightarrow{W}(\mathbf{r},\mathbf{r};\omega)]}}, \tag{3.5}$$

where Det stands for the determinant of the matrix.

The state of polarization of the polarized portion of the beam can be represented in terms of the polarization ellipse

$$C(\mathbf{r};\omega)\epsilon^{(r)2}_x-2\,\mathrm{Re}\,D(\mathbf{r};\omega)\epsilon^{(r)}_x\epsilon^{(r)}_y+B(\mathbf{r};\omega)\epsilon^{(r)2}_y=[\mathrm{Im}\,D(\mathbf{r};\omega)]^2, \tag{3.6}$$

where $\epsilon^{(r)}_x$ and $\epsilon^{(r)}_y$ are time-independent components of the "equivalent monochromatic electric field" at point $\mathbf{r}$ and oscillating with frequency ω. Here quantities $B(\mathbf{r};\omega)$, $C(\mathbf{r};\omega)$, and $D(\mathbf{r};\omega)$ are related to the elements of the cross-spectral density matrix by the formulas [6]

$$B(\mathbf{r};\omega)=\tfrac{1}{2}(W_{xx}-W_{yy}+\sqrt{(W_{xx}-W_{yy})^2+4|W_{xy}|^2}),$$

$$C(\mathbf{r};\omega)=\tfrac{1}{2}(W_{xx}-W_{yy}-\sqrt{(W_{xx}-W_{yy})^2+4\,Re[W_{xy}]^2}),$$

$$D(\mathbf{r};\omega)=W_{xy}, \tag{3.7}$$

where we have dropped the arguments $(\mathbf{r};\omega)$ of the elements of the cross-spectral density matrix for brevity. It will be convenient to characterize the orientation and the shape of the ellipse by the following two quantities: The orientation angle $0\leq\theta<\pi$ defined as the smallest positive angle between the positive x direction and the direction of the major axis of the ellipse is given by the formula

$$\theta(\mathbf{r},\omega)=\frac{1}{2}\arctan\left[\frac{2\,\mathrm{Re}[W_{xy}(\mathbf{r},\omega)]}{W_{xx}(\mathbf{r},\omega)-W_{yy}(\mathbf{r},\omega)}\right]. \tag{3.8}$$

The shape of the polarization ellipse can be determined from the value of the degree of ellipticity $0\leq\epsilon\leq 1$, given by the expression

$$\epsilon(\mathbf{r},\omega)=\frac{A_{minor}}{A_{major}}, \tag{3.9}$$

where A_{minor} and A_{major} are the semiaxes of the ellipse given by

$$A^2_{major/minor}=\tfrac{1}{2}[\sqrt{(W_{xx}-W_{yy})^2+4|W_{xy}|^2}\pm\sqrt{(W_{xx}-W_{yy})^2+4\,\mathrm{Re}[W_{xy}]^2}]. \tag{3.10}$$

The degree of coherence of the beam at a pair of spatial arguments propagating in the atmosphere can be calculated by the formula [11]

$$\eta(\mathbf{r}_1,\mathbf{r}_2;\omega)=\frac{\mathrm{Tr}[\overleftrightarrow{W}(\mathbf{r}_1,\mathbf{r}_2;\omega)]}{\sqrt{\mathrm{Tr}[\overleftrightarrow{W}(\mathbf{r}_1,\mathbf{r}_1;\omega)]}\sqrt{\mathrm{Tr}[\overleftrightarrow{W}(\mathbf{r}_2,\mathbf{r}_2;\omega)]}}. \tag{3.11}$$

B. Random Fields

For the beams generated by random electromagnetic sources, the calculations of the second-order statistical properties in the plane of the source and in the field are the same: They are calculated from the elements of the cross-spectral density matrix; i.e., formulas (3.4)–(3.6) and (3.11) can be used.

Unlike for monochromatic beams, for partially coherent beams the degree of polarization and the state of polarization of its fully polarized part should be calculated both in the source plane and in the field, since, as was recently shown [7,8], these properties of the beam can generally change on propagation.

4. EXAMPLES

In order to illustrate the usefulness of our method, we will consider several examples relating to the behavior of model random electromagnetic fields propagating in an isotropic and homogeneous atmosphere. The power spectrum of atmospheric turbulence is taken to be von Karman, i.e.,

$$\Phi_n(\kappa)=0.033C_n^2\frac{\exp(-\kappa^2/\kappa_m^2)}{(\kappa^2+\kappa_0^2)^{11/6}},$$

where $C_n^2=10^{-14}\ \mathrm{m}^{-2/3}$, $\kappa_m=5.92/l_0$, and $\kappa_0=1/L_0$ with inner scale $l_0=5$ mm and outer scale $L_0=10$ m.

A. Two Tilted Correlated Plane Waves

Let us first consider the field that consists of two mutually orthogonal infinite polychromatic plane waves $E_1(\omega)=e^{i\mathbf{k}_1\cdot\mathbf{r}}$ and $E_2(\omega)=e^{i\mathbf{k}_2\cdot\mathbf{r}}$, propagating along directions $\mathbf{k}_1$ and $\mathbf{k}_2$, respectively, i.e., the field of the form

$$\mathbf{E}(\omega)=E_1(\omega)\bar{x}+E_2(\omega)\bar{y}, \tag{4.1}$$

where $\bar{x}$ and $\bar{y}$ are unit vectors in the Cartesian coordinate system.

Suppose that the average intensities of the plane waves are $\langle|E_1(\omega)|\rangle^2=I_1(\omega)$, $\langle|E_2(\omega)|\rangle^2=I_2(\omega)$, respectively, and that the plane waves are mutually correlated, i.e., $\langle E_1^*(\omega)E_2(\omega)\rangle=\mu(\omega)\sqrt{I_1(\omega)I_2(\omega)}$, where $0\leq|\mu(\omega)|\leq 1$ is the correlation coefficient.

On substituting from Eq. (4.1) into Eq. (2.6), we find that the cross-spectral density matrix of this model field at any two points in space is given by the expression

$$\overleftrightarrow{W}^{(0)}(\mathbf{r}_1,\mathbf{r}_2;\omega)=\begin{bmatrix} I_1(\omega)e^{i\mathbf{k}_1\cdot(\mathbf{r}_2-\mathbf{r}_1)} & \mu(\omega)\sqrt{I_1(\omega)I_2(\omega)}e^{i(\mathbf{k}_2\cdot\mathbf{r}_2-\mathbf{k}_1\cdot\mathbf{r}_1)} \\ \mu^*(\omega)\sqrt{I_1(\omega)I_2(\omega)}e^{i(\mathbf{k}_1\cdot\mathbf{r}_2-\mathbf{k}_2\cdot\mathbf{r}_1)} & I_2(\omega)e^{i\mathbf{k}_2\cdot(\mathbf{r}_2-\mathbf{r}_1)} \end{bmatrix}. \tag{4.2}$$

On substituting the elements of the matrix (4.2) into Eqs. (3.5), (3.8), and (3.9), we find that the polarization properties of this field are given by the formulas

$$\mathcal{P}^{(0)}(\omega)=\frac{\sqrt{(I_1(\omega)-I_2(\omega))^2+4|\mu(\omega)|^2I_1(\omega)I_2(\omega)}}{I_1(\omega)+I_2(\omega)}, \tag{4.3}$$

$$\theta^{(0)}(\omega)=\frac{1}{2}\arctan\left[\frac{2\,\mathrm{Re}[\mu(\omega)\sqrt{I_1(\omega)}\sqrt{I_2(\omega)}]}{I_1(\omega)-I_2(\omega)}\right], \tag{4.4}$$

$$\epsilon^{(0)}(\omega)=\frac{\sqrt{(I_1-I_2)^2+4|\mu(\omega)|^2I_1(\omega)I_2(\omega)}-\sqrt{(I_1-I_2)^2+4\,\mathrm{Re}[\mu(\omega)]^2I_1(\omega)I_2(\omega)}}{\sqrt{(I_1-I_2)^2+4|\mu(\omega)|^2I_1(\omega)I_2(\omega)}+\sqrt{(I_1-I_2)^2+4\,\mathrm{Re}[\mu(\omega)]^2I_1(\omega)I_2(\omega)}}. \tag{4.5}$$

When the field propagates through the turbulent atmosphere, its cross-spectral density matrix has the form [see Eq. (2.9)]

$$\overleftrightarrow{W}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\begin{bmatrix} I_1 e^{i\mathbf{k}_1\cdot(\mathbf{r}_2-\mathbf{r}_1)}\langle P_1^T(\mathbf{r}_2)P_1^{T*}(\mathbf{r}_1)\rangle & \mu\sqrt{I_1 I_2}e^{i(\mathbf{k}_2\cdot\mathbf{r}_2-\mathbf{k}_1\cdot\mathbf{r}_1)}\langle P_2^T(\mathbf{r}_2)P_1^{T*}(\mathbf{r}_1)\rangle \\ \mu^*\sqrt{I_1 I_2}e^{i(\mathbf{k}_1\cdot\mathbf{r}_2-\mathbf{k}_2\cdot\mathbf{r}_1)}\langle P_1^T(\mathbf{r}_2)P_2^{T*}(\mathbf{r}_1)\rangle & I_2 e^{i\mathbf{k}_2(\mathbf{r}_2-\mathbf{r}_1)}\langle P_2^T(\mathbf{r}_2)P_2^{T*}(\mathbf{r}_1)\rangle \end{bmatrix}, \tag{4.6}$$

where we have dropped ω dependence on the right-hand side and

$$\langle P_1^T(\mathbf{r}_2)P_1^{T*}(\mathbf{r}_1)\rangle = P_1^*(\mathbf{r}_1)P_1(\mathbf{r}_2)\exp[2H_{11}^{(1)}+H_{11}^{(2)}],$$

$$\langle P_1^{T*}(\mathbf{r}_1)P_2^{T*}(\mathbf{r}_2)\rangle = P_1^*(\mathbf{r}_1)P_2(\mathbf{r}_2)\exp[2H_{12}^{(1)}+H_{12}^{(2)}],$$

$$\langle P_2^{T*}(\mathbf{r}_1)P_1^{T*}(\mathbf{r}_2)\rangle = P_2^*(\mathbf{r}_1)P_1(\mathbf{r}_2)\exp[2H_{21}^{(1)}+H_{21}^{(2)}],$$

$$\langle P_2^{T*}(\mathbf{r}_1)P_2^{T}(\mathbf{r}_2)\rangle = P_2^*(\mathbf{r}_1)P_2(\mathbf{r}_2)\exp[2H_{22}^{(1)}+H_{22}^{(2)}]. \tag{4.7}$$

Here subscripts 1 and 2 of the statistical moments $H^{(1)}$ and $H^{(2)}$ refer to the plane waves $\mathbf{E}_1(\omega)$ and $\mathbf{E}_2(\omega)$, respectively.

For the equal spatial arguments, i.e., in the case $\mathbf{r}_1=\mathbf{r}_2\equiv\mathbf{r}$, the cross-spectral density matrix (4.6) reduces to the form

$$\overleftrightarrow{W}(\mathbf{r},\mathbf{r},\omega)=\begin{bmatrix} I_1 & \mu\sqrt{I_1 I_2}e^{i(\mathbf{k}_2-\mathbf{k}_1)\cdot\mathbf{r}}\exp[2H_{12}^{(1)}+H_{12}^{(2)}] \\ \mu^*\sqrt{I_1 I_2}e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}}\exp[2H_{12}^{(1)}+H_{12}^{(2)}] & I_2 \end{bmatrix}, \tag{4.8}$$

where we omitted the ω dependence on the right-hand side again.

On substituting from Eq. (4.8) into Eqs. (3.5), (3.8), and (3.9), we obtain the following expressions for the polarization properties of the field propagating through the atmosphere:

$$\mathcal{P}(\omega)=\frac{\sqrt{(I_1-I_2)^2+4|\mu|^2 I_1 I_2\exp[4H_{12}^{(1)}+2H_{12}^{(2)}]}}{I_1+I_2}, \tag{4.9}$$

$$\theta(\omega)=\frac{1}{2}\arctan\left[\frac{2\,Re[\mu\sqrt{I_1}\sqrt{I_2}\exp[4H_{12}^{(1)}+2H_{12}^{(2)}]]}{I_1-I_2}\right], \tag{4.10}$$

$$\epsilon(\omega)=\frac{\sqrt{(I_1-I_2)^2+4|\mu\exp[4H_{12}^{(1)}+2H_{12}^{(2)}]|^2 I_1 I_2}-\sqrt{(I_1-I_2)^2+4\,\mathrm{Re}[\mu\exp[4H_{12}^{(1)}+2H_{12}^{(2)}]]^2 I_1 I_2}}{\sqrt{(I_1-I_2)^2+4|\mu\exp[4H_{12}^{(1)}+2H_{12}^{(2)}]|^2 I_1 I_2}+\sqrt{(I_1-I_2)^2+4\,\mathrm{Re}[\mu\exp[4H_{12}^{(1)}+2H_{12}^{(2)}]]^2 I_1 I_2}}. \tag{4.11}$$

In particular, if the intensities of the two plane waves are the same, i.e., if $I_1(\omega)=I_2(\omega)$, we then have

$$\mathcal{P}(\omega)=2|\mu(\omega)|\exp[2H_{12}^{(1)}+H_{12}^{(2)}]. \tag{4.12}$$

Figure 2 shows the behavior of the degree of polarization of two plane waves of equal intensity as the distance z from the source increases. It can be seen that, regardless of the initial degree of polarization of the two-plane-wave system, the degree of polarization decreases continuously to zero on propagation. Furthermore, the rate of decrease grows as the directional separation of the plane waves Δk increases. This is physically reasonable, as plane waves propagating in different directions "see" different realizations of the turbulence.

As can be seen from Eq. (4.10), the angle of orientation is trivially constant for plane waves of equal intensity; similarly, it can be seen from Eq. (4.11) that the ellipticity will be identically zero for a real-valued degree of coherence μ. Figure 3 shows the behavior of the degree of polarization, the angle of orientation, and the ellipticity of a pair of plane waves of unequal intensity and complex degree of coherence. All three behaviors can be explained by considering the limiting form of the cross-spectral density matrix on propagation,

$$\lim_{L\to\infty}\overleftrightarrow{W}(\mathbf{r},\mathbf{r},\omega)=\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}. \tag{4.13}$$

With $I_1=1$ and $I_2=2$, this matrix can be decomposed into an unpolarized part (diagonal matrix) and a part predominantly polarized along the y axis. This leads to a nonzero degree of polarization and a definite angle of orientation for the polarized part of the field. With $\mu=0.5\exp[i\pi/4]$, the polarized part of the field starts with a slight degree of ellipticity, but as the coherence between the plane waves is degraded, the field evolves into a linearly polarized state ($\epsilon=0$).

B. Electromagnetic Gaussian Schell-Model Beam

Recently, the so-called electromagnetic Gaussian Schell-model beam was employed [7] in order to demonstrate the complicated behavior of the degree of polarization of random beams propagating in the atmosphere. The analysis in that paper was carried out with the help of the ex-

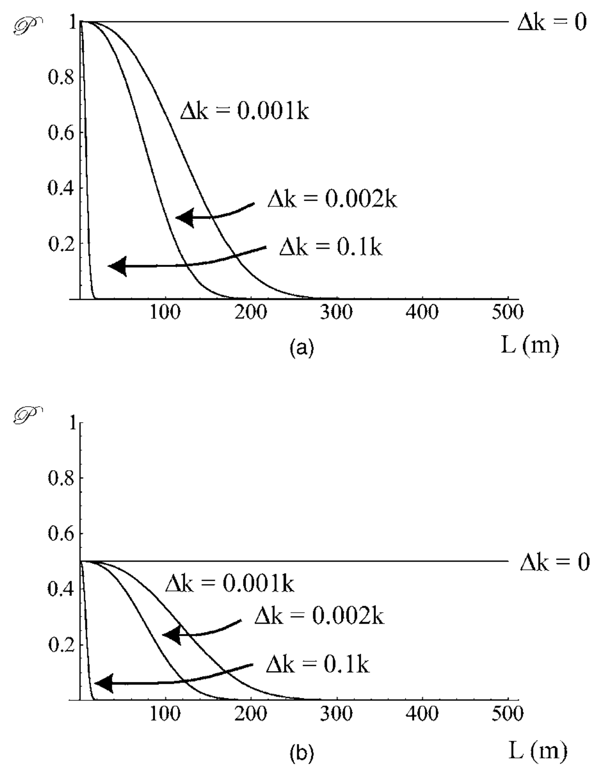

Fig. 2. Evolution of the degree of polarization for a pair of tilted, partially correlated plane waves of equal intensity. (a) $\mu=1$; (b) $\mu=0.5$. Here $\lambda=1$ μm, $C_n^2=10^{-14}$ m$^{-2/3}$, and $\Delta k=k|\mathbf{u}_2-\mathbf{u}_1|$.

tended Huygens–Fresnel integral. We will now show that this phenomenon can also be illustrated with the help of our approach, i.e., on the basis of Rytov's perturbation technique.

We consider for simplicity an electromagnetic Gaussian Schell-model beam ([6], Section 5.6.4) whose cross-spectral density matrix in the source plane has diagonal form, viz.,

$$W_{ij}^{(0)}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \begin{cases} I_i \exp\left[-\dfrac{r_1^2+r_2^2}{2\sigma_I^2}\right]\exp\left[-\dfrac{(\mathbf{r}_2-\mathbf{r}_1)^2}{2\delta_{ii}^2}\right], & i=j \\ 0, & i\neq j \end{cases}, \tag{4.14}$$

where I_i is the ith component of the on-axis intensity, σ_I is the rms. width of the intensity, and δ_{ii} is the correlation length of the ith component of the field. On substituting from Eq. (4.14) into Eq. (2.11), one obtains for the angular correlation functions the expression

$$A_{ii}(\mathbf{u}_1,\mathbf{u}_2,\omega) = \frac{1}{8\pi^2} I_i \sigma_I^2 \sigma_{ii}^2 \exp\left[-\frac{k^2(\mathbf{u}_2-\mathbf{u}_1)^2\sigma_I^2}{4}\right] \times \exp\left[-\frac{k^2(\mathbf{u}_2^2+\mathbf{u}_1^2)\sigma_{ii}^2}{8}\right], \tag{4.15}$$

where

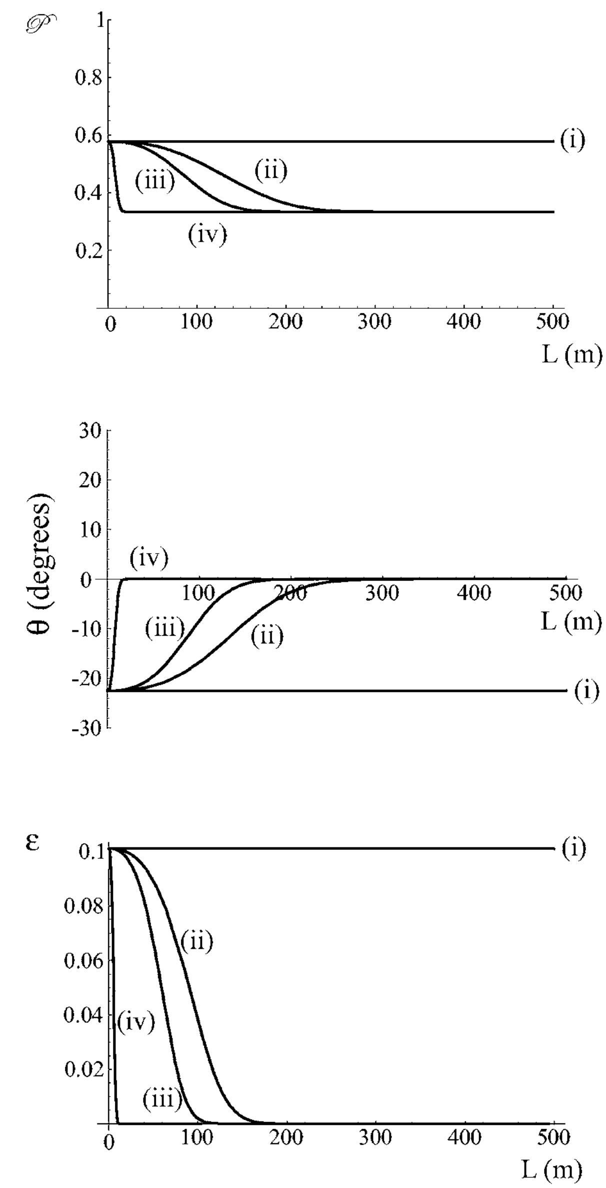

Fig. 3. Evolution of the degree of polarization for a pair of tilted, partially correlated plane waves with $I_1=1$, $I_2=2$, and $\mu=0.5\exp[i\pi/4]$. Here (i) represents $\Delta k=0$, (ii) represents $\Delta k=0.001k$, (iii) represents $\Delta k=0.002k$, and (iv) represents $\Delta k=0.1k$. All other parameters are as in Fig. 2.

$$\sigma_{ii}^2 = \frac{1}{2}\left(\frac{1}{4\sigma_I^2}+\frac{1}{2\delta_{ii}^2}\right)^{-1}. \tag{4.16}$$

We now can substitute from Eqs. (4.15) and (4.16) into Eqs. (2.12)–(2.15) and numerically evaluate the resulting integrals to determine the elements of the cross-spectral density matrix of the field propagating in the atmosphere at a point $\mathbf{r}$, i.e., $\overleftrightarrow{W}(\mathbf{r},\mathbf{r},\omega)$. Then, using Eq. (3.5), we find the degree of polarization of the beam at that point.

Figure 4 shows the behavior of the degree of polarization along the optical axis of several typical electromagnetic Gaussian Schell model beams, generated by the source with cross-spectral density matrix given by Eq. (4.14). Comparison shows that the results pertaining to

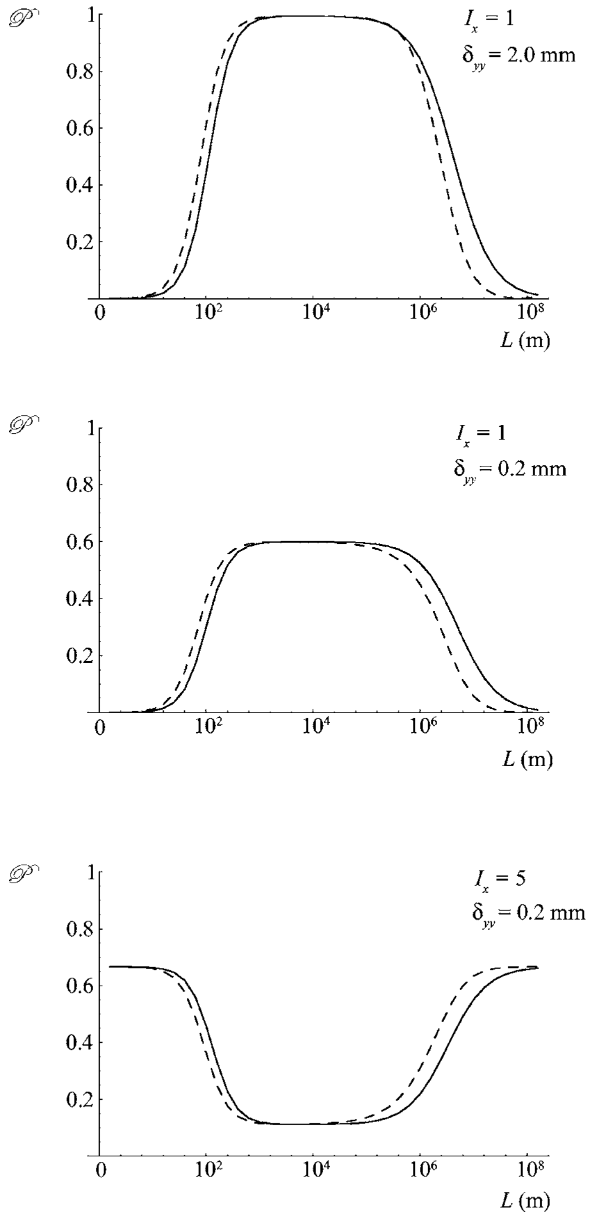

Fig. 4. On-axis degree of polarization of a Gaussian Schell-model beam as a function of propagation distance L for different values of spatial coherence and source degree of polarization. In all cases, $I_y=1$, $\delta_{xx}=0.1$ mm, $\sigma_I=5$ cm, and $C_n^2=10^{-14}$ m$^{-2/3}$.

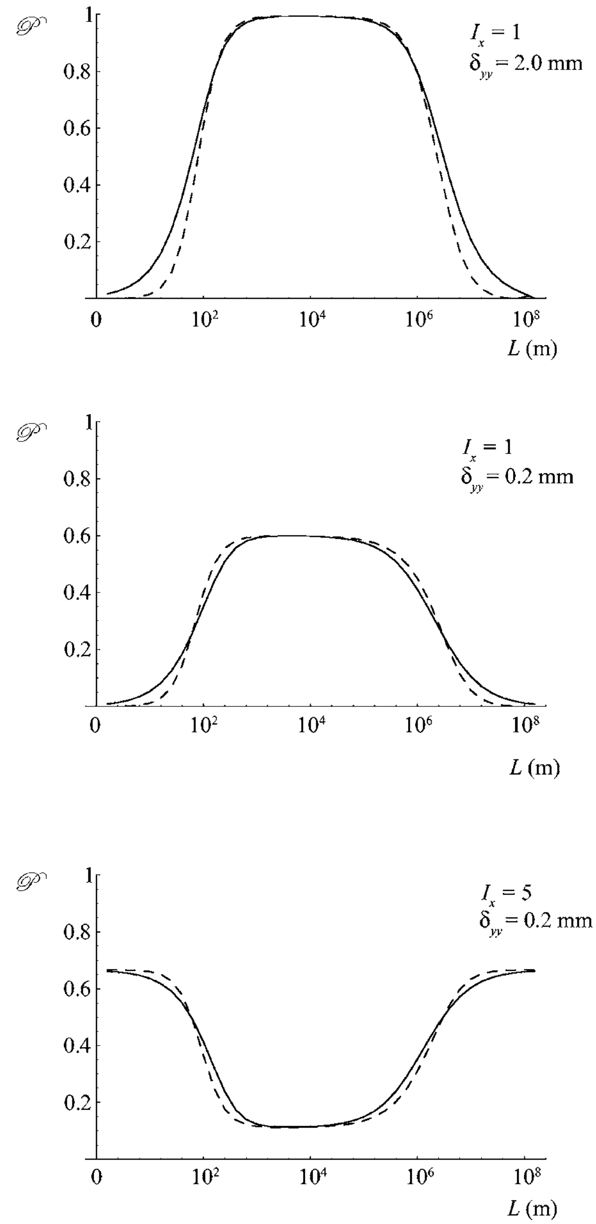

Fig. 5. On-axis degree of polarization of an exponentially correlated beam as a function of propagation distance L for different values of spatial coherence and source degree of polarization. In all cases, $I_y=1$, $\delta_{xx}=0.1$ mm, $\sigma_I=5$ cm, and $C_n^2=10^{-14}$ m$^{-2/3}$. The dashed curve indicates the results for a Gaussian Schell-model beam with the same parameters.

our new method (solid curves) and the extended Huygens–Fresnel integral (dashed curves) (see [7]) are in very good agreement.

C. Electromagnetic Exponentially Correlated Beam and Mixed-Correlation Beam

As a final example, we consider a field with Gaussian intensity profile and exponential correlation,

$$W_{ij}^{(0)}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \begin{cases} I_i \exp\left[-\dfrac{r_1^2+r_2^2}{2\sigma_I^2}\right]\exp\left[-\dfrac{|\mathbf{r}_2-\mathbf{r}_1|}{2\delta_{ii}}\right], & i=j \\ 0, & i \neq j \end{cases} . \tag{4.17}$$

Figure 5 shows the evolution of the degree of polarization for several typical configurations; the dashed curve indicates the result for a Gaussian Schell-model beam with the same parameters. It can be seen that the qualitative behavior of the fields is the same; however, the quantitative behavior in the transition regions ($L \sim 10^2$ m and $L \sim 10^7$ m) is appreciably different. The first of these regions results from the different free-space diffraction behaviors of the x and y components of the field, while the second of these regions results from the influence of turbulence on the different field components. It is clear that the behavior in both of these regions will depend on the specific forms of the correlation function.

We may also consider a "mixed" beam, for which the x component of the field is the Gaussian Schell-model and the y component of the field is exponentially correlated.

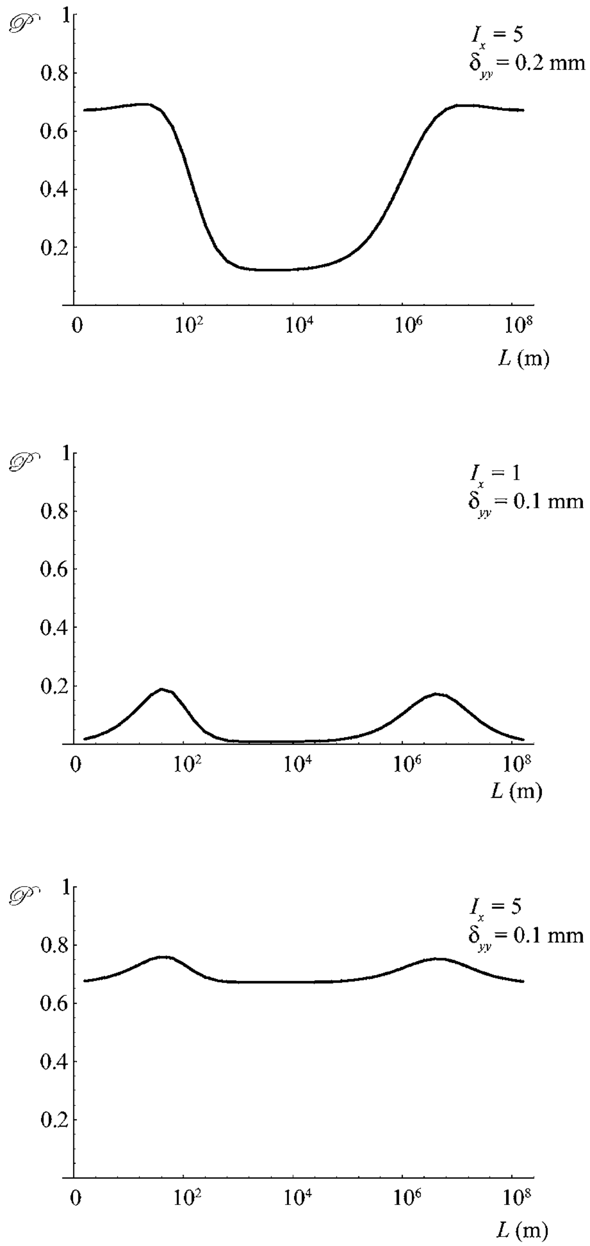

Fig. 6. On-axis degree of polarization of a beam of mixed-correlation type as a function of propagation distance L for different values of spatial coherence and source degree of polarization. In all cases, $I_y=1$, $\delta_{xx}=0.1$ mm, $\sigma_I=5$ cm, and $C_n^2=10^{-14}$ m$^{-2/3}$.

The results are shown in Fig. 6. The middle and bottom plots illustrate that even when the length scales of the x and y components of the field are the same (i.e., $\delta_{xx}=\delta_{yy}$), one can still see significant polarization changes in the transition regions. These changes can be attributed to the functional differences between the correlation functions, rather than to the widths of the correlation functions.

5. CONCLUSIONS

In this paper we have developed an angular spectrum technique for the propagation of general partially coherent electromagnetic beams in weak atmospheric turbulence. We have demonstrated how various polarization properties of the electromagnetic field, the degree of polarization, the angle of orientation of the polarization ellipse, and the ellipticity of the polarization ellipse may be calculated once the effect of turbulence on a pair of partially correlated plane waves is known.

We note here that our results do not contradict well-established estimates relating to depolarization of beams propagating in atmospheric turbulence due to refractive index irregularities (see [13–15]). In those works the analysis was made for fields generated by completely coherent sources, and it was shown that in this case the depolarization is negligible. Our theory also predicts practically no changes in the polarization properties in the limiting case of a fully coherent field.

This technique has been applied to study the behavior of a pair of correlated plane waves, as well as Gaussian Schell-model beams, exponentially correlated beams, and mixed-correlation beams. The Gaussian Schell-model results agree well with results calculated using well-established methods.

It is expected that this angular spectrum technique will prove useful in the study of new and nontrivial beam classes and their propagation in turbulent atmosphere.

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