Complete transverse representation
of a correlation singularity
# of a partially coherent field

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An understanding of correlation singularities is fundamentally important in imaging science. Until now spatial coherence studies have examined a two-dimensional projection of the four-dimensional correlation function, finding so-called correlation vortices or correlation ring dislocations in a given transverse plane. Here we describe the properties and symmetries of the full four-dimensional correlation function. The general solution is found to be a hyperbola in two reduced dimensions. For perfect coherence this reduces to crossed straight lines, whereas in the incoherent limit it reduces to parallel lines. These results elucidate a number of previous experimental and theoretical observations regarding correlation singularities and suggest other behaviors of such singularities. © 2008 Optical Society of America

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1. INTRODUCTION

In recent years there has been increasing interest in the study of phase singularities of coherent wave functions in optics, Bose–Einstein condensates, and other wave systems. These singularities are characterized by lines or surfaces where the field amplitude vanishes and the phase is therefore undefined [1,2]. It has been shown that the structure of the field around such points follows well-defined and consistent behaviors analogous to crystal dislocations [3]. The study of such "singular behavior" is now a subfield of optics, known as *singular optics* [4]. In scalar fields, these singularities may manifest in the form of lines of zero amplitude in three-dimensional space, around which the phase has a circulating or helical structure, and they are consequently referred to as *optical vortices*.

Beyond purely topological interests, studies in singular optics present a new way to look at wave fields and have led to a number of actual and prospective applications. Among these are the use of vortex fields in pattern recognition [5], optical spanners and tweezers [6], spatial coherence filtering [7], temporal coherence filtering [8], the development of an optical vortex coronagraph for astronomical planet-finding [9,10], phase contrast microscopy [11], and the use of vortices as information carriers in free-space optical communication [12]. The properties of vortices are also of paramount importance in studies of phase retrieval [13–15].

Most of these applications of singular optics necessarily involve the use of partially coherent light fields; for instance, light coming from a distant star is partially coherent. In recent years, a number of authors have investigated the phase singularities of the correlation functions of wave fields [16–20], and connections have been made between these phase singularities and the coherence properties of the light field. For instance, the correlation function may be used to characterize the coherence of a beam of light: the area enclosed by a line of zero correlation has been shown to be a measure of the coherence area [18]. Inside (outside) this region the field is positively (negatively) correlated. An understanding of correlation singularities is therefore fundamentally important in imaging science where partially coherent illumination is often desirable. Other theoretical [21] and experimental [22] work has been reported on correlation singularities, including a study of the propagation of the singularities [23] and the development of coherence conservation laws [24].

Intriguingly, it has been shown that there is a connection between the optical vortices created by a linear optical system and the correlation singularities produced by the same system [19,25,26]. A good physical understanding of this connection is still lacking, due at least in part to the complicated nature of correlation singularities.

Because the correlation function is a measure of the statistical correlations between two arbitrary points in a field, a complete description of the function in a plane requires the specification of the correlation function in four variables (six if one includes the propagation of the field). Owing to experimental considerations, previous work on correlation singularities has examined the projections of this function onto a two-variable manifold. For example, a wavefront folding interferometer reveals a correlation ring dislocation [18] whereas a Mach–Zehnder interferometer reveals correlation vortices [26]. Other experimental configurations are possible, and hence, a full four-

dimensional description of correlation singularities is required. To our knowledge an analysis of the full un-projected transverse correlation function has not yet been reported.

In this paper we undertake such a theoretical description, using a simple model of a correlation singularity. In Section 2 we introduce some mathematical preliminaries concerning phase singularities and correlation singularities. In Section 3 we describe the geometric form of correlation singularities in the full four-variable correlation space. In Section 4 we look at selected projections of these correlation singularities, and Section 5 presents some concluding remarks. In an appendix we consider the simpler case of a one-dimensional partially coherent source (with a two-variable correlation function) to identify some of the main features of correlation singularities.

2. PHASE SINGULARITIES AND CORRELATION SINGULARITIES

A complex-valued, quasi-monochromatic scalar wave field $U(\mathbf{r}, t)$ of frequency ω_0 may be expressed in terms of the product of its space-dependent part, $U(\mathbf{r}, \omega_0)$, and a complex exponential time dependence, $\exp[-i\omega_0 t]$. Although ω_0 is simply a parameter in the spatial dependence of a monochromatic field, we write it as a functional argument in anticipation of the discussion of partially coherent fields to follow.

By use of the Madelung transform the spatially dependent factor can be separated further into the product of a complex exponential containing the spatial phase $\varphi(\mathbf{r}, \omega_0)$ of the field, and a real-valued amplitude $A(\mathbf{r}, \omega_0) = \sqrt{I(\mathbf{r}, \omega_0)}$, where $I(\mathbf{r}, \omega_0)$ is the instantaneous spectral intensity:

$$U(\mathbf{r}, \omega_0) = A(\mathbf{r}, \omega_0) \exp[i\varphi(\mathbf{r}, \omega_0)]. \qquad (1)$$

This factorization is well-defined at all spatial points except the origin where $A(\mathbf{r}, \omega_0) = 0$. At such points, the defi-

nition of the phase is ambiguous or *singular*. Typically the region of zero amplitude manifests itself as a line, about which the phase takes on a circulating or helical structure, commonly referred to as an *optical vortex*. Phase singularities such as vortices have long been recognized as loci of zero amplitude in coherent wave functions [27,28]. Later it was recognized that vortices occur not only as modes of wave equations but also as zeros in coherent speckle fields [3,29].

An example of a field containing a vortex mode is a Laguerre–Gauss beam [30,31] of order $n = 0$, $m = 1$. Isophase lines and the amplitude profile of the cross-section of the beam are illustrated in Fig. 1. The beam contains an amplitude zero at the origin, $(x, y) = (0, 0)$, for all propagation distances, z, giving the beam a characteristic "doughnut-like" profile. The lines of constant phase are rays that radiate from the origin, meeting at the central zero of amplitude. Following a counterclockwise closed path around the singularity, the phase increases continuously by 2π.

Although amplitude zeros of a monochromatic wave field are common, it has been shown that intensity zeros of a time averaged partially coherent wave field are rare [25]. However, zeros of the two-point correlation functions of partially coherent wave fields do exist, and the phase of the correlation function is well defined.

In the time domain, the mutual coherence function of a statistically stationary, fluctuating wave field can be written in the form [[32], Section 4.3]

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = \langle U^*(\mathbf{r}_1, t) U(\mathbf{r}_2, t + \tau) \rangle, \qquad (2)$$

where the angle brackets represent a time average or, equivalently, an ensemble average.

It will be more convenient for our purposes to work with the temporal Fourier transform of the mutual coherence function, known as the cross-spectral density [[32], Section 4.3.2]:
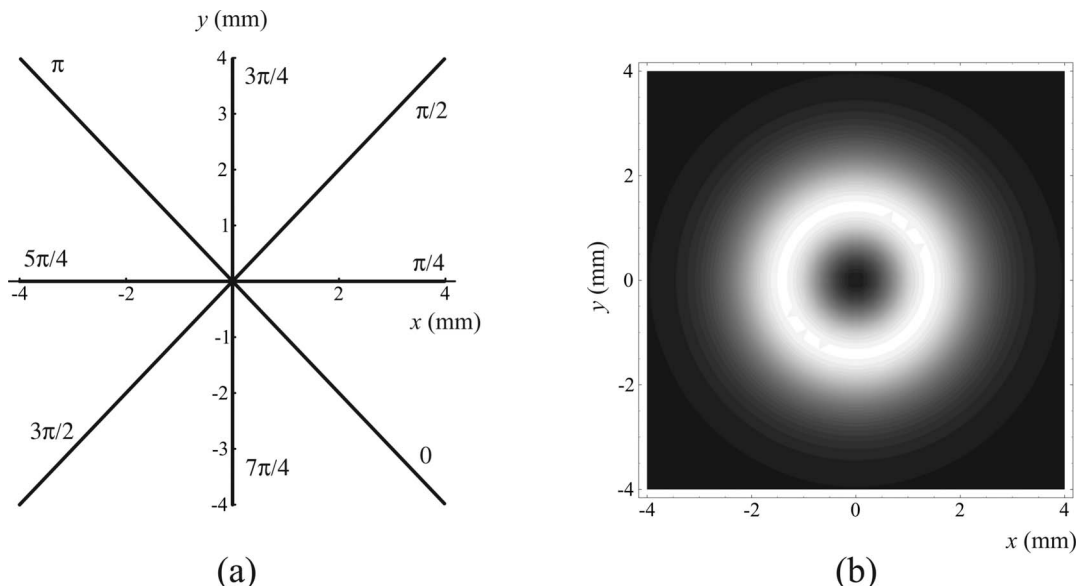

Fig. 1. Illustration of the (a) phase and (b) amplitude of a Laguerre–Gaussian beam of order $m = 1$, $n = 0$ in the source plane where $w_0 = 2$ mm.

$$W(\mathbf{r}_1,\mathbf{r}_2,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)\exp[i\omega\tau]\mathrm{d}\tau. \tag{3}$$

The cross-spectral density characterizes the intensity and spatial coherence properties of the field at frequency ω, and it contains all of the information of the mutual coherence function. For a quasi-monochromatic field of frequency ω_0, the spatial coherence properties are well-approximated by the value of the cross-spectral density at ω_0.

It has been shown that the cross-spectral density of an arbitrary partially coherent wave field at frequency ω may always be expressed as the average of an ensemble of monochromatic realizations of the field [33], i.e.,

$$W(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle U^*(\mathbf{r}_1,\omega)U(\mathbf{r}_2,\omega)\rangle_\omega, \tag{4}$$

where $U(\mathbf{r},\omega)$ is a monochromatic realization of the partially coherent field and the subscript ω denotes averaging with respect to this special ensemble. The advantage of this representation is that it allows one to construct models of partially coherent fields in the frequency domain directly without resorting first to finding the more complicated mutual coherence function. We will use this below to construct a model of a quasi-monochromatic, partially coherent field.

In the absence of sources, the cross-spectral density is known to satisfy a pair of Helmholtz wave equations [32], i.e.,

$$[\nabla_i^2 + k^2]W(\mathbf{r}_1,\mathbf{r}_2,\omega) = 0, \quad i = 1,2. \tag{5}$$

From this, it follows that, with one observation point fixed, the cross-spectral density behaves exactly like a monochromatic wave field and can possess vortices with the same topological and phase properties as ordinary monochromatic fields. Such vortices are now known as *correlation vortices* (or *coherence vortices*).

Such vortices are distinct from ordinary optical vortices, both in their behaviors and their interpretation. Correlation vortices are phase singularities of the *two-point* correlation function of the field, in contrast to optical vortices, which are phase singularities of a *one-point* monochromatic wave field. The location of a correlation vortex is therefore dependent upon the choice of the fixed observation point.

The formalism described above is, in principle, exact: correlation singularities can exist independently at each frequency of a partially coherent wave field. For the remainder of the paper, we will restrict ourselves to a quasi-monochromatic field of center frequency ω, and further expression of this frequency will be suppressed.

A simple model for a correlation vortex was proposed in [25]. We imagine a Laguerre–Gauss beam of order $n=0$, $m=1$, whose central vortex core is a random function of position. The cross-spectral density of such a field may be written as

$$W(\mathbf{r}_1,\mathbf{r}_2) = \int f(\mathbf{r}_0)U^*(\mathbf{r}_1-\mathbf{r}_0)U(\mathbf{r}_2-\mathbf{r}_0)\mathrm{d}^2r_0, \tag{6}$$

with

$$U(\mathbf{r}) = \sqrt{2}U_0\exp[i\varphi]\frac{r}{w_0}\exp[-r^2/w_0^2]. \tag{7}$$

Here $\mathbf{r}=(r,\varphi)$, and $f(\mathbf{r}_0)$ is the probability density for the position $\mathbf{r}_0$ of the central vortex core, taken to be Gaussian:

$$f(\mathbf{r}_0) = \frac{1}{\sqrt{\pi}\delta}\exp[-r_0^2/\delta^2]. \tag{8}$$

The quantity δ is an arbitrary diffusion scale that represents the average wander of the vortex core.

In the limit $\delta\to 0$, the beam does not wander at all and is therefore fully coherent. Increasing δ represents decreasing spatial coherence. Some examples of this correlation vortex are presented in Fig. 2.

Although we are using a seemingly very specialized model source, other research [26] suggests that all generic correlation singularities have a similar local form.

3. TWO-DIMENSIONAL SOURCES AND FOUR-VARIABLE CORRELATION SINGULARITIES

The integral of Eq. (6) can be evaluated analytically and has been shown to take the form [25]

$$\begin{aligned}
W(\mathbf{r}_1,\mathbf{r}_2) = {} & \frac{2\sqrt{\pi}|U_0|^2}{w_0^6A^3\delta}\exp[-(\mathbf{r}_1-\mathbf{r}_2)^2/w_0^4A] \\
& \times\exp[-(r_1^2+r_2^2)/\delta^2 w_0^2A] \\
& \times\{[\gamma^2(x_1-iy_1)+(x_1-x_2)+i(y_1-y_2)] \\
& \times[\gamma^2(x_2+iy_2)-(x_1-x_2)+i(y_1-y_2)]+w_0^4A\},
\end{aligned} \tag{9}$$

where $\gamma\equiv w_0/\delta$, $\mathbf{r}\equiv(x,y)$, and

$$A \equiv \left(\frac{2}{w_0^2}+\frac{1}{\delta^2}\right). \tag{10}$$

The manifold of singular phase is defined by two equations: the real and imaginary parts of the term in curved brackets above must vanish, i.e.,

$$\begin{aligned}
&[(\gamma^2+1)x_1-x_2][(\gamma^2+1)x_2-x_1] \\
&\quad +[(\gamma^2+1)y_1-y_2][(\gamma^2+1)y_2-y_1]+w_0^4A = 0, \quad (11)
\end{aligned}$$

$$\begin{aligned}
[(\gamma^2+1)y_1&-y_2][(\gamma^2+1)x_2-x_1] \\
&-[(\gamma^2+1)x_1-x_2][(\gamma^2+1)y_2-y_1] = 0. \quad (12)
\end{aligned}$$

The first equation is the equation for the real part; the second is for the imaginary part. The equation for the imaginary part may be simplified straightforwardly to the form

$$\frac{x_1}{x_2} = \frac{y_1}{y_2}. \tag{13}$$

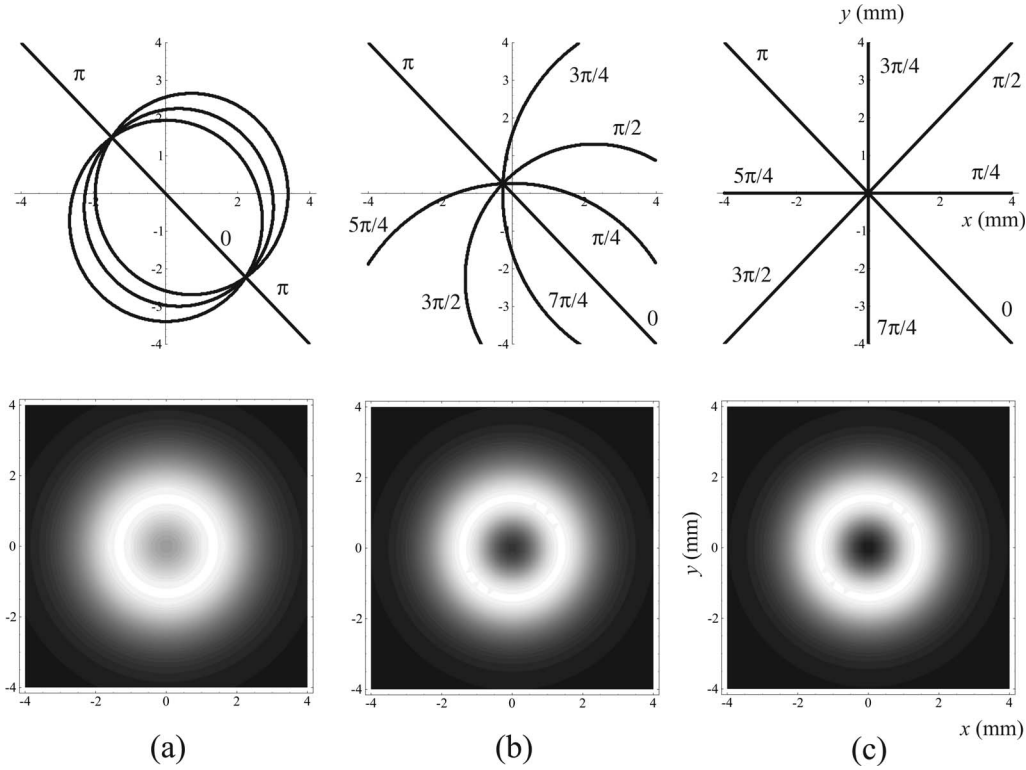Let us consider a transformation to a modified polar coordinate system, so that

Fig. 2. Illustrating the evolution of a coherence vortex to an optical vortex, for the partially coherent field defined by Eqs. (6)–(8), with $w_0 = 2$ mm. The phase of the cross-spectral density and the intensity of the field are plotted for (a) $\delta = 0.8$ mm, (b) $\delta = 0.3$ mm, and (c) $\delta \to 0$. The observation point $\mathbf{r}_1$ is taken to be $(x_1, y_1) = (0.1\ \text{mm}, -0.1\ \text{mm})$.

$$x_1 = \rho_1 \cos \varphi_1, \tag{14}$$

$$y_1 = \rho_1 \sin \varphi_1, \tag{15}$$

and similarly for (x_2, y_2), but $-\infty < \rho_i < \infty$ and $0 \le \varphi_i < \pi$, with $i = 1, 2$. The modified system is used for later plotting convenience. Equation (13) takes on the simple form

$$\tan \varphi_2 = \tan \varphi_1, \tag{16}$$

which in turn suggests that zeros exist only for observation points such that $\varphi_1 = \varphi_2$. In other words, phase singularities exist only when the observation points are collinear with the central axis of the beam.

In polar coordinates, Eq. (11) simplifies to the form

$$\alpha \rho_1^2 - (\alpha^2 + 1)\rho_1 \rho_2 + \alpha \rho_2^2 = w_0^4 A, \tag{17}$$

where

$$\alpha = \gamma^2 + 1, \tag{18}$$

which is the equation for a hyperbola whose foci lie along a line rotated $-\pi/4$ from the x axis. This can be seen by making the coordinate transformation

$$\rho_1 = \frac{w_1 + w_2}{\sqrt{2}}, \quad \rho_2 = \frac{w_2 - w_1}{\sqrt{2}}. \tag{19}$$

In terms of the rotated coordinates w_1 and w_2, Eq. (17) reduces to

$$\frac{(\alpha + 1)^2}{w_0^4 A} w_1^2 - \frac{(\alpha - 1)^2}{w_0^4 A} w_2^2 = 1. \tag{20}$$

We can compare this to the standard form of a hyperbola,

$$\frac{X^2}{a^2} - \frac{Y^2}{c^2 - a^2} = 1, \tag{21}$$

where a is the intercept of the hyperbola and c is the focal distance. Our zero manifold in ρ_1, ρ_2-space (with $\mathbf{r}_1$ and $\mathbf{r}_2$ collinear with the origin) is a hyperbola with intercept

$$a \equiv \sqrt{\frac{w_0^4 A}{(\alpha + 1)^2}} = 1/\sqrt{A}. \tag{22}$$

In the limit as $\delta \to 0$ (i.e., the field becomes fully coherent), $A \to \infty$ and $a \to 0$. The hyperbola becomes a pair of crossed straight lines on the ρ_1 and ρ_2 axes. The behavior of the zero manifold in ρ_1, ρ_2-space for various values of δ is shown in Fig. 3.

Equations (17) and (16) completely define the zero manifold of a correlation singularity for a two-dimensional source.

4. PROJECTIONS OF THE CORRELATION SINGULARITIES

We first demonstrate that the results here produce the correlation vortex of [25] and the ring dislocation of [18]. A correlation screw dislocation is observed by studying the phase of the cross-spectral density with one observation point fixed. Let us fix $x_1 = x_0$, $y_1 = y_0$, and $\rho_0 = \sqrt{x_0^2 + y_0^2}$.
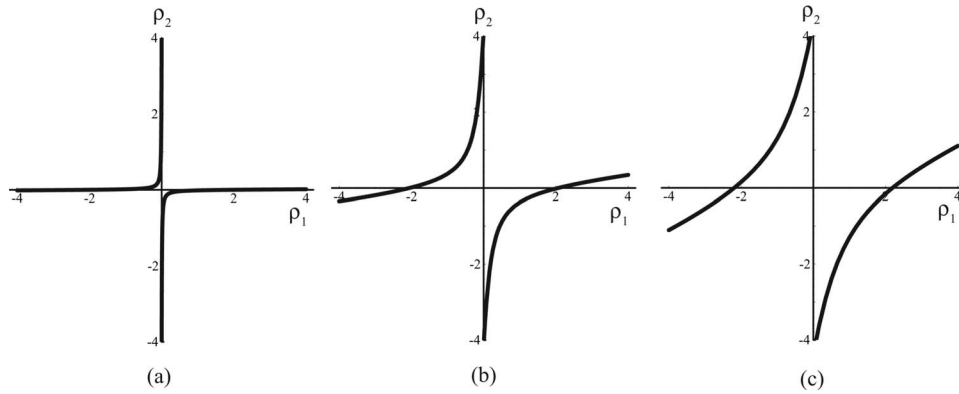
Fig. 3. Plot of the zero manifolds of the four-variable correlation function defined by Eq. (9), for various values of the diffusion parameter δ. In all plots, $w_0 = 2$ mm. The zero manifolds are plotted for (a) $\delta = 0.1$ mm, (b) $\delta = 0.5$ mm, (c) $\delta = 1.0$ mm.

And Eq. (17) can be written entirely in terms of ρ_2. The roots of this quadratic equation can be immediately found to be

$$\rho_2 = \left(\frac{\alpha^2 + 1}{2\alpha^2}\right)\rho_0 \pm \sqrt{\left(\frac{\alpha^2 + 1}{2\alpha^2}\right)^2 \rho_0^2 - \rho_0^2 + \frac{w_0^4 A}{\alpha^2}}. \quad (23)$$

In agreement with the results of [25], we find that a pair of correlation singularities are present for a fixed value of $\mathbf{r}_1$. This is not surprising, since we are starting with the same model for a correlation singularity. What is more useful is that we are able to quantify the location of these singularities: their locations are always on a line connecting the origin to the point $\mathbf{r}_1$, and they are symmetrically located around a central point $\mathbf{r}'$ defined by the formula

$$\mathbf{r}' \equiv \mathbf{r}_0\left(\frac{\alpha^2 + 1}{2\alpha^2}\right). \quad (24)$$

In [18], a projection of the general four-variable correlation function is taken by considering cross-correlations of the vector field, i.e., $\mathbf{r}_1 = -\mathbf{r}_2$. We therefore take $x_1 = -x_2$, $y_1 = -y_2$. In this case Eq. (13) is automatically satisfied. We are left with a single equation defining the zero manifold for this projection; from Eq. (17),

$$\alpha\rho_1^2 + (\alpha^2 + 1)\rho_1^2 + \alpha\rho_1^2 = w_0^4 A, \quad (25)$$

which may be written in terms of ρ_1 as

$$\rho_1^2 = \frac{w_0^4 A}{(\alpha + 1)^2} = \frac{1}{A}. \quad (26)$$

This is clearly the equation of a circle; the cross-correlation of the cross-spectral density automatically results in a ring dislocation of radius

$$r_0^2 = \frac{1}{A} = \frac{1}{2/w_0^2 + 1/\delta^2}. \quad (27)$$

For $w_0 \gg \delta$, the radius may be written in the approximate form

$$r_0 \approx \frac{\delta}{\sqrt{2}}. \quad (28)$$

As δ is effectively proportional to the inverse of the correlation length, we find that the radius of the ring dislocation is inversely related to the correlation length of the field, in agreement with the observations of [18].

We can illustrate these projections on the zero manifold defined by Eqs. (17) and (16); this is done in Fig. 4. The cross-correlation function is defined by taking $\varphi_1 = \varphi_2$ and looking at $\rho_2 = -\rho_1$. It can be seen that this results in a pair of zero points in ρ_1, ρ_2 space, which become a ring when all angles are taken into account. The correlation vortex projection involves a fixed value of ρ_1. The vertical line will intersect the manifold at two points.

The asymptotes of the hyperbola of Fig. 4 are given by the equations

$$\rho_2 = \alpha\rho_1, \quad \rho_2 = \frac{\rho_1}{\alpha}, \quad (29)$$

where δ is an arbitrary diffusion parameter. It is to be noted that no correlation singularities exist at all when
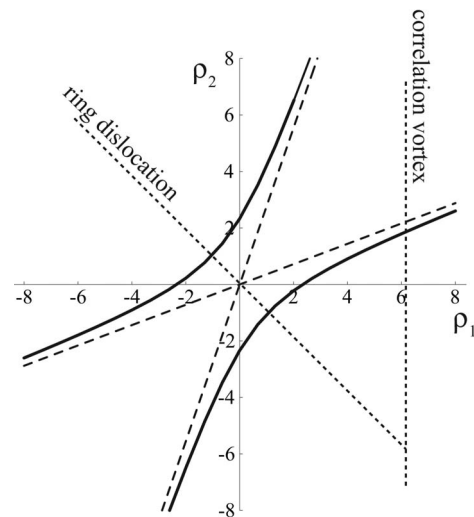

Fig. 4. Illustrating the standard projections of a correlation vortex and their relationship to the full zero manifold. Here $w_0 = 2$ mm and $\delta = 0.1$ mm. The thick dashed lines represent the asymptotes of the hyperbola.

points ρ_1 and ρ_2 are chosen between the two asymptotes. In the coherent limit, $\alpha \to \infty$ and the asymptotes coincide with the axes of the ρ_1, ρ_2 coordinate system. In the incoherent limit, $\alpha \to 1$ and the asymptotes both lie along the line $\rho_1 = \rho_2$. In this case the hyperbola reduces to a pair of lines parallel to the asymptotes, separated by perpendicular distance $\sqrt{2}w_0$. A pair of correlation singularities will, in this limit, be separated by a distance $2w_0$.

5. CONCLUSIONS

In this paper we have derived the complete mathematical representation of a correlation singularity in four variables. This structure is surprisingly simple, considering the four-variable space it resides in. The general solution is found to be a hyperbola in two reduced dimensions, ρ_1 and ρ_2, with correlation singularities existing only for observation points collinear with the origin of the field. In the limit of complete coherence this hyperbola reduces to crossed straight lines, whereas in the incoherent limit it reduces to parallel lines. In all cases, a region completely devoid of correlation singularities is defined by the asymptotes of the hyperbola.

   This new understanding of coherence singularities will provide experimentalists with new insights for generating beams with designed coherence structures. Further work in this area could include polarization considerations, non-Gaussian beam profiles, complex vortex patterns, and vortices having multiple charges.

APPENDIX A: ONE-DIMENSIONAL SOURCES AND TWO-VARIABLE CORRELATION SINGULARITIES

The general four-variable correlation function considered in this paper is potentially difficult to visualize and interpret. In this appendix, we consider the simpler case of a one-dimensional scalar source $U(x)$, illustrated in Fig. 5. This source is taken to have a phase singularity in its center,

$$U(x) = U_0 \exp[-x^2/2w_0^2]x/w_0, \qquad \text{(A1)}$$

but is assumed to have a singularity core that is a random function of position, resulting in a cross-spectral density of the form
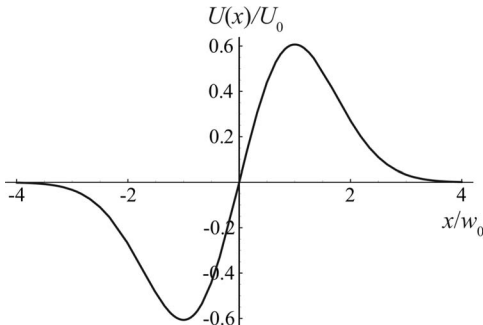

Fig. 5.   Plot of the one-dimensional source function defined by Eq. (A1) with a phase singularity at the origin.

$$W(x_1, x_2) = \int_{-\infty}^{\infty} f(x_0)U^*(x_1 - x_0)U(x_2 - x_0), \qquad \text{(A2)}$$

where

$$f(x_0) = \frac{1}{\sqrt{\pi}\delta} \exp[-x_0^2/\delta^2]. \qquad \text{(A3)}$$

It is to be noted that there is no such thing as a vortex for the one-dimensional source described here. This integral may be readily evaluated to find a specific form for the cross-spectral density,

$$W(x_1, x_2) = \frac{U_0^2 A}{w_0^3} \exp[-x_1^2/2w_0^2]\exp[-x_2^2/2w_0^2]$$

$$\times \exp[(x_1 + x_2)^2 A^2/4w_0^4]$$

$$\times \left[x_1 x_2 + B(x_1 + x_2)^2 + \frac{1}{2}A^2 \right], \qquad \text{(A4)}$$

where

$$\frac{1}{A^2} = \frac{1}{\delta^2} + \frac{1}{w_0^2} \qquad \text{(A5)}$$

and

$$B = \left[\frac{A^4}{4w_0^4} - \frac{A^2}{2w_0^2} \right]. \qquad \text{(A6)}$$

From this equation, it is clear that zeros of the correlation function satisfy the formula

$$\left[(2B + 1)x_1 x_2 + Bx_1^2 + Bx_2^2 + \frac{1}{2}A^2 \right] = 0. \qquad \text{(A7)}$$

This is a standard quadratic form, whose determinant may be shown to be

$$4D + 1 = 4\left(\frac{A^2}{2w_0^2} - \frac{1}{2} \right)^2 > 0, \qquad \text{(A8)}$$

which indicates that the zeros of the correlation function form a hyperbola. By making the coordinate transformation,

$$x_1 = y_2 - y_1, \qquad \text{(A9)}$$

$$x_2 = y_2 + y_1, \qquad \text{(A10)}$$

we may write the hyperbola in a standard form:

$$\frac{y_1^2}{(A^2/2)} - \frac{(4B + 1)y_2^2}{(A^2/2)} = 1. \qquad \text{(A11)}$$

The equation for a hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1, \qquad \text{(A12)}$$

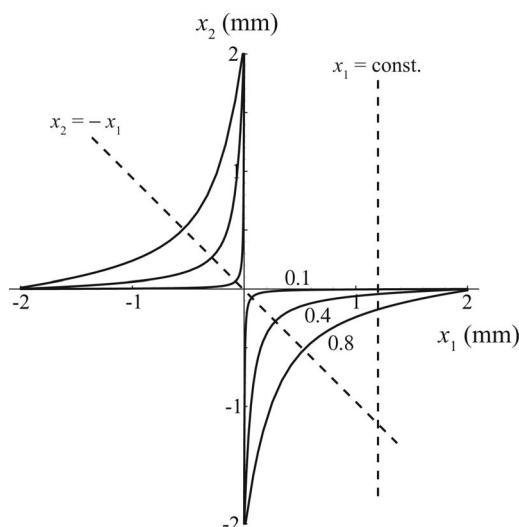where a is the intercept of the hyperbola and c is the focal distance. These quantities may be written as

Fig. 6.   Plot of the zero lines of the two-variable correlation function defined by Eq. (A4), for values of the diffusion parameter δ =0.8 mm, δ=0.4 mm, and δ=0.1 mm. In all plots the characteristic beam size is w_0=2 mm. The standard projections x_1 =constant (correlation vortex) and $x_1 = -x_2$ (ring dislocation) are illustrated by dashed lines.

$$a = A, \tag{A13}$$

$$c = \left(\frac{4D+2}{4D+1} \right)^{1/2} A. \tag{A14}$$

The correlation singularity of a one-dimensional partially coherent source therefore manifests itself as a hyperbola in x_1,x_2 space. Several of these hyperbolas are illustrated in Fig. 6 for various values of δ. In the coherent limit, it can be seen that the hyperbolas "press against" the x_1 and x_2 axes and reduce to the expected field singularities.

The asymptotes of a hyperbola are defined by

$$y = \pm \frac{\sqrt{c^2 - a^2}}{a} x = \pm \frac{1}{4D+1} x = \pm \frac{1}{1 + \delta^2/w_0^2} x. \tag{A15}$$

This simpler one-dimensional source with a correlation singularity illustrates what to look for in the case of a two-dimensional source containing a correlation singularity.

ACKNOWLEDGMENTS

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