Directional, nonpropagating, and polychromatic excitations in one-dimensional wave systems

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It has been demonstrated that, in a one-dimensional wave system, monochromatic waves may be generated which are completely localized to the region of excitation or which propagate in only one direction. We further the discussion of such nonpropagating and directional excitations and demonstrate that they can be extended to excitations of an arbitrary finite number of frequencies. Two techniques for mathematically constructing these excitations are discussed. Furthermore, the relation between nonpropagating excitations and nonscattering scatterers is discussed. The results presented here may be useful in the development of devices for one-dimensional and quasi-one-dimensional wave systems.

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I. INTRODUCTION

Well-established results in classical physics can still occasionally yield new surprises; not too long ago it was demonstrated that unusual interference effects can arise in one-dimensional wave systems such as vibrating strings. In particular, it was shown [1] that under certain conditions, a monochromatic driving force applied to a localized region of a string will produce no excitation outside the region of the applied force. Such *nonpropagating excitations* are closely related to so-called *nonradiating sources* and other "invisible" objects [2], and invisibility has recently received renewed attention in the literature [3,4].

Later work on nonpropagating excitations demonstrated their existence even in the presence of damping or excitations of finite bandwidth [5,6]. Furthermore, a number of other curious effects relating to nonpropagating excitations have been studied numerically [7], among them the existence of *directional excitations*, which propagate only in one direction away from the region of applied force.

It is well known that nonradiating sources are mathematically closely related to their scattering counterparts, known as *nonscattering scatterers* [2]. However, the one-dimensional versions of nonscattering scatterers have not been studied in detail, although some research has been done on reflectionless stratified media [8].

In this article we further the discussion of nonpropagating and directional excitations and demonstrate that they can be extended to the excitations of an arbitrary finite number of frequencies. Two techniques for mathematically constructing such excitations are described. The relevance to one-dimensional nonscattering scatterers is discussed.

II. NONPROPAGATING AND UNIDIRECTIONAL EXCITATIONS

We first briefly review the basic principles of directional and nonpropagating excitations, and then discuss two methods of constructing such excitations at a single frequency of excitation ω. In the next section we will consider extending these results to multiple frequencies.

We consider a general one-dimensional wave system with wave amplitude $y(x,t)$ excited by a driving force $q(x,t)$ localized to the region $a \leq x \leq b$ of an otherwise infinite domain. This system satisfies the wave equation

$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} = q(x,t), \tag{1}$$

where v is the velocity at which the wave propagates. If we restrict ourselves to harmonic driving forces, $q(x,t) = q(x)\exp[-i\omega t]$, the steady-state wave amplitude will also be monochromatic—i.e., $y(x,t)=y(x)\exp[-i\omega t]$—and the problem is reduced to studying solutions of the one-dimensional Helmholtz equation

$$\frac{d^2 y(x)}{dx^2} + k^2 y(x) = q(x), \tag{2}$$

where the wave number $k=\omega/v$.

The general solution to this equation has been shown to be ([9], Chap. 4)

$$y(x) = \frac{1}{2ik}\int_a^b q(x')\exp[ik|x-x'|]dx'. \tag{3}$$

Outside of the region of applied force, to its right and left, Eq. (3) is reduced to the forms

$$y(x)|_R = \frac{\exp[ikx]}{2ik}\int_a^b q(x')\exp[-ikx']dx', \tag{4}$$

$$y(x)|_L = \frac{\exp[-ikx]}{2ik}\int_a^b q(x')\exp[ikx']dx'. \tag{5}$$

The integrals in these two equations are proportional to the Fourier transform of the force density, given by the equation

$$\tilde{q}(K) = \int_{-\infty}^{\infty} q(x')e^{-i2\pi Kx'}dx'. \tag{6}$$

Of particular interest are circumstances in which the force distribution satisfies one or both of the following conditions:

$$\tilde{q}(k') = 0, \quad \tilde{q}(-k') = 0, \tag{7}$$

where $k'=k/2\pi$. As can be seen from Eqs. (4) and (5), a system for which both conditions are satisfied will produce no wave amplitude outside the region of applied force: the result is an excitation which is localized to this region—i.e., a *nonpropagating excitation*. If only one of the conditions (7) is satisfied, the excitation will propagate in only one direction; i.e., it will be a *directional excitation*.

It is not obvious at first glance how to construct examples of such localized excitations. We consider here two mathematical techniques of construction, each with their own individual advantages and disadvantages.

A. Amplitude construction

For the rest of this paper, we consider a coordinate system such that $a=-x_0$ and $b=x_0$. This allows us to easily consider, without any loss of generality, nonpropagating excitations of even and odd symmetry.

It is well known that the solution of the Helmholtz equation must be continuous and possess continuous first derivatives. From this knowledge, it has been shown previously [1] that all nonpropagating excitations satisfy the boundary conditions

$$y(-x_0)=y(x_0)=0, \qquad \left.\frac{dy}{dx}\right|_{x=-x_0} = \left.\frac{dy}{dx}\right|_{x=x_0}=0. \tag{8}$$

In other words, the wave amplitude and its first derivative must vanish on the boundary of the region of applied force.

This observation suggests a straightforward way of constructing nontrivial examples of nonradiating sources: *any* function $y(x)$ which is continuous, possesses continuous first derivatives, and satisfies the boundary conditions (8) represents within $-x_0 \leqslant x \leqslant x_0$ the field generated by a nonpropagating source. Once a function $y(x)$ is chosen, the force distribution $q(x)$ may be found by an application of Eq. (2). Similar constructions of nonradiating sources in three dimensions based on the boundary conditions have also been investigated by several authors [10–13].

We may use a similar method to construct directional excitations by replacing one pair of the boundary conditions (8). It can be seen from Eq. (5) that every wave propagating to the left of the region of applied force has the general form

$$y(x)|_L = A_0 \exp[-ikx], \tag{9}$$

where A_0 is in general a constant, complex, number. This suggests that we may construct a left-going directional excitation by finding a function $y(x)$ such that

$$y(x_0)=0, \qquad \left.\frac{dy}{dx}\right|_{x=x_0}=0, \quad y(-x_0)=A_0\exp[ikx_0], \tag{10}$$

$$\left.\frac{dy}{dx}\right|_{x=-x_0} = -ikA_0\exp[ikx_0],$$

with a similar set of conditions for right-going excitations.

A number of strategies may be used to determine functions $y(x)$ which represent nonpropagating or directional excitations. Perhaps the simplest is to assume that $y(x)$ is a polynomial with complex coefficients c_n—i.e.,

$$y(x)=\sum_{n=0}^{N} c_n x^n. \tag{11}$$

How many polynomial terms are required to satisfy the boundary conditions for a nonpropagating excitation? Let us restrict ourselves to solutions which are even (n even only) or odd (n odd only), in which case the left and right boundary conditions are redundant. With two boundary conditions, we might try an even solution of the form

$$y(x)=c_m x^m + c_{m+2}x^{m+2}, \tag{12}$$

which results in the following pair of complex homogeneous equations

$$c_m x_0^m + c_{m+2}x_0^{m+2}=0, \tag{13}$$

$$mc_m x_0^{m-1} + (m+2)c_{m+2}x_0^{m+1}=0. \tag{14}$$

This set of equations will only have a nontrivial solution for c_m and c_{m+2} if the determinant of the system of equations vanishes—i.e.,

$$2x_0^{2m+1}=0. \tag{15}$$

This condition cannot be satisfied; it is readily shown, however, that three or more polynomial terms will result in a solvable system of equations.

With this in mind, we now consider the application of a simple harmonic force distribution $q(x)$ within the domain $-x_0 \leqslant x \leqslant x_0$ of an infinitely long string. Introducing the dimensionless unit $u=kx$ for convenience, we consider for a nonpropagating excitation the series solution

$$y(u)=\begin{cases} Au^m + Bu^{(m+2)} + Cu^{(m+4)} & \text{for } u_0 \leqslant u \leqslant u_1, \\ 0 & \text{otherwise}, \end{cases} \tag{16}$$

where $u_0=-kx_0$, $u_1=kx_0$, and the coefficients A, B, and C replace the c_n's of Eq. (11) for the sake of notational convenience. Solving for the coefficients, we have $B=-2Cu_1^2$ and $A=Cu_1^4$, with the condition that $m \geqslant 1$. Examples with $m=1,2$ and $C=1$ are shown in Fig. 1, where the wave number k has been arbitrarily taken to be $1/x_0$—i.e., $u=x/x_0$ and $u_1=-u_0=1$. Clearly, we see that the solutions exhibit odd or even symmetry, depending on whether m is odd or even. The solutions in Fig. 1 can also be shifted to any $u_0' \leqslant u \leqslant u_1'$ through the appropriate translation $y(u-u')$, where $u'=\frac{u_0'+u_1'}{2}$.

To construct a directional excitation, we take a slightly different approach. We attempt to construct a directional excitation propagating to the right of the string such that $y(u_1)=Ce^{iu_1}$. The solution we consider is

$$y(u) = \begin{cases} 0 & \text{for } u \leqslant u_0, \\ A(u-u_1)^m + B(u-u_1)^{(m+2)} + Ce^{iu} & \text{for } u_0 \leqslant u \leqslant u_1, \\ Ce^{iu} & \text{for } u \geqslant u_1, \end{cases} \tag{17}$$

with $m \geqslant 2$ and $C=A_0$, which is the desired amplitude of the

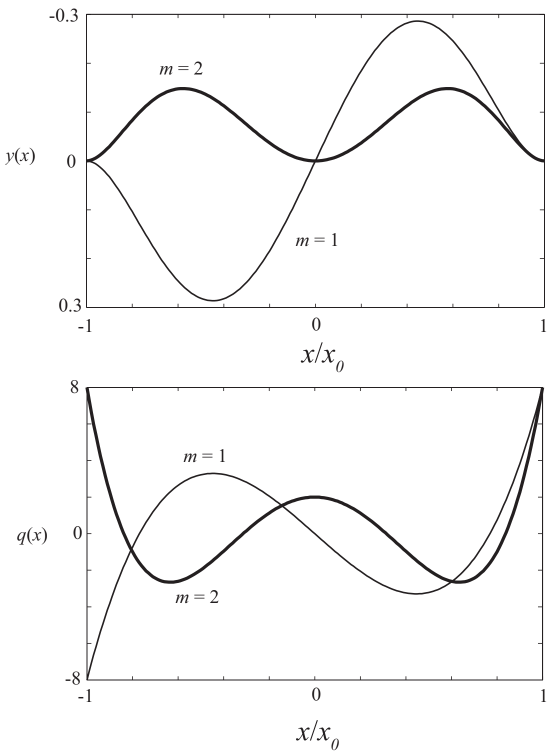

FIG. 1. The displacement $y(x)$ and force distribution $q(x)$, with $k=1/x_0$, for the nonpropagating excitation of Eq. (16), with $m=1,2$ and $C=1$.

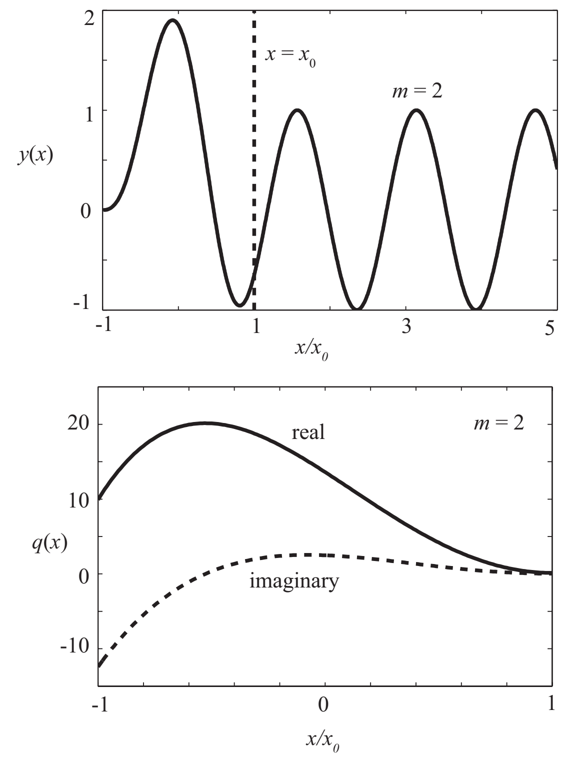

FIG. 2. The displacement $y(x)$ and force distribution $q(x)$, with $k=4/x_0$, for the unidirectional excitation of Eq. (17), with $m=2$ and $C=1$. To illustrate the wave propagating to the right, $y(x)$ is plotted from $-x_0 \leqslant x \leqslant 5x_0$, corresponding to $-4 \leqslant u \leqslant 20$.

propagating wave as defined in Eq. (9). Matching the boundary conditions for the unidirectional excitation at u_0 and u_1, we find that the coefficients A and B (complex in this case) are given in terms of C as

$$A_R = \frac{C[2\Delta u \sin u_0 - (m+2)\cos u_0]}{2(-2\Delta u)^m}, \tag{18a}$$

$$A_I = -\frac{C[2\Delta u \cos u_0 + (m+2)\sin u_0]}{2(-2\Delta u)^m}, \tag{18b}$$

$$B_R = \frac{C[m \cos u_0 - 2\Delta u \sin u_0]}{2(-2\Delta u)^{(m+2)}}, \tag{18c}$$

$$B_I = \frac{C[m \sin u_0 + 2\Delta u \cos u_0]}{2(-2\Delta u)^{(m+2)}}, \tag{18d}$$

where $A=A_R+\mathrm{i}A_I$, $B=B_R+\mathrm{i}B_I$, and $\Delta u=\frac{u_1-u_0}{2}\neq 0$. An example with wave number $k=4/x_0$, again an arbitrary choice, is shown in Fig. 2, where it is taken that $m=2$, and $C=1$.

B. Force construction

The amplitude construction technique provides an easy method of constructing localized excitations, but the force distributions generated will be smoothly varying, possibly complex, functions of position which will be difficult to generate experimentally. We may approach the problem in reverse and consider the development of simple force distributions which result in complicated localized excitations.

The simplest distributions, and the ones we will consider, are piecewise constant force distributions. Let us first define a single step $S(u)$ by the formula

$$S(u) = \begin{cases} 1 & |u| \leqslant 1, \\ 0 & |u| > 1. \end{cases} \tag{19}$$

A piecewise constant force distribution may be written in terms of a collection of steps in the form

$$q(x) = \sum_{n=1}^{N} a_n S\left[\frac{(x-x_n)}{\sigma_n}\right], \tag{20}$$

where N is the total number of steps in the distribution, x_n is the center of the nth step, a_n is the height of the nth step, and σ_n is the half-width of the nth step (Fig. 3). The steps are assumed to be nonoverlapping, though overlapping steps still result in a piecewise constant distribution.

The simplest force distribution with $N=1$—i.e., a single step function of half-width σ_1—results in nonpropagating excitation provided that

$$k\sigma_1 = m\pi, \tag{21}$$

where m is a nonzero integer [1]. For a fixed step size, only excitations with wave number satisfying Eq. (21) will produce nonpropagating excitations. To produce an excitation for an arbitrary source size and wave number, we must consider multiple steps.

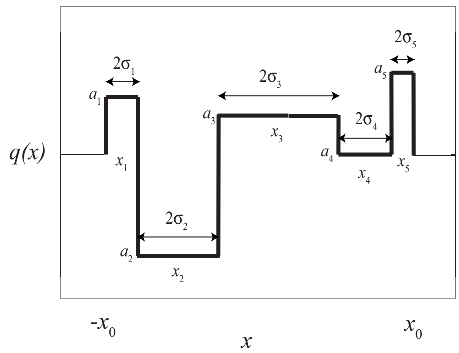

FIG. 3. An example of a force distribution $q(x)$ of Eq. (20) with five steps. It is taken that $q(x)$ is a real function. The constants a_n's and σ_n's represent the height and half-width of each of the steps, respectively. The x_n's are the midpoints for each of the steps.

We restrict our investigations of the nonpropagating excitations to real-valued source distributions which are of even symmetry, $q_e(x)$ and odd symmetry, $q_o(x)$. Such distributions, with $2N$ steps, may be written in the form

$$q_{e/o}(x) = \sum_{n=1}^{N} a_n \left\{ S\left[\frac{(x-x_n)}{\sigma_n}\right] \pm S\left[\frac{(x+x_n)}{\sigma_n}\right]\right\}, \quad (22)$$

where the sign is "+" or "−" for even or odd distributions, respectively. The Fourier transforms of such distributions result in the equations

$$\tilde{q}_e(\pm k') = \sum_{n=1}^{N} 4a_n\sigma_n \cos(2\pi k' x_n)\mathrm{sinc}[2k'\sigma_n], \quad (23)$$

$$\tilde{q}_o(\pm k') = \pm \sum_{n=1}^{N} \mathrm{i}4a_n\sigma_n \sin(2\pi k' x_n)\mathrm{sinc}[2k'\sigma_n], \quad (24)$$

where $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$. Two special cases should be noted here. First, if $k'\sigma_n = m/2$, the nth step satisfies Eq. (21) and is by itself a nonpropagating excitation. In fact, one could create a multistep nonpropagating excitation as a collection of steps which are individually nonpropagating. Second, if (for an even source) $k'x_n = m$ or (for an odd source) $k'x_n = (2m+1)/2$, the two symmetric (antisymmetric) steps of order n combine by themselves to form a nonpropagating excitation. This effect is a generalization of an example given in [14], in which an appropriately spaced pair of point sources result in nonpropagating waves. In both of these special cases, the nth-order steps do not contribute to the total field emitted by the source, and we restrict ourselves to situations when neither case is satisfied. It can then be seen that a minimum of two nonzero a_n terms must be used to satisfy the nonpropagating condition; this results in a total of four steps for the odd case or three (distinct) steps for the even case.

The symmetry properties of the Fourier transform make finding a nonpropagating excitation a somewhat straightforward process. For a real $q(x)$, $\tilde{q}(K) = \tilde{q}^*(-K)$. Therefore, the condition $\tilde{q}(k') = 0$ automatically ensures that $\tilde{q}(-k') = 0$. Let us consider, for illustration purposes, a real even force distribution consisting of three steps,

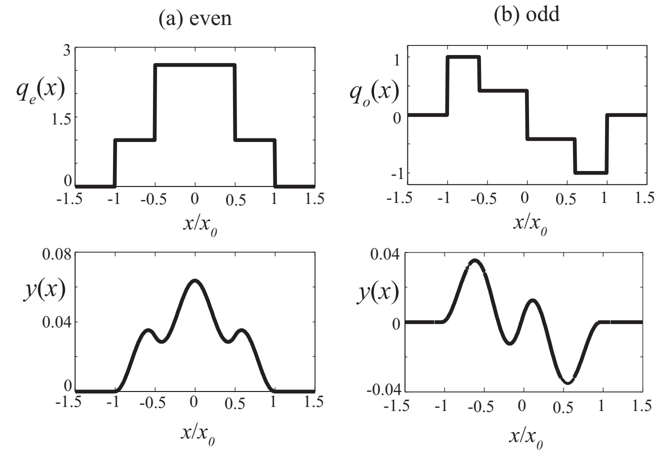

FIG. 4. The force distribution and displacement, for the nonpropagating excitation of Eq. (25) (even) and Eq. (26) (odd), with $k'x_0 = 1.2$ and $a_1 = 1$. The half-width σ_1 was taken to be $\frac{x_0}{4}$ and $\frac{x_0}{5}$ for the even and odd cases, respectively.

$$q_e(x) = \begin{cases} a_1 & \text{for } -x_0 \leq |x| \leq -x_0 + 2\sigma_1, \\ a_2 & \text{for } -x_0 + 2\sigma_1 \leq |x| \leq 0, \\ 0 & \text{otherwise}, \end{cases} \quad (25)$$

and a real odd force distribution consisting of four steps,

$$q_o(x) = \begin{cases} -a_1 \,\mathrm{sgn}(x) & \text{for } -x_0 \leq |x| \leq -x_0 + 2\sigma_1, \\ -a_2 \,\mathrm{sgn}(x) & \text{for } -x_0 + 2\sigma_1 \leq |x| \leq 0, \\ 0 & \text{otherwise}, \end{cases} \quad (26)$$

where sgn denotes the signum function and $\sigma_1 < \frac{x_0}{2}$. Imposing the condition that $\tilde{q}(k') = 0$ and solving for a_2 in terms of a_1, we find for $q_e(x)$

$$a_2 = a_1\left[1 + \frac{\sin(2\pi k' x_0)}{\sin[2\pi k'(2\sigma_1 - x_0)]}\right] \quad (27)$$

and for $q_o(x)$

$$a_2 = a_1\left[1 + \frac{1 - \cos(2\pi k' x_0)}{\cos[2\pi k'(2\sigma_1 - x_0)] - 1}\right]. \quad (28)$$

Examples for these two cases are shown in Fig. 4, with $k'x_0 = 1.2$ and $a_1 = 1$. The half-width σ_1 is taken to be $\frac{x_0}{4}$ and $\frac{x_0}{5}$ for the even and odd cases, respectively.

For the unidirectional propagating excitation, we require complex coefficients Q_n. More specifically, we take the real and imaginary parts of $q(x)$ to be even and odd, respectively, so that $\tilde{q}(K)$ is a sum of real even and odd functions. Let us consider the complex force distribution

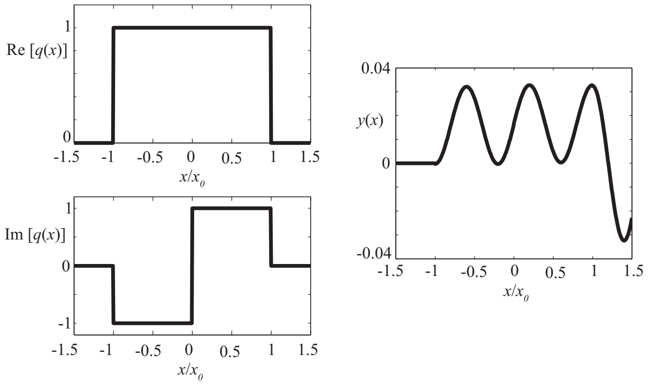

FIG. 5. The force distribution $q(x)$ and displacement $y(x)$, for the unidirectional propagating excitation of Eq. (29), with $k'x_0=1.25$ and $a_1=1$.

$$q(x) = \begin{cases} a_1 + \mathrm{i}a_2\,\mathrm{sgn}(x) & \text{for } -x_0 \leqslant x \leqslant x_0, \\ 0 & \text{otherwise}. \end{cases} \tag{29}$$

Imposing the condition that $\tilde{q}(-k')=0$ (excitation propagating to the right) and solving for a_2 in terms of a_1, we obtain

$$a_2 = \frac{a_1 \sin(2\pi k' x_0)}{\cos(2\pi k' x_0) - 1}. \tag{30}$$

An example is shown in Fig. 5 for the case $k'x_0=1.25$ and $a_1=1$.

It is clear now that increasing the number of coefficients serves to increase the flexibility in the choice of the wave number. In fact, we will simulate nonpropagating polychromatic excitations through the force construction method by employing more coefficients in the next section. Of course, a practical force distribution will not be perfectly discontinuous. Nevertheless, it is expected, from previous results [5] and the linearity of Eq. (3), that nonradiating force distributions that are "nearly" piecewise constant will be "nearly" nonpropagating.

III. POLYCHROMATIC EXCITATIONS

Building upon the mathematical constructs of Sec. II, we will now extend our discussion to polychromatic excitations. Let us examine excitations that involve two frequencies, $\omega_1=k_1 v$ and $\omega_2=k_2 v$. The task is to find $q(x)$ which will produce nonpropagating excitation of both frequencies, unidirectional excitation of one of the frequencies, or bidirectional excitation where the two frequencies propagate in opposite directions. We restrict our investigations to the force construction method. The amplitude construction method, because of the factor k^2 operating on the wave $y(x)$ in the Helmholtz equation, will produce different force distributions for different frequencies; such a technique, although in principle possible, would be prohibitively difficult experimentally.

Clearly, to satisfy the conditions $\tilde{q}(-k_1')=\tilde{q}(k_1')=0$ and $\tilde{q}(-k_2')=\tilde{q}(k_2')=0$, more steps in $q(x)$ are needed than in the last section [cf. Eq. (25)]. Let us consider the real and even force distribution

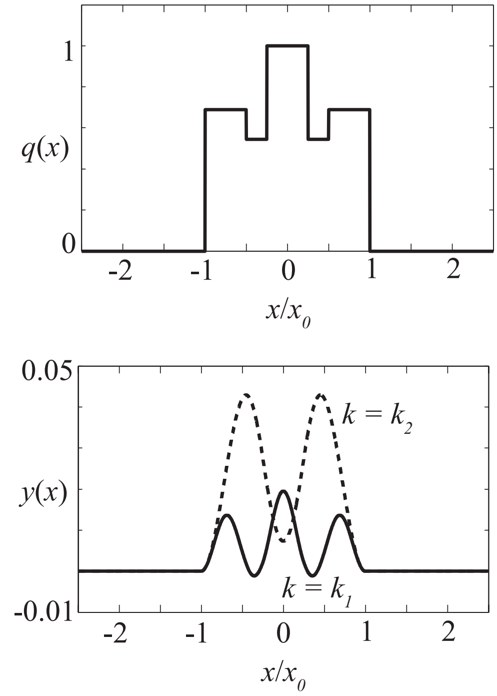

FIG. 6. The polychromatic force distribution $q(x)$ and displacement for the nonpropagating excitation of Eq. (31), with $k_1'x_0=1.6$ and $k_2'x_0=0.9$, for $\sigma_1=1/4, \sigma_2=1/8$, and $a_3=1$.

$$q(x) = \begin{cases} a_1 & \text{for } -x_0 \leqslant |x| \leqslant -x_0 + 2\sigma_1, \\ a_2 & \text{for } -x_0 + 2\sigma_1 \leqslant |x| \leqslant -x_0 + 2\sigma_1 + 2\sigma_2, \\ a_3 & \text{for } -x_0 + 2\sigma_1 + 2\sigma_2 \leqslant |x| \leqslant 0, \\ 0 & \text{otherwise}, \end{cases} \tag{31}$$

with $\sigma_1+\sigma_2<\frac{x_0}{2}$. Solving for a_1 and a_2 in terms of a_3, we find

$$a_1 = a_3\left[\frac{\delta\phi_1 + (\Psi_{11}\phi_2\rho_1 - \Psi_{12}\phi_1\rho_2)(\Psi_{21}\xi_1)}{2\Psi_{11}\delta\rho_1}\right], \tag{32a}$$

$$a_2 = a_3\left[\frac{\Psi_{12}\phi_1\rho_2 - \Psi_{11}\phi_2\rho_1}{2\delta}\right], \tag{32b}$$

where

$$\Psi_{np} = \sin[2\pi k_p' \sigma_n], \tag{33a}$$

$$\phi_n = \sin[2\pi k_n'(2\sigma_1 + 2\sigma_2 - x_0)], \tag{33b}$$

$$\rho_n = \cos[2\pi k_n'(x_0 - \sigma_1)], \tag{33c}$$

$$\xi_n = \cos[2\pi k_n'(x_0 - 2\sigma_1 - \sigma_2)], \tag{33d}$$

with $\delta=[\Psi_{12}\Psi_{21}\rho_2\xi_1 - \Psi_{11}\Psi_{22}\rho_1\xi_2]$ and $n,p=1,2$. An example with $k_1'x_0=1.6$ and $k_2'x_0=0.9$, for $\sigma_1=x_0/4$, $\sigma_2=x_0/8$, and $a_3=1$ is shown in Fig. 6.

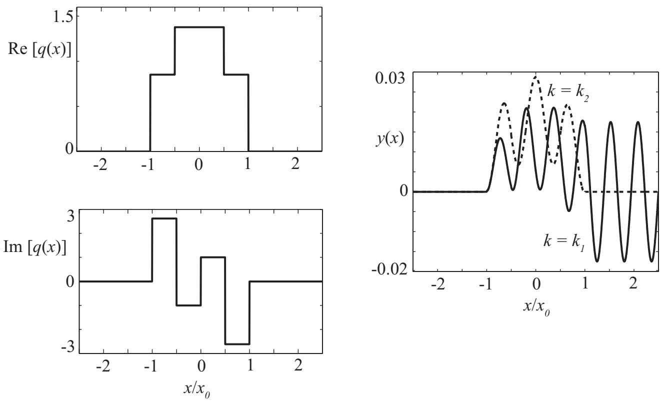

FIG. 7. The polychromatic force distribution $q(x)$ and displacement for k_1 propagating to the right and k_2 nonpropagating. It is taken that $k_1' x_0 = 1.8$, $k_2' x_0 = 1.4$, $\sigma_1 = 1/4$, and $a_4 = 1$.

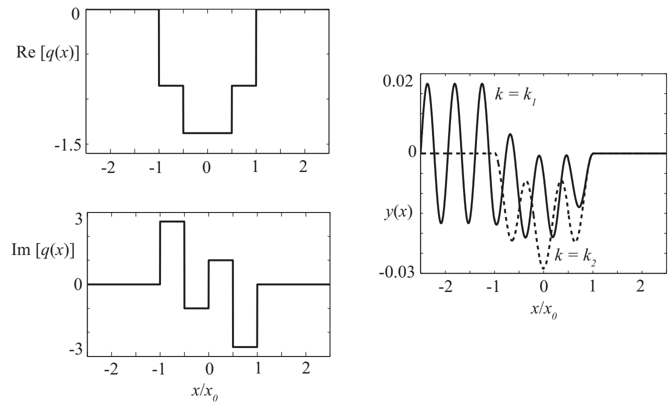

FIG. 8. The polychromatic force distribution $q(x)$ and displacement for k_1 propagating to the left and k_2 nonpropagating. It is taken that $k_1' x_0 = 1.8$, $k_2' x_0 = 1.4$, $\sigma_1 = 1/4$, and $a_4 = 1$.

To achieve unidirectional propagation of one of the two frequencies or bidirectional propagation of the two frequencies, we have seen from the last section that it is necessary to employ a complex force distribution $q(x) = \mathrm{Re}[q(x)] + i\mathrm{Im}[q(x)]$. This requirement of a complex $q(x)$ has also been observed when we constructed a unidirectional propagation with the amplitude construction method (Fig. 2). Let us now consider the following complex force distribution with real and imaginary parts given, respectively, as

$$\mathrm{Re}[q(x)] = \begin{cases} a_1 & \text{for } -x_0 \leq |x| \leq -x_0 + 2\sigma_1, \\ a_2 & \text{for } -x_0 + 2\sigma_1 \leq |x| \leq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{34}$$

$$\mathrm{Im}[q(x)] = \begin{cases} a_3\,\mathrm{sgn}(x) & \text{for } -x_0 \leq |x| \leq -x_0 + 2\sigma_1, \\ a_4\,\mathrm{sgn}(x) & \text{for } -x_0 + 2\sigma_1 \leq |x| \leq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{35}$$

For excitations involving $k_1 = 2\pi k_1'$ and $k_2 = 2\pi k_2'$, we look at the cases where (i) k_1 propagates to the right, and k_2 is nonpropagating; (ii) k_1 propagates to the left, and k_2 is nonpropagating; (iii) k_1 propagates to the right, and k_2 propagates to the left. In terms of the force construction method, these three cases are specified by the conditions

$$\tilde{q}(-k_1') = \tilde{q}(-k_2') = \tilde{q}(k_2') = 0 \quad [\text{case(i)}: k_1 \rightarrow], \tag{36a}$$

$$\tilde{q}(k_1') = \tilde{q}(-k_2') = \tilde{q}(k_2') = 0 \quad [\text{case(ii)}: k_1 \leftarrow], \tag{36b}$$

$$\tilde{q}(-k_1') = \tilde{q}(k_2') = 0 \quad [\text{case(iii)}: k_1 \rightarrow, k_2 \leftarrow]. \tag{36c}$$

It is noted that there is one fewer condition to satisfy for case (iii), compared to the other two cases. As such, we can expect that one fewer coefficient will be needed in the solutions for this case. This situation can be easily accommodated in a number of ways—for instance, by setting any two of the coefficients to be equal (e.g., $a_3 = a_4$) or by setting one of the two coefficients a_3 and a_4 to zero. In anticipation of the solutions to the coefficients in Eqs. (34) and (35), we define the terms

$$\psi_n = \sin 2\pi k_n' x_0, \tag{37a}$$

$$\varphi_n = \sin 2\pi k_n'(x_0 - 2\sigma_1), \tag{37b}$$

$$\varrho_n = \cos 2\pi k_n' x_0, \tag{37c}$$

$$\chi_n = \cos 2\pi k_n'(x_0 - 2\sigma_1), \tag{37d}$$

where $n = 1, 2$. For cases (i) and (ii), we have solved for the coefficients a_1, a_2, and a_3 in terms of a_4, which yield the results

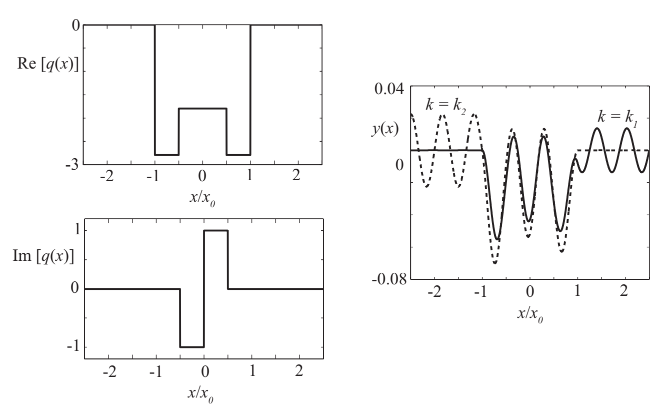

FIG. 9. The polychromatic force distribution $q(x)$ and displacement for k_1 propagating to the right and k_2 propagating to the left. It is taken that $k_1' x_0 = 1.6$, $k_2' x_0 = 1.5$, $\sigma_1 = 1/4$, $a_3 = 0$, and $a_4 = 1$.

$$a_1 = (-)a_4 \frac{\varphi_2[(1-\chi_1)(\varrho_2-\chi_2)+(1-\chi_2)(\chi_1-\varrho_1)]}{(\varphi_1\psi_2-\psi_1\varphi_2)(\chi_2-\varrho_2)}, \tag{38a}$$

$$a_2 = (-)a_4 \frac{(\varphi_2-\psi_2)[(1-\chi_1)(\varrho_2-\chi_2)+(1-\chi_2)(\chi_1-\varrho_1)]}{(\varphi_1\psi_2-\psi_1\varphi_2)(\chi_2-\varrho_2)}, \tag{38b}$$

$$a_3 = a_4 \frac{\chi_2-1}{\chi_2-\varrho_2}, \tag{38c}$$

where the minus sign in parentheses applies to case (ii). We show an example for cases (i) and (ii) with $k_1'x_0=1.8$, $k_2'x_0=1.4$, $\sigma_1=1/4$, and $a_4=1$ in Figs. 7 and 8.

For case (iii), we have solved for the coefficients a_1 and a_2 in terms of a_3 and a_4. The results are

$$a_1 = \frac{a_3[\varphi_1(\varrho_2-\chi_2)+\varphi_2(\varrho_1-\chi_1)]+a_4[\varphi_1(\chi_2-1)+\varphi_2(\chi_1-1)]}{(\varphi_1\psi_2-\psi_1\varphi_2)}, \tag{39a}$$

$$a_2 = \frac{a_3[\gamma_2(\varrho_1-\chi_1)+\gamma_1(\varrho_2-\chi_2)]+a_4[\gamma_2(\chi_1-1)+\gamma_1(\chi_2-1)]}{(\varphi_1\psi_2-\psi_1\varphi_2)}, \tag{39b}$$

with $\gamma_n=(\varphi_n-\psi_n)$ and $n=1,2$. An example with $k_1'x_0=1.6$, $k_2'x_0=1.5$, $\sigma_1=1/4$, $a_3=0$, and $a_4=1$ is shown in Fig. 9.

IV. CONCLUSIONS

We have numerically constructed and demonstrated nonpropagating and unidirectional excitations in a one-dimensional, monochromatic system, with illustrative examples of the amplitude and force construction methods. While the amplitude construction technique provides an easy method of constructing localized excitations, it is seen that the resulting force distributions can be difficult to generate experimentally. Extending our discussion to excitations involving two frequencies, we employed the force distribution technique to generate nonpropagating, unidirectional, and bidirectional excitations. For polychromatic excitations that include more than two frequencies, the boundary conditions can still be satisfied if more coefficients (c_n's for amplitude construction and a_n's for force construction) are used, resulting in more complex force distributions $q(x)$. It is to be noted that these excitations are localized as a result of complete destructive interference of the outgoing radiation.

In the study of so-called nonradiating sources in three dimensions, it is well known that there is a close mathematical relationship between the properties of a nonradiating source and a weakly scattering object which produces no scattered field for a finite number of illumination directions [2,15,16]. Within the accuracy of the first Born approximation, the scattered field is related to the Fourier transform of the scattering potential, just as the radiation field from a primary source is related to the Fourier transform of the charge distribution. This relationship holds for one-dimensional source and scattering problems as well, and suggests that one can create one-dimensional scattering potentials which are strongly scattering for some frequencies and nonscattering for others. Unlike the familiar Bragg grating used in fiber optics, however, a "nonscattering" or "directional" scatterer could be constructed for any set of frequencies, at the cost of increasing complexity of the scatterer as the number of frequencies is increased. Such a construction may be useful in designing novel devices for one-dimensional and quasi-one-dimensional wave systems.

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