Momentum conservation in partially coherent wave fields

S. M. Kim
Department of Physics, Yonsei University, Wonju, Kangwon-Do 220-710, Korea

Greg Gbur
Department of Physics and Optical Science, University of North Carolina at Charlotte, Charlotte, North Carolina 28223, USA

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Momentum flow in electromagnetic wave systems has become a topic of considerable importance in recent years with the development of optical tweezers and spanners. Although momentum conservation has been explored for deterministic wave fields, the corresponding laws for partially coherent wave fields have yet to be completely determined. In this paper we derive the frequency-domain representation for the Maxwell stress tensor for partially coherent fields and sources.

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I. INTRODUCTION

It is well known that electromagnetic waves carry momentum, and it is standard for electromagnetics textbooks to develop the corresponding momentum conservation laws alongside energy conservation laws (Sec. 6.8 of [1]). Studies of momentum flow in electromagnetic systems gained increased significance with the discovery of "optical tweezing," in which a small dielectric particle of high refractive index is trapped through radiation pressure [2]. Later research has demonstrated the trapping of low-index particles in the central minimum of a dark hollow beam [3], and a dark hollow beam has also been used to simultaneously trap high-index and low-index particles [4]. Beams carrying spin or orbital angular momentum have been shown to apply a torque to absorptive microscopic particles [5,6].

Recently, a number of authors have demonstrated that dark hollow regions can be produced in the region of focus by only varying the spatial coherence of the light field [7–9]; this work has also been done in the fully electromagnetic case [10]. It has been suggested that the ability to change the focus from a high intensity to a low intensity region by spatial coherence effects might be useful in creating "tunable" optical tweezing systems which can be readily changed from high-index to low-index trapping. In order to investigate such a possibility, though, the Maxwell stress tensor for partially coherent electromagnetic fields must be properly defined and the momentum conservation law for said fields must be elucidated.

In modern coherence theory, it is most convenient to work in the space-frequency domain rather than the space-time domain; in fact, energy conservation laws for scalar [11] and electromagnetic fields [12] in the space-frequency domain have been derived in recent years in the context of understanding correlation-induced spectral changes [13]. Momentum laws for partially coherent fields have only been developed in the space-time domain [14]; possibly due to their complexity, though, little use has been made of them.

In this paper we derive the Maxwell stress tensor and the momentum conservation law in the space-frequency domain for partially coherent electromagnetic fields. In Sec. II we derive the tensor and the momentum conservation law as it applies to fields and sources. In Sec. III we derive a few special cases of these momentum formulas, and in Sec. IV we present simple examples of its application. Section V presents concluding remarks.

II. MAXWELL STRESS TENSOR AND MOMENTUM CONSERVATION IN THE SPACE-FREQUENCY DOMAIN

We consider a region of space which contains time-fluctuating electric and magnetic fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$. The fields are assumed to be statistically stationary, at least in the wide sense (Sec. 2.2 of [15]). We initially follow the derivation by Jackson (Sec. 6.8 of [1]) for the stress tensor, making the appropriate modifications for partially coherent fields along the way. It is to be noted that we work exclusively with the microscopic Maxwell's equations, as the proper definition for momentum for the macroscopic equations, if any, is still a source of controversy, usually referred to as the Minkowski-Abraham controversy [16]. The total change in mechanical momentum of a continuous distribution of real-valued charges $\rho(\mathbf{r},t)$ and currents $\mathbf{J}(\mathbf{r},t)$ is given by

$$\frac{d\mathbf{P}_{\text{mech}}}{dt}(t) = \int_D \left[\rho(\mathbf{r}',t)\mathbf{E}(\mathbf{r}',t) + \frac{1}{c}\mathbf{J}(\mathbf{r}',t) \times \mathbf{B}(\mathbf{r}',t)\right] d^3r', \tag{1}$$

where D is a closed domain containing the source. In working with partially coherent fields, however, it is more appropriate to use an analytic signal representation of the field (Sec. III A of [15]), in which case all source and field quantities are complex and the expression for the (real-valued) momentum is

$$\frac{d\mathbf{P}_{\text{mech}}}{dt}(t) = \text{Re}\left\{\int_D \left[\rho(\mathbf{r}',t)\mathbf{E}^*(\mathbf{r}',t) + \frac{1}{c}\mathbf{J}(\mathbf{r}',t) \times \mathbf{B}^*(\mathbf{r}',t)\right] d^3r'\right\}. \tag{2}$$

This quantity may be written entirely in terms of field quantities by the use of Maxwell's equations in Gaussian units,

$$\rho(\mathbf{r},t) = \frac{1}{4\pi}\nabla\cdot\mathbf{E}(\mathbf{r},t), \tag{3}$$

$$\mathbf{J}(\mathbf{r},t) = \frac{c}{4\pi}\left[\nabla\times\mathbf{B}(\mathbf{r},t) - \frac{1}{c}\frac{\partial\mathbf{E}(\mathbf{r},t)}{\partial t}\right]. \tag{4}$$

The integrand of Eq. (2) then takes on the form

$$\rho\mathbf{E}^* + \frac{1}{c}\mathbf{J}\times\mathbf{B}^*(\mathbf{r}',t) = \frac{1}{4\pi}\left[\mathbf{E}^*(\nabla\cdot\mathbf{E}) + \frac{1}{c}\mathbf{B}^*\times\frac{\partial\mathbf{E}}{\partial t} - \mathbf{B}^*\times(\nabla\times\mathbf{B})\right], \tag{5}$$

where we have, for the moment, suppressed the functional dependencies for brevity. By use of the product rule of derivatives and Maxwell's equations, the entire integral for momentum may be written in the form

$$\frac{d\mathbf{P}_{\text{tot}}}{dt}(t) = \frac{1}{4\pi}\mathrm{Re}\left\{\int_D [\mathbf{E}^*(\nabla\cdot\mathbf{E}) + \mathbf{B}^*(\nabla\cdot\mathbf{B}) = \mathbf{E}\times(\nabla\times\mathbf{E}^*) - \mathbf{B}^*\times(\nabla\times\mathbf{B})]d^3r'\right\}, \tag{6}$$

where we have defined

$$\frac{d\mathbf{P}_{\text{tot}}}{dt}(t) = \frac{d\mathbf{P}_{\text{mech}}}{dt}(t) + \frac{1}{4\pi c}\mathrm{Re}\left\{\frac{d}{dt}\int_D \mathbf{E}\times\mathbf{B}^* d^3r'\right\}. \tag{7}$$

We have identified the *total* change in momentum within the volume D as consisting of the change in mechanical momentum plus the change in the momentum of the enclosed electromagnetic fields.

The quantity in the integral in Eq. (6) can be simplified using elementary vector calculus identities; the result is

$$\frac{d\mathbf{P}_{\text{tot}}}{dt}(t) = \mathrm{Re}\left\{\int_D \nabla\cdot\hat{T}(\mathbf{r}',t)d^3r'\right\}, \tag{8}$$

where $\hat{T}$ is defined as the stress tensor of the complex analytic field, which in tensor notation appears as

$$\hat{T}_{ij}(\mathbf{r},t) = \frac{1}{4\pi}\left\{E_i^*E_j + B_i^*B_j - \frac{1}{2}\delta_{ij}[\mathbf{E}^*\cdot\mathbf{E} + \mathbf{B}^*\cdot\mathbf{B}]\right\}. \tag{9}$$

This is a generalization of the ordinary Maxwell stress tensor to the case where the field is represented as a complex analytic signal. By use of the divergence theorem, Eq. (8) can be written as

$$\frac{d\mathbf{P}_{\text{tot}}}{dt}(t) = \mathrm{Re}\left\{\int_S \hat{T}(\mathbf{r}',t)\cdot\mathbf{n}da\right\}, \tag{10}$$

where S is a surface enclosing the volume D, da is an area element of that surface, and $\mathbf{n}$ is the unit vector normal to that surface.

So far we have introduced what amounts to a straightforward extension of the standard theory of momentum flow in electromagnetic systems. The total net change in momentum in the volume is equal to the flow of the stress tensor through the surface of the volume.

To generalize to partially coherent fields, we consider an ensemble average of the stress tensor. We define the complex correlation tensors of the electric and magnetic fields as follows:

$$\Gamma_{ij}^E(\mathbf{r}_1,\mathbf{r}_2,\tau) \equiv \langle E_i^*(\mathbf{r}_1,t)E_j(\mathbf{r}_2,t+\tau)\rangle, \tag{11}$$

$$\Gamma_{ij}^B(\mathbf{r}_1,\mathbf{r}_2,\tau) \equiv \langle B_i^*(\mathbf{r}_1,t)B_j(\mathbf{r}_2,t+\tau)\rangle, \tag{12}$$

where we have assumed the fields are statistically stationary. We may then write the stress tensor in terms of these correlation functions in the form

$$\langle\hat{T}_{ij}(\mathbf{r},t)\rangle = \frac{1}{4\pi}\left[\Gamma_{ij}^E(\mathbf{r},\mathbf{r},0) - \frac{1}{2}\delta_{ij}\Gamma_{ll}^E(\mathbf{r},\mathbf{r},0)\right] + \frac{1}{4\pi}\left[\Gamma_{ij}^B(\mathbf{r},\mathbf{r},0) - \frac{1}{2}\delta_{ij}\Gamma_{ll}^B(\mathbf{r},\mathbf{r},0)\right], \tag{13}$$

where we use the Einstein summation convention in that repeated indices are summed. It is to be noted that the average value of the stress tensor is independent of the origin of time, which means that the average momentum flow is the same at all points in time. This is a simple consequence of the stationarity of the wave fields.

It is generally more convenient to work in the space-frequency domain instead of the space-time domain; for this purpose, the cross-spectral density of a partially coherent wave field can be defined as

$$W_{ij}^E(\mathbf{r}_1,\mathbf{r}_2,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma_{ij}^E(\mathbf{r}_1,\mathbf{r}_2,\tau)e^{-i\omega\tau}d\tau, \tag{14}$$

and this Fourier expression can be inverted to write

$$\Gamma_{ij}^E(\mathbf{r}_1,\mathbf{r}_2,\tau) = \int_{-\infty}^{\infty}W_{ij}^E(\mathbf{r}_1,\mathbf{r}_2,\omega)e^{i\omega\tau}d\omega. \tag{15}$$

With this expression, the stress tensor may be written in the form

$$\langle\hat{T}_{ij}(\mathbf{r},t)\rangle = \int_{-\infty}^{\infty}\left\{\frac{1}{4\pi}\left[W_{ij}^E(\mathbf{r},\mathbf{r},\omega) - \frac{1}{2}\delta_{ij}W_{ll}^E(\mathbf{r},\mathbf{r},\omega)\right] + \frac{1}{4\pi}\left[W_{ij}^B(\mathbf{r},\mathbf{r},\omega) - \frac{1}{2}\delta_{ij}W_{ll}^B(\mathbf{r},\mathbf{r},\omega)\right]\right\}d\omega. \tag{16}$$

It can be seen from this equation that the total stress tensor may be written as the linear sum of the individual frequency contributions. Because different frequency components of a statistically stationary field are uncorrelated, we are justified in writing the stress tensor in the frequency domain in the form

$$\langle \hat{T}_{ij}(\mathbf{r},\omega)\rangle = \frac{1}{4\pi}\left[W^{E}_{ij}(\mathbf{r},\mathbf{r},\omega) - \frac{1}{2}\delta_{ij}W^{E}_{ll}(\mathbf{r},\mathbf{r},\omega)\right] + \frac{1}{4\pi}\left[W^{B}_{ij}(\mathbf{r},\mathbf{r},\omega) - \frac{1}{2}\delta_{ij}W^{B}_{ll}(\mathbf{r},\mathbf{r},\omega)\right]. \quad (17)$$

Equation (17), together with Eq. (16), describes how the component of the field with frequency ω contributes to the total stress tensor of the field. For a broadband field, expression (17) can be used frequency by frequency to determine the total momentum of a field. If the field is quasimonochromatic, a single component of the stress tensor at the center frequency serves as an approximation to the total stress tensor. This equation will be used to derive a number of useful expressions relating to momentum flow in partially coherent fields.

It is to be noted that, for the case of partially coherent fields, the momentum conservation law [Eq. (10)] may be written as

$$\left\langle \frac{d\mathbf{P}_{\text{tot}}}{dt}\right\rangle = \mathrm{Re}\left\{ \int_0^\infty \left[\int_S \langle \hat{T}(\mathbf{r},\omega)\rangle \cdot \mathbf{n} da \right] d\omega \right\}. \quad (18)$$

III. SPECIAL CASES OF THE MOMENTUM FORMULAS

The general formulas for momentum flow and momentum conservation are quite complicated; for certain special cases, they simplify considerably. We consider two of these cases here.

A. Momentum flow from primary polarization sources

We first consider the flow of momentum from partially coherent *primary* polarization distributions. The electric field produced by a monochromatic electric field has the well known form [12]

$$E_i(\mathbf{r},\omega) = [k^2 + \partial_i\partial_j]\int_D P_j(\mathbf{r}',\omega)\frac{e^{ikR}}{R}d^3r', \quad (19)$$

where, in the Cartesian coordinates, $\partial_j \equiv \partial/\partial x_j$ and $R \equiv |\mathbf{r}-\mathbf{r}'|$. To find the cross-spectral density of the electric field, we apply a result from coherence theory in the space-frequency domain [17], which states that the cross-spectral density may always be written as the average of an ensemble of *monochromatic* fields,

$$W^{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle E_i^*(\mathbf{r}_1,\omega)E_j(\mathbf{r}_2,\omega)\rangle_\omega, \quad (20)$$

where the subscript ω indicates that this is an average over an ensemble of monochromatic realizations of the field. It is to be noted that for statistically stationary fields such as those we consider, different frequency components of the field are uncorrelated. Mathematically, this means that one can perform a space-frequency average which treats the field at frequency ω as an ensemble of monochromatic fields. The wave number k in this case is therefore a constant, and the wave propagator is a deterministic function. A similar expression may be written for the cross-spectral density of the polarization, W^P. We may substitute from Eq. (19) into Eq. (20) to write the cross-spectral density of the electric field in the form

$$W^{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2) = \int\int_D W^{P}_{lm}(\mathbf{r}',\mathbf{r}'')[k^2\delta_{il} + \partial_{1i}\partial_{1l}] \times [k^2\delta_{jm} + \partial_{2j}\partial_{2m}]\frac{e^{-ikR_1}}{R_1}\frac{e^{ikR_2}}{R_2}d^3r_1'd^3r_2', \quad (21)$$

where $R_a \equiv |\mathbf{r}_a - \mathbf{r}_a'|$, with $a=1,2$. We have suppressed further expression of the variable ω for brevity. We may simplify this expression by considering points in the far zone of the source, in which case we may write

$$\frac{e^{ikR}}{R} \sim \frac{e^{ikr}}{r}e^{-ik\mathbf{u}\cdot\mathbf{r}'}, \quad (22)$$

where $\mathbf{u}$ is a unit vector pointing from the origin to the point of observation in the far zone. With some work we may write the cross-spectral density of the electric field in the compact form

$$W^{E}_{ij}(r_1\mathbf{u}_1,r_2\mathbf{u}_2) \sim (2\pi)^6k^4\frac{e^{ik(r_2-r_1)}}{r_1r_2}\widetilde{W}^{P}_{lm}(-k\mathbf{u}_1,k\mathbf{u}_2)[\delta_{il} - u_{1i}u_{1l}] \times [\delta_{jm} - u_{2j}u_{2m}], \quad (23)$$

where $\widetilde{W}^P$ is the sixfold Fourier transform of the polarization cross-spectral density W^P, i.e.,

$$\widetilde{W}^P(\mathbf{K}_1,\mathbf{K}_2) = \frac{1}{(2\pi)^6}\int\int_D W^P(\mathbf{r}_1',\mathbf{r}_2')e^{i(\mathbf{K}_1\cdot\mathbf{r}_1'+\mathbf{K}_2\cdot\mathbf{r}_2')}d^3r_1'd^3r_2'. \quad (24)$$

In a similar manner, the cross-spectral density for the magnetic field in the far-zone may be written in the form

$$W^{B}_{ij}(r_1\mathbf{u}_1,r_2\mathbf{u}_2) \sim (2\pi)^6k^4\frac{e^{ik(r_2-r_1)}}{r_1r_2}\epsilon_{ipq}\epsilon_{jrs}u_{1p}u_{2r} \times \widetilde{W}^{P}_{qs}(-k\mathbf{u}_1,k\mathbf{u}_2), \quad (25)$$

where ϵ_{ijk} is the Levi-Civita tensor.

On a sphere of fixed radius r centered on the origin, we may write the momentum flow as a function of normal direction $\mathbf{u}$ in the form

$$\mathbf{u}\cdot\langle\hat{T}(\mathbf{r})\rangle = -\frac{(2\pi)^6}{4\pi r^2}\mathbf{u}k^4[\delta_{lm} - u_lu_m]\widetilde{W}^{P}_{lm}(-k\mathbf{u},k\mathbf{u}). \quad (26)$$

This result demonstrates that the momentum flow depends not only on the distribution of polarization sources, but also their correlation properties, as expressed in $\widetilde{W}^P_{ij}$.

B. Momentum flow in a general scattering formalism

More useful for potential applications in optical trapping is the change in momentum when partially coherent light is scattered from a microscopic particle. We now derive an expression for momentum flow as applied in a general scattering formalism.

Let us suppose a partially coherent field with electric field $\mathbf{E}^{(i)}(\mathbf{r})$ is incident upon a scattering object localized within a domain D. The interaction produces a scattered electric field $\mathbf{E}^{(s)}(\mathbf{r})$, and the total electric field throughout space is the sum of these two terms,

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{(i)}(\mathbf{r}) + \mathbf{E}^{(s)}(\mathbf{r}). \tag{27}$$

By the use of Eq. (20), we can therefore write the stress tensor of the total electromagnetic field in the form

$$\hat{T}_{ij}(\mathbf{r}) = \hat{T}_{ij}^{(i)}(\mathbf{r}) + \hat{T}_{ij}^{(s)}(\mathbf{r}) + \hat{S}_{ij}(\mathbf{r}), \tag{28}$$

where $\hat{T}_{ij}^{(i)}$ is the stress tensor of the incident field alone, $\hat{T}_{ij}^{(s)}$ is the stress tensor of the scattered field alone, and $\hat{S}_{ij}$ is an interference term which is given by

$$\hat{S}_{ij}(\mathbf{r}) \equiv \langle \hat{T}_{ij}(\mathbf{r},\omega)\rangle = \frac{1}{4\pi}\left[W_{ij}^{is,E}(\mathbf{r},\mathbf{r},\omega) - \frac{1}{2}\delta_{ij}W_{ll}^{is,E}(\mathbf{r},\mathbf{r},\omega)\right] + \frac{1}{4\pi}\left[W_{ij}^{is,B}(\mathbf{r},\mathbf{r},\omega) - \frac{1}{2}\delta_{ij}W_{ll}^{is,B}(\mathbf{r},\mathbf{r},\omega)\right], \tag{29}$$

and we define a mixed incident field/scattered field tensor by

$$W_{ij}^{is,E} \equiv \langle E_i^{(s)*}(\mathbf{r})E_j^{(i)}(\mathbf{r}) + E_i^{(i)*}(\mathbf{r})E_j^{(s)}(\mathbf{r})\rangle, \tag{30}$$

with a similar expression for $W_{ij}^{is,B}$. It is to be noted that the mixed tensor is Hermitian, i.e., that

$$W_{ji}^{is,E*} = W_{ij}^{is,E}. \tag{31}$$

The net momentum flow into the volume D can then be determined by the use of Eq. (10). This formula may be simplified considerably by noting that the incident field, i.e., the field in the absence of any scatterer, will produce no net momentum flow into or out of the volume D. Its contribution vanishes and the flow reduces to two terms,

$$\frac{d\mathbf{P}_{\mathrm{tot}}}{dt}(\omega) = \mathrm{Re}\left\{\int_S [\hat{T}_{ij}^{(s)}(\mathbf{r}) + \hat{S}_{ij}(\mathbf{r})]\cdot\mathbf{n}da\right\}. \tag{32}$$

This equation suggests that the scattered field depends not only on the radiation pattern of the scattered field but also on the spatial correlation between the incident and scattered field.

IV. EXAMPLES

We now apply the results of the previous sections to a few simple examples. We first consider momentum conservation as it applies to certain classes of model sources.

We focus on the momentum flow from certain classes of partially coherent primary sources. The net momentum flow at frequency ω from a spherical surface of radius r may be defined as

$$\frac{d\mathbf{P}_{\mathrm{tot}}}{dt}(\omega) \equiv \mathrm{Re}\{\mathbf{T}\}, \tag{33}$$

where $\mathbf{T}$ is a complex vector with components T_j specified by

$$T_j \equiv \int_S \hat{T}_{ij}u_i da = -\frac{(2\pi)^6k^4}{4\pi}\int_\Omega \widetilde{W}_{lm}^P(-k\mathbf{u},k\mathbf{u},\omega) \times(\delta_{lm} - u_lu_m)u_jd\Omega \tag{34}$$

and $d\Omega$ indicates an infinitesimal solid angle.

First, we consider the class of homogeneous and isotropic sources. The cross-spectral density for this case can be represented by

$$W_{ij}^P(\mathbf{r}_1,\mathbf{r}_2,\omega) = S_P\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2},\omega\right)\mu_{ij}^P(\mathbf{r}_2-\mathbf{r}_1,\omega), \tag{35}$$

where S_P represents the spectral density of the polarization and μ_{ij}^P represents the spectral degree of coherence of the polarization. It is well known (Sec. 3.3 and 3.4 of [18]) that, for an isotropic source, the spectral degree of coherence may be written in the form

$$\mu_{ij}^P(\mathbf{r},\omega) = \delta_{ij}A(r,\omega) + r_ir_jB(r,\omega). \tag{36}$$

The Fourier transform of such a source is readily found to be given by the expression

$$\widetilde{W}_{ij}^P(-k\mathbf{u},k\mathbf{u},\omega) = \widetilde{S}_P(0,\omega)\left[\delta_{ij}\left\{\widetilde{A}(k,\omega) - \frac{1}{k}\frac{d}{dk}\widetilde{B}(k,\omega)\right\} + u_iu_j\left\{\frac{1}{k}\frac{d}{dk}\widetilde{B}(k,\omega) - \frac{d^2}{dk^2}\widetilde{B}(k,\omega)\right\}\right]. \tag{37}$$

The resulting net force from the corresponding stress tensor becomes

$$T_j = -\frac{2(2\pi)^6k^4}{4\pi}\int_\Omega \widetilde{S}_P(0,\omega)\left[\widetilde{A}(k,\omega) - \frac{1}{k}\frac{d}{dk}\widetilde{S}\widetilde{B}(k,\omega)\right]u_jd\Omega = 0. \tag{38}$$

Therefore, there is no net momentum flow for quasihomogeneous sources.

We can partially remove the isotropy of the source polarization and consider an example of a quasihomogeneous source with fixed polarization direction taken (arbitrarily) to lie in the z direction. In this case the cross-spectral density and related physical quantities can be expressed as

$$W_{ij}^P(\mathbf{r}_1,\mathbf{r}_2,\omega) = S_P\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2},\omega\right)\mu_0(|\mathbf{r}_2-\mathbf{r}_1|,\omega)\delta_{i3}\delta_{j3}, \tag{39}$$

$$\widetilde{W}_{ij}^P(-k\mathbf{u},k\mathbf{u},\omega) = \widetilde{S}_P(0,\omega)\widetilde{\mu}_0(k,\omega)\delta_{i3}\delta_{j3}, \tag{40}$$

$$(\delta_{ij} - u_iu_j)\widetilde{W}_{ij}^P(-k\mathbf{u},k\mathbf{u},\omega) = (1-u_3^2)\widetilde{S}_P(0,\omega)\widetilde{\mu}_0(k,\omega). \tag{41}$$

The corresponding net momentum flow becomes

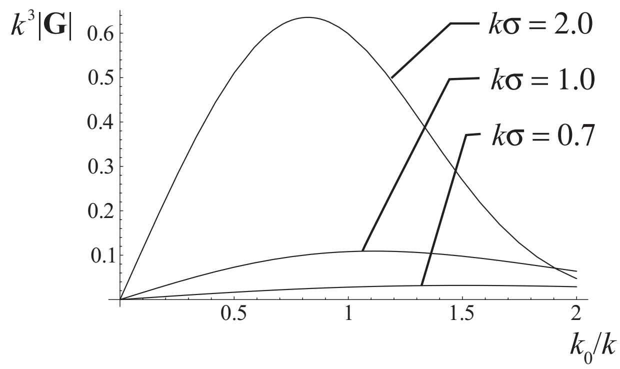

FIG. 1. The (normalized) net momentum flow is shown in terms of the normalized anisotropy parameter k_0/k and the normalized correlation length $k\sigma$.

$$T_j = -\frac{2(2\pi)^6 k^4}{4\pi}\tilde{S}_P(0,\omega)\int_\Omega (1-u_3^2)u_j\tilde{\mu}_0(k,\omega)d\Omega. \quad (42)$$

Because $\tilde{\mu}_0$ is independent of direction $\mathbf{u}$, it can be readily seen that the net momentum flow also vanishes in this case.

One can readily produce examples of anisotropic momentum flow by completely removing the isotropy of the correlations. We now assume a cross-spectral density of the form,

$$W_{ij}^P(\mathbf{r}_1,\mathbf{r}_2,\omega) = S_P\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2},\omega\right)\mu_0(|\mathbf{r}_2-\mathbf{r}_1|,\omega)e^{-i\mathbf{k}_0\cdot(\mathbf{r}_2-\mathbf{r}_1)}\delta_{ij}, \quad (43)$$

where $\mathbf{k}_0$ is a nonzero vector. This source is unpolarized at every point in the source domain. The corresponding net force becomes

$$T_j = -\frac{2(2\pi)^6 k^4}{4\pi}\tilde{S}_P(0,\omega)\int_\Omega \tilde{\mu}_0(|k\mathbf{u}-\mathbf{k}_0|,\omega)u_j d\Omega. \quad (44)$$

This expression has no symmetry with respect to $\mathbf{u}$, and in general will be nonzero. We consider the special case of the Gaussian correlation,

$$\mu_0(\mathbf{r},\omega) = \exp[-r^2/2\sigma^2], \quad (45)$$

and choose $\mathbf{k}_0 = k_0\hat{\mathbf{z}}$. Looking only at the integral of Eq. (44), we find that

$$\begin{aligned}\mathbf{G} &\equiv \int_\Omega \tilde{\mu}_0(|k\mathbf{u}-k_0\hat{\mathbf{z}}|)\mathbf{u}d\Omega \\ &= \frac{\sigma^3}{(2\pi)^{1/2}}e^{-k^2\sigma^2/2}e^{-k_0^2\sigma^2/2}\hat{\mathbf{z}}\int_0^\pi e^{2kk_0\cos\theta\sigma^2/2}\cos\theta\sin\theta d\theta \\ &= \frac{\sigma^3}{(2\pi)^{1/2}}e^{-k^2\sigma^2/2}e^{-k_0^2\sigma^2/2}\hat{\mathbf{z}}\frac{2\sinh\alpha-2\alpha\cosh\alpha}{\alpha^2}, \end{aligned} \quad (46)$$

where

$$\alpha \equiv kk_0\sigma^2. \quad (47)$$

The behavior of the momentum is shown in Fig. 1, as a function of the normalized correlation length $k\sigma$ and as a function of the normalized anisotropy, k_0/k. As one would expect, it can be seen that the net momentum flow vanishes for low anisotropy and for low correlation length and increases as these parameters are increased. Beyond a certain critical anisotropy value which depends upon the correlation length, however, the net flow decreases again. This arises because the oscillations of the exponential for $k_0 > k$ begin to produce primarily evanescent waves.

Even when the net momentum flow of a source is zero, the distribution of momentum flow can depend strongly on the degree of coherence. As a final example, we consider the simple case of two partially coherent dipoles. Let us assume that two statistically identical dipoles $\mathbf{P}=(0,0,P)$ position at $\mathbf{r}_+=(0,0,a/2)$ and $\mathbf{r}_-=(0,0,-a/2)$, respectively. Then the cross-spectral density at frequency ω may be written in the form

$$\mathbf{W}^P = \left\langle \sum_{c,d=+,-}\mathbf{P}_c^*(\mathbf{r}_1)\mathbf{P}_d(\mathbf{r}_2)\right\rangle_\omega = \sum_{c,d}\mu_{cd}\mathbf{P}_c^*(\mathbf{r}_1)\mathbf{P}_d(\mathbf{r}_2), \quad (48)$$

where μ_{cd} is the degree of coherence between dipoles c and d and

$$\mathbf{P}_+(\mathbf{r}) \equiv \mathbf{P}\delta^{(3)}(\mathbf{r}-\mathbf{r}_+), \quad (49)$$

$$\mathbf{P}_-(\mathbf{r}) \equiv \mathbf{P}\delta^{(3)}(\mathbf{r}-\mathbf{r}_-). \quad (50)$$

With this system of dipoles, the momentum flow through a large sphere of radius r and in direction $\mathbf{u}$ may be written, according to Eq. (26), as

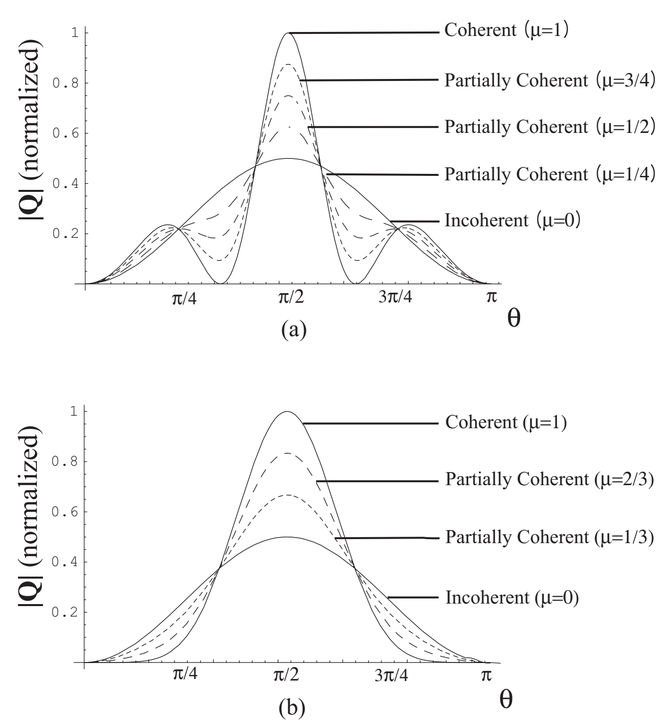

FIG. 2. (Normalized) distribution of momentum flow for a pair of partially coherent dipoles as a function of angle θ from the axis of the dipoles. (a) $ka=2\pi$. (b) $ka=\pi$.

$$\mathbf{u}\cdot\langle\hat{T}(\mathbf{r})\rangle = -\frac{k^4}{4\pi r^2}\sum_{c,d}\mu_{cd}[\mathbf{P}\cdot\mathbf{P}-(\mathbf{u}\cdot\mathbf{P}^*)(\mathbf{u}\cdot\mathbf{P})]\mathbf{u}e^{-ik\mathbf{u}\cdot(\mathbf{r}_c-\mathbf{r}_d)}, \tag{51}$$

where μ_{cd} has the property of $\mu_{cd}=\mu_{dc}^*$ and $\mu_{cc}=1$ for $c,d=+,-$.

This may be simplified to the form

$$\mathbf{u}\cdot\langle\hat{T}(\mathbf{r})\rangle = -\frac{k^4P^2}{4\pi r^2}(1-\cos^2\theta)(2+2\,\mathrm{Re}\{\mu_{+-}e^{-ika\cos\theta}\})\mathbf{u}. \tag{52}$$

For $\mu_{+-}\equiv\mu_0\in\mathrm{Re}$, this may be further simplified to the form

$$\mathbf{Q}(\mathbf{u}) \equiv \mathbf{u}\cdot\langle\hat{T}(\mathbf{r})\rangle = -\frac{k^4P^2}{4\pi r^2}(1-\cos^2\theta) \times[2+2\mu_0\cos(ka\cos\theta)]\mathbf{u}. \tag{53}$$

The angular distribution of momentum flow depends not only upon the separation of the dipoles and the wavelength of emission but also upon the spectral degree of coherence between them. The resulting angular distribution of the change in the momentum can be found in Fig. 2 for various values of μ_0 and two values of ka. It can be seen that in the limit of incoherently radiating dipoles, the distribution of momentum flow becomes independent of ka.

V. CONCLUSIONS

In this paper we have explicitly calculated the formulas relating to momentum flow and momentum conservation for partially coherent fields in the space-frequency representation. The general formulas were applied to the more specialized cases of primary radiation sources and general scattering. Examples of momentum flow for partially coherent sources were given, and these examples demonstrated that the momentum flow of an electromagnetic field depends significantly upon the state of coherence of the source.

These formulas can be used to analyze the net momentum transfer from fields to particles in problems of optical trapping with partially coherent fields. Partial coherence represents an additional "degree of freedom" in the design of electromagnetic systems, and it has been suggested that fields in unusual states of coherence can produce novel trapping devices. We intend to explore the effects of partial coherence on trapping forces in detail in future research.

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