Nonradiating Sources and the Inverse Source Problem

by

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For my parents,

John Gbur

and

Patricia Gbur.

Curriculum Vitae

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Abstract

The existence of radiationless solutions to the inhomogeneous wave equation is demonstrated and important results relating to such solutions are reviewed and proven. Three open questions regarding such *nonradiating sources* are introduced: (1) the experimental confirmation of the existence of nonradiating sources, (2) consequences of their existence, and (3) the existence of unique solutions to the inverse source problem. Methods of experimentally producing nonradiating sources are investigated by examining nonpropagating excitations on a vibrating string. Some unusual consequences of the existence of nonradiating sources are found. Finally, it is shown that no nonradiating quasi-homogeneous sources exist – the inverse source problem for quasi-homogeneous sources is therefore uniquely solvable, and methods of performing the inversion are described.

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Chapter 1

Introduction and overview

> *"Here is a solid body which we touch, but which we cannot see. The fact is so unusual that it strikes us with terror. Is there no parallel, though, for such a phenomenon? ... It is not* theoretically impossible*, mind you, to make a glass which shall not reflect a single ray of light – a glass so pure and homogeneous in its atoms that the rays from the sun will pass through it as they do through the air, refracted but not reflected ..."*
>
> *"That's all very well, Hammond, but these are inanimate substances. Glass does not breathe, air does not breathe.* This *thing has a heart that palpitates – a will that moves it – lungs that play, and inspire and respire."*
>
> from *What Was It?*, by Fitz-James O'Brien (1828–1862) [1]

The possibility of invisible objects has intrigued both scientists and nonscientists for well over a century. Though only the animate variety seems to capture the imagination of fiction writers, the existence of any such objects has important practical and physical consequences.

We will distinguish between two types of "invisible" objects. The first class consists of scatterers (objects with an inhomogeneous index of refraction) which do not scatter incident plane waves for one or several directions of incidence.[1] The second class are primary radiation sources (radiating atoms, for instance, or time-varying charge-current distributions) which do not, in fact, radiate power. It is the latter class with which this thesis will be primarily concerned with.

Such *nonradiating sources*[2] have had a colorful history. Their origins lay in the theory of the extended rigid electron, initiated by Sommerfeld ([2],[3]) and others[3] in the early 1900s. Evidently Ehrenfest [5] was the first researcher to recognize explicitly that radiationless motions of such extended charge distributions are possible. Much later, in 1933, Schott [6] demonstrated that it is possible for a rigid charged spherical shell to move in a periodic orbit without radiating. Soon after (in 1937) Schott [7] also showed that, under certain additional constraints, it is possible for the charged shell to move periodically in the absence of external forces. In 1948 Bohm and Weinstein [8] extended Schott's treatment of radiationless modes and self-oscillating modes to other spherically symmetric charge distributions. Goedecke [9] later (in 1964) demonstrated that at least one asymmetric, spinning, extended charge distribution exists which does not radiate.

Many of these early authors postulated that nonradiating charge distributions might be used as models for elementary particles. Schott suggested that such objects might provide a stable model of the neutron and possibly of other atomic nuclei;

[1] These so-called 'nonscattering scatterers' will be discussed in section 2.4.

[2] In three-dimensional wave problems, the term *nonradiating source* is used to describe sources which generate no power and produce no field outside their domain of support. In the theory of the vibrating string which we will discuss in chapter 3, the term *nonpropagating excitation* has come to be used in describing such phenomena. Occasionally we will use the term *localized excitation* to describe both these cases, and others similar to them.

[3] A nice description of classical electron models is given by Pearle in [4].

Bohm and Weinstein suggested that a nonradiating source might explain the muon as an excited self-oscillating state of the electron. Goedecke suggested that such distributions might lead to a "theory of nature" in which all stable particles or aggregates are described as nonradiating charge-current distributions.

In more modern times, the existence or nonexistence of nonradiating sources has been shown to be of fundamental importance to the solution of the inverse source problem.[4] An early paper by Friedlander in 1973 [10] explored the mathematical properties of nonradiating sources and discussed some circumstances under which the inverse problem is unique. At about the same time, Devaney and Wolf [11] investigated monochromatic nonradiating classical current distributions. In 1977 Bleistein and Cohen [12] explicitly showed how the existence of nonradiating sources implies the nonuniqueness of the inverse source problem. Nevertheless, controversy over this result lingered for some time.[5]

About the same time that the inverse source problem was found to be nonunique for monochromatic sources, the same problem for partially coherent sources came to be investigated. In 1979 Hoenders and Baltes [14] developed a criterion for nonradiating stochastic sources, and determined a mathematical method to construct such sources. In 1984 Devaney and Wolf [15] obtained a simpler criterion for nonradiating stochastic sources. In 1985 LaHaie [16] explicitly investigated the inverse source problem for partially coherent sources and determined that it is, in fact, nonunique. Two papers, one in 1979 by Devaney [17] and another in 1986 by LaHaie [18], sug-

[4]The inverse source problem may be loosely defined as the problem of determining the spatial structure of a source from measurements of the field radiated by that source. It will be discussed in section 2.3.

[5]For a particularly heated argument on the question of uniqueness, see [13] and the comments which follow it.

gested that the inverse source problem was, however, unique for quasi-homogeneous sources, but this work seems to have received little attention.

No doubt because of the nonuniqueness of the inverse source problem, most of the papers on nonradiating sources since the early 1980s have focused on the mathematical properties of such sources. Of these, the work of Kim and Wolf [19] and Gamliel, Kim, Nachman and Wolf [20] have provided some of the most worthwhile results. Much work has focused on the description of the "radiating" and "nonradiating" parts of a source ([21], [22], [23]). Other researchers have examined so-called "minimum energy" solutions to the inverse source problem ([24],[25]).

Despite the large amount of research that has been done on nonradiating sources, several problems remain unsolved. Foremost among these is that, as yet, no nonradiating source has been produced experimentally. Such a confirmation seems desirable to justify the steady stream of theoretical papers on this subject that continue to be published. Furthermore, beyond the few papers mentioned already, very little has been done to describe the properties of partially coherent nonradiating sources. Perhaps most important, it is still unclear for what realistic classes of sources, if any, the inverse source problem is unique.

This thesis attempts to answer and investigate each of the three questions just mentioned. First, a review of some important results in the theory of nonradiating sources is described in chapter 2.

One-dimensional nonradiating sources, which are analogous to nonpropagating excitations on a vibrating string, are examined in chapter 3. Such nonpropagating excitations would presumably be easier to produce experimentally than a three-dimensional nonradiating source. The simplicity of the one-dimensional problem makes it possible to treat in a straightforward way complicating effects, such as damping, that would appear in a realistic experimental setup.

In chapter 4, some unusual coherence effects are considered which may be considered a consequence of the existence of partially coherent nonradiating sources. Among these are the generation of a fully coherent field by a partially coherent source, and the generation of an almost completely polarized field by an unpolarized electromagnetic source.

Finally, in chapter 5, the quasi-homogeneous approximation, used often in coherence theory, is examined in detail for three-dimensional, primary sources. As a result of this investigation it is shown that there can be no nonradiating quasi-homogeneous sources. This result broadens greatly the class of sources for which the inverse source problem is unique, and possible inversion methods are described.

Chapter 2

Summary of results from the theory of nonradiating sources

A vast amount of information has been amassed about the properties of nonradiating sources. This chapter is intended to serve as a review of those results necessary for the understanding of the rest of this thesis, as well as a justification of why research on such sources is interesting and important. Furthermore, some description is given of other objects which may be said to be "invisible" or to "localize light", and the differences between these objects and the traditional nonradiating sources are discussed.

2.1 Monochromatic nonradiating sources

We consider a scalar source $Q(\mathbf{r}, t)$, confined to a domain D, which produces a field $U(\mathbf{r}, t)$. $Q(\mathbf{r}, t)$ may represent an acoustical wave source or, with a slight modification to the theory, an electromagnetic source. The field and the source are related by the scalar wave equation,

$$\nabla^2 U(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}U(\mathbf{r},t) = -4\pi Q(\mathbf{r},t). \tag{2.1}$$

Let us suppose that the source $Q(\mathbf{r},t)$, and hence the field $U(\mathbf{r},t)$, is monochromatic, i.e. that

$$Q(\mathbf{r},t) = Re\left\{q(\mathbf{r})e^{-\mathrm{i}\omega t}\right\}, \tag{2.2a}$$

$$U(\mathbf{r},t) = Re\left\{u(\mathbf{r})e^{-\mathrm{i}\omega t}\right\}, \tag{2.2b}$$

where $q(\mathbf{r})$ and $u(\mathbf{r})$ are, in general, complex functions of position, and Re denotes the real part. On substitution of Eqs. (2.2) into the wave equation, Eq. (2.1), the wave equation reduces to the inhomogeneous Helmholtz equation,

$$\nabla^2 u(\mathbf{r}) + k^2 u(\mathbf{r}) = -4\pi q(\mathbf{r}), \tag{2.3}$$

where

$$k = \frac{\omega}{c}. \tag{2.4}$$

The solution to this equation is well-known to be ([26], section 6.6, or [27], section 2.1)

$$u(\mathbf{r}) = \int_D q(\mathbf{r}')\frac{e^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3r'. \tag{2.5}$$

We are interested in finding source distributions $q(\mathbf{r})$ which do not radiate, i.e. which do not generate power in the far zone. Far away from the source,

$$\frac{e^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \sim \frac{e^{\mathrm{i}kr}}{r}e^{-\mathrm{i}k\mathbf{s}\cdot\mathbf{r}'}, \quad (kr \to \infty), \tag{2.6}$$

where $r \equiv |\mathbf{r}|$ and $\mathbf{s} \equiv \mathbf{r}/r$ is a unit vector in the direction of $\mathbf{r}$ (see Fig. 2.1). On substituting from Eq. (2.6) into Eq. (2.5), we find that the field in the far zone of the source may be expressed in the form

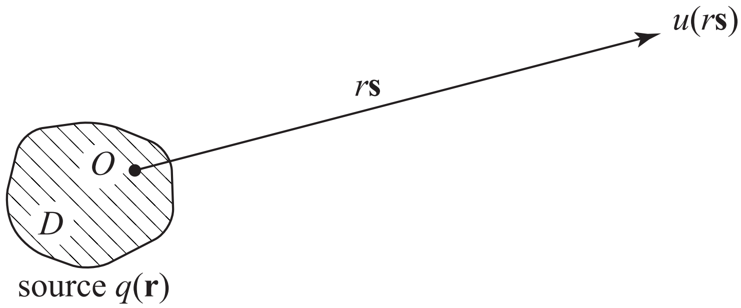


Figure 2.1: Illustrating notation relating to monochromatic radiation sources.

$$u(r\mathbf{s}) \sim (2\pi)^3 \frac{e^{ikr}}{r} \tilde{q}(k\mathbf{s}), \tag{2.7}$$

where

$$\tilde{q}(\mathbf{K}) = \frac{1}{(2\pi)^3} \int_D q(\mathbf{r}') e^{-i\mathbf{K}\cdot\mathbf{r}'} \mathrm{d}^3 r' \tag{2.8}$$

is the three-dimensional Fourier transform of the source distribution $q(\mathbf{r})$. The function $\tilde{q}(k\mathbf{s})$ is often referred to as the *radiation pattern* of the source. From Eq. (2.7), we obtain the following theorem:

Theorem 2.1 *A source will be nonradiating, i.e. it will not produce any power in the far zone of the source, if and only if*

$$\tilde{q}(k\mathbf{s}) = 0 \quad \textit{for all directions } \mathbf{s}. \tag{2.9}$$

Condition (2.9) is the most-often mentioned requirement that a nonradiating source distribution $q(\mathbf{r})$ must satisfy. It states that the three-dimensional Fourier transform of the source distribution must vanish on a sphere of radius equal to the wavenumber k of the incident radiation.

As a simple example,[1] we consider a homogeneous source of radius a, i.e.

$$q(\mathbf{r}) = \begin{cases} q_0 & \text{when } r \leq a, \\ 0 & \text{when } r > a. \end{cases} \tag{2.10}$$

On substituting from Eq. (2.10) into Eq. (2.9) and carrying out the integration, the nonradiating condition (2.9) takes on the simple form

$$j_1(ka) = 0, \tag{2.11}$$

where $j_1(x)$ is the first order spherical Bessel function. A homogeneous spherical source is therefore nonradiating only if ka is a zero of the first order spherical Bessel function.

The origin of the nonradiating phenomenon may be described as an unusual interference effect involving the field generated by every element of the source. We may see this explicitly for the homogeneous sphere by considering it to consist of two pieces arranged concentrically, a sphere $q_1(\mathbf{r})$ of radius a_1 and a spherical shell $q_2(\mathbf{r})$ of inner radius a_1 and outer radius a_2 (see Fig. 2.2). Let us assume that ka_2 is the first zero of the first spherical Bessel function, $j_1(x)$. It can be seen by use of Eq. (2.7) that the far zone fields of the individual pieces are given by

$$u_1(\mathbf{r}) = q_0 \frac{e^{\mathrm{i}kr}}{r} \frac{a_1^2}{2\pi^2 k} j_1(ka_1), \tag{2.12a}$$

$$u_2(\mathbf{r}) = q_0 \frac{e^{\mathrm{i}kr}}{r} \left(\frac{a_2^2}{2\pi^2 k} j_1(ka_2) - \frac{a_1^2}{2\pi^2 k} j_1(ka_1) \right) = -u_1(\mathbf{r}), \tag{2.12b}$$

where the last equation was simplified by the fact that ka_2 is a zero of the function j_1. Because ka_2 is the *first* zero of the function j_1, it is clear from Eq. (2.12a) that the individual fields u_1 and u_2 are nonzero, though their sum is zero, i.e. the fields

[1] This example was originally described in [19].

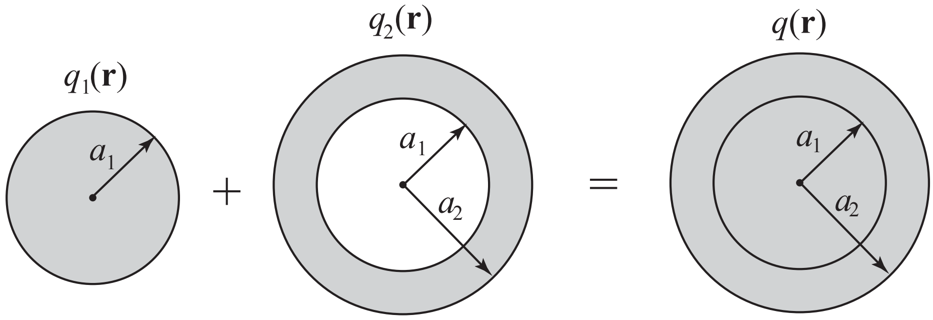


Figure 2.2: Showing the two pieces of the nonradiating source $q(\mathbf{r})$ and their arrangement.

u_1 and u_2 destructively interfere in the far zone of the source. It is to be noted, however, that the choice of a_1 is arbitrary, and the above argument is valid for any a_1 satisfying $a_1 < a_2$. It is not quite appropriate, therefore, to speak of the field of piece 1 of the source destructively interfering with the field of piece 2 of the source; the nonradiating effect is produced by the mutual interference of the fields from all points within the source domain.

We will now briefly describe a few results from the theory of monochromatic nonradiating sources which we will find useful later.

Theorem 2.2 *A necessary and sufficient condition for a source distribution to be nonradiating is that the equation*

$$\int_D q(\mathbf{r}')j_0(k|\mathbf{r}-\mathbf{r}'|)\mathrm{d}^3r' = 0 \tag{2.13}$$

be satisfied for all values of $\mathbf{r}$ *[12, 10].*

We first show that Eq. (2.13) follows from the nonradiating condition, Eq. (2.9). To see this, we multiply Eq. (2.9) by $e^{ik\mathbf{s}\cdot\mathbf{r}}$, and integrate the resulting equation over all real directions $\mathbf{s}$. This gives

$$\int_{(4\pi)} \int_D q(\mathbf{r}') e^{\mathrm{i}k\mathbf{s}\cdot(\mathbf{r}-\mathbf{r}')} \mathrm{d}^3 r' \mathrm{d}\Omega = 0, \tag{2.14}$$

where $\mathrm{d}\Omega$ is an element of solid angle, and the Ω-integration is over the complete 4π solid angle. By using the well-known identity ([28], footnote on p. 123),

$$j_0(k|\mathbf{r}-\mathbf{r}'|) = \frac{1}{4\pi} \int_{(4\pi)} e^{\mathrm{i}k\mathbf{s}\cdot(\mathbf{r}-\mathbf{r}')} \mathrm{d}\Omega, \tag{2.15}$$

Eq. (2.13) follows immediately. To demonstrate that Eq. (2.9) follows from Eq. (2.13), we substitute from Eq. (2.15) into Eq. (2.13), which results again in Eq. (2.14). Let us multiply this equation by its complex conjugate, giving the new equation

$$\iint_{(4\pi)} \iint_D q^*(\mathbf{r}') q(\mathbf{r}'') e^{-\mathrm{i}k\mathbf{s}'\cdot(\mathbf{r}-\mathbf{r}')} e^{\mathrm{i}k\mathbf{s}''\cdot(\mathbf{r}-\mathbf{r}'')} \mathrm{d}^3 r' \mathrm{d}^3 r'' \mathrm{d}\Omega' \mathrm{d}\Omega'' = 0. \tag{2.16}$$

We choose $\mathbf{r}$ to lie in the x, y-plane of some Cartesian coordinate system. Let (s'_x, s'_y, s'_z) and (s''_x, s''_y, s''_z) represent the components of $\mathbf{s}'$, $\mathbf{s}''$, respectively, in this coordinate system. Equation (2.16) may then be expressed in the form

$$\iint_{(4\pi)} \iint_D q^*(\mathbf{r}') e^{\mathrm{i}k\mathbf{s}'\cdot\mathbf{r}'} q(\mathbf{r}'') e^{-\mathrm{i}k\mathbf{s}''\cdot\mathbf{r}''} e^{-\mathrm{i}k[(s'_x - s''_x)x + (s'_y - s''_y)y]} \mathrm{d}^3 r' \mathrm{d}^3 r'' \mathrm{d}\Omega' \mathrm{d}\Omega'' = 0. \tag{2.17}$$

We now integrate with respect to $\mathbf{r}$ over the entire x, y-plane; using the Fourier representation of a Dirac delta function,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}(a-b)x} \mathrm{d}x = \delta(a-b), \tag{2.18}$$

equation (2.17) becomes

$$\iint_{(4\pi)} \iint_D q^*(\mathbf{r}') e^{\mathrm{i}k\mathbf{s}'\cdot\mathbf{r}'} q(\mathbf{r}'') e^{-\mathrm{i}k\mathbf{s}''\cdot\mathbf{r}''} \delta(ks'_x - ks''_x) \delta(ks'_y - ks''_y) \mathrm{d}^3 r' \mathrm{d}^3 r'' \mathrm{d}\Omega' \mathrm{d}\Omega'' = 0. \tag{2.19}$$

Integrating over all directions of $\mathbf{s}'$, $\mathbf{s}''$ equates the x and y components of those vectors. Because $\mathbf{s}'$ and $\mathbf{s}''$ are unit vectors, this implies that $\mathbf{s}' = \mathbf{s}''$. Then, using Eq. (2.8), Eq. (2.19) may be further simplified, becoming

$$\int_{(4\pi)} |\tilde{q}(k\mathbf{s}'')|^2 \mathrm{d}\Omega'' = 0. \tag{2.20}$$

The integrand of Eq. (2.20) is nonnegative. This equation therefore implies that $\tilde{q}(k\mathbf{s})$ must satisfy Eq. (2.9), and Theorem 2.2 follows.

It is interesting to note that Theorem 2.2 implies that the retarded and advanced fields[2] of a nonradiating source are equal. From the definition of j_0 it follows that

$$j_0(kr) = \frac{e^{\mathrm{i}kr} - e^{-\mathrm{i}kr}}{2\mathrm{i}kr}. \tag{2.21}$$

On substituting from this equation into Eq. (2.13), the latter equation may be rewritten as

$$\int_D q(\mathbf{r}') \frac{e^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \mathrm{d}^3 r' = \int_D q(\mathbf{r}') \frac{e^{-\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \mathrm{d}^3 r'. \tag{2.22}$$

This expression demonstrates that the retarded and advanced fields of a nonradiating source $q(\mathbf{r})$ are equal.

Theorem 2.3 *The field of a nonradiating source vanishes everywhere outside the domain of the source.*[3]

To prove this result, we return to the solution to the inhomogeneous Helmholtz equation given by (2.5), i.e.

$$u(\mathbf{r}) = \int_D q(\mathbf{r}') \frac{e^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \mathrm{d}^3 r'. \tag{2.23}$$

[2]See [26], section 6.6 for a discussion of retarded and advanced fields.

[3]This result was first proven in [10], Theorem 3.1, although it was indirectly stated in [29]. Similar results relating to the field scattered by a scattering potential of finite support were described in [30]. We follow closely the derivation of the latter.

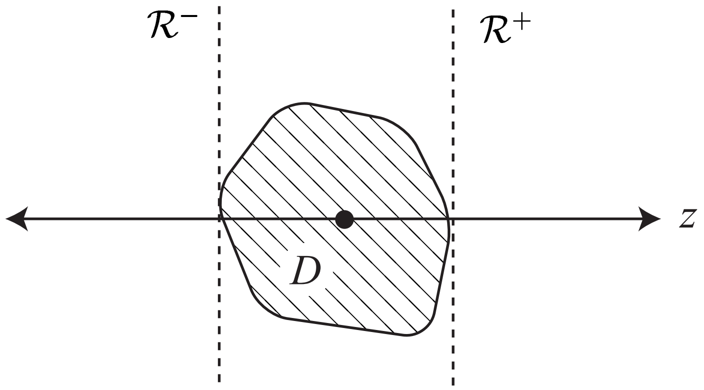


Figure 2.3: Illustrating the regions $\mathcal{R}^+$ and $\mathcal{R}^-$ defined in the proof of Theorem 2.3.

Let us choose a direction to be the z-axis, and denote by $\mathcal{R}^+$ and $\mathcal{R}^-$ the regions to the right and left of the source domain D with respect to the z-axis, respectively (see figure 2.3). Now we may expand the Green's function in the regions $\mathcal{R}^+$, $\mathcal{R}^-$ by use of the Weyl representation ([28], section 3.2.4),

$$\frac{e^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mathrm{i}k}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\beta\int_{C^\pm}\mathrm{d}\alpha\sin\alpha e^{\mathrm{i}[k\mathbf{s}\cdot(\mathbf{r}-\mathbf{r}')]}, \tag{2.24}$$

where

$$s_x = \sin\alpha\cos\beta \quad , \quad s_y = \sin\alpha\sin\beta \quad , \quad s_z = \cos\alpha, \tag{2.25}$$

and the integration over α is taken on the *complex* curve C^+ if $\mathbf{r}$ lies in $\mathcal{R}^+$ and on the curve C^- if $\mathbf{r}$ lies in $\mathcal{R}^-$ (see figure 2.4). On substituting from the Weyl expansion (2.24) into Eq. (2.23), we may express the field $u(\mathbf{r})$ in the regions $\mathcal{R}^+$ and $\mathcal{R}^-$ in the form

$$u(\mathbf{r}) = \frac{\mathrm{i}k}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\beta\int_{C^\pm}\mathrm{d}\alpha\sin\alpha A(\alpha,\beta)e^{\mathrm{i}k\mathbf{s}\cdot\mathbf{r}}, \tag{2.26}$$

where

$$A(\alpha,\beta) = \int_D q(\mathbf{r}')e^{-\mathrm{i}k\mathbf{s}\cdot\mathbf{r}'}\mathrm{d}^3r'. \tag{2.27}$$

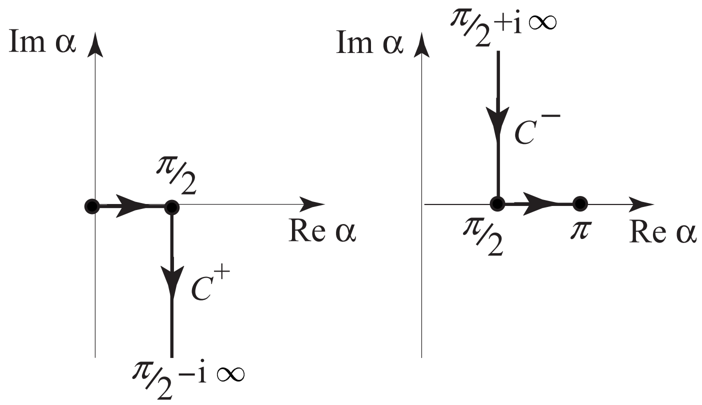


Figure 2.4: Depicting the complex paths of integration C^+ and C^-.

Let us compare Eq. (2.27) with the nonradiating condition (2.9). It is seen that if the source is nonradiating, then

$$A(\alpha, \beta) = 0 \quad \text{for all} \quad 0 \leq \alpha \leq \pi/2, \quad 0 \leq \beta < 2\pi. \tag{2.28}$$

Equation (2.28) demonstrates that if the source is nonradiating, then the function $A(\alpha, \beta)$ vanishes over a two-dimensional region of (α, β) space. We will use this property shortly. The function $A(\alpha, \beta)$ may be expressed explicitly in the form

$$A(\alpha, \beta) = \int_D q(\mathbf{r}') e^{-\mathrm{i}k[(\sin\alpha\cos\beta)x' + (\sin\alpha\sin\beta)y' + (\cos\alpha)z']} \mathrm{d}^3 r'. \tag{2.29}$$

Let us consider the behavior of this function for all complex values of α, β. It is not difficult to show that, for any β, $A(\alpha, \beta)$ is an entire analytic function of α, and that, for any α, $A(\alpha, \beta)$ is an entire analytic function of β. It therefore follows from a theorem of analysis in several complex variables ([31], p. 143) that $A(\alpha, \beta)$ is an entire analytic function in two complex variables, α and β. It therefore cannot vanish over a continuous region of dimension equal to or greater than that of a surface, unless it vanishes identically. We have seen, though, that if the source is nonradiating, Eq. (2.28) is satisfied, and $A(\alpha, \beta)$ therefore vanishes for all values of α, β. Therefore, by Eq. (2.26), the field vanishes identically in the half-spaces $\mathcal{R}^+$ and $\mathcal{R}^-$.

We may repeat this process with different choices of the z-axis; in this way, it is possible to demonstrate that the field vanishes up to a convex surface enclosing the source domain. If the source domain has concavities within it, analytic continuation methods can be used to demonstrate that the field vanishes also within these concavities (see [32], Theorem 3.5). Hence Theorem 2.3 follows.

We have shown that the field of a nonradiating source vanishes everywhere outside the domain of the source. Some authors have suggested that nonradiating sources might be probed by the methods of near-field optics [33]. Theorem 2.3 demonstrates that this is, in fact, not possible – nonradiating sources possess neither evanescent nor homogeneous external fields. Moreover, it is clear that the radiated field must contain *both* evanescent and homogeneous waves, if it does radiate. Thus, within the scalar theory, purely evanescent radiation sources are not possible.[4]

Furthermore, our derivation has shown that the radiation pattern of a source in the far zone cannot vanish over a continuous finite solid angle of directions unless it vanishes identically. Thus a source must radiate in "almost all" directions (in the sense of measure theory) or not radiate at all.

Theorem 2.4 *A bounded nonradiating source distribution* $q(\mathbf{r})$ *of finite support and the field* $u(\mathbf{r})$ *that it generates are related by the inhomogeneous Helmholtz equation,*

[4]However, this is possible for electromagnetic sources. See the discussion of such sources at the end of this section.

Eq. (2.3), subject to the boundary conditions

$$u(\mathbf{r})|_{\mathbf{r}\in S} = 0, \tag{2.30a}$$

$$\left.\frac{\partial u(\mathbf{r})}{\partial n}\right|_{\mathbf{r}\in S} = 0, \tag{2.30b}$$

where S is the boundary of the source domain and $\partial/\partial n$ denotes differentiation along the outward normal [20, 19].

This theorem is perhaps the most valuable theorem regarding nonradiating sources, for it gives an easy-to-apply method of constructing such sources for any domain: one only needs to determine a function $f(\mathbf{r})$ which is continuous, has a continuous first derivative, and satisfies the boundary conditions (2.30). The function $f(\mathbf{r})$ is then the field of a nonradiating source, and the source itself can be readily determined by the use of the inhomogeneous Helmholtz equation (2.3), as

$$q(\mathbf{r}) = -\frac{1}{4\pi}\left(\nabla^2 + k^2\right) f(\mathbf{r}). \tag{2.31}$$

Theorem 2.4 also demonstrates the existence of nonradiating sources for any reasonably well-behaved, connected source domain, for it is always possible to create a function $f(\mathbf{r})$ which satisfies the continuity requirements and the boundary conditions (2.30). Nonradiating cubes, pyramids, and toroids are therefore derivable, and explicit expressions may be found for their source distributions.

Because of the complexity of the derivation, we will not explicitly prove Theorem 2.4; its proof follows from Theorem 2.3 and from the continuity of the field and its gradient. We will, however, show that it is true for the homogeneous source given by Eq. (2.10).[5] The field within the homogeneous source can be found by the use

[5]This derivation follows from [19].

of the multipole expansion ([26], p. 742),

$$\frac{e^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = 4\pi\mathrm{i}k \sum_{l=0}^{\infty}\sum_{m=-l}^{l} j_l(kr_<)h_l^{(1)}(kr_>)Y_l^m(\theta,\phi)Y_l^{m*}(\theta',\phi'), \tag{2.32}$$

where j_l is the lth order spherical Bessel function, $h_l^{(1)}$ is the lth order spherical Hankel function of the first kind, and Y_l^m are the spherical harmonics. The variables $r_<$ and $r_>$ denote the smaller and larger of the distances $r = |\mathbf{r}|$ and $r' = |\mathbf{r}'|$, respectively. By substituting from Eqs. (2.32) and (2.10) into (2.5), the field within the homogeneous sphere (when the nonradiating condition Eq. (2.11) is satisfied) is readily found to be

$$u(\mathbf{r}) = \frac{4\pi q_0}{k^2}\left\{\frac{a}{r}\cos[k(a-r)] - \frac{1}{kr}\sin[k(a-r)] - 1\right\}, \tag{2.33}$$

and its normal derivative is

$$\begin{aligned}\frac{\mathrm{d}u(\mathbf{r})}{\mathrm{d}r} = \frac{4\pi q_0}{k^2} & \left\{-\frac{a}{r^2}\cos[k(a-r)] + \frac{ka}{r}\sin[k(a-r)]\right. \\ & \left. + \frac{1}{kr^2}\sin[k(a-r)] + \frac{1}{r}\cos[k(a-r)]\right\}. \end{aligned} \tag{2.34}$$

On setting $r = a$ in Eqs. (2.33) and (2.34), it is clear that Theorem 2.4 is satisfied.

In concluding this section we note that, with little modification, the results demonstrated above also apply to electromagnetic sources and fields [9, 11, 34]. If there exists a monochromatic current distribution $\mathbf{j}(\mathbf{r})$ localized to a domain D, the space-dependent parts of the electric field $\mathbf{E}(R\mathbf{s})$ and the magnetic field $\mathbf{H}(R\mathbf{s})$ in a direction $\mathbf{s}$ and at a distance R in the far zone of the source are given by ([35], Eqs. (4.4), (4.5) and (4.19))

$$\mathbf{E}(R\mathbf{s}) = -(2\pi)^3\frac{\mathrm{i}k}{c}\left(\mathbf{s}\times[\mathbf{s}\times\tilde{\mathbf{j}}(k\mathbf{s})]\right)\frac{e^{\mathrm{i}kR}}{R}, \tag{2.35a}$$

$$\mathbf{H}(R\mathbf{s}) = (2\pi)^3\frac{\mathrm{i}k}{c}\mathbf{s}\times\tilde{\mathbf{j}}(k\mathbf{s})\frac{e^{\mathrm{i}kR}}{R}, \tag{2.35b}$$

where

$$\tilde{\mathbf{j}}(\mathbf{K}) = \frac{1}{(2\pi)^3}\int_D \mathbf{j}(\mathbf{r}')e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}'}\mathrm{d}^3r' \tag{2.36}$$

is the three-dimensional spatial Fourier transform of the current density. From Eqs. (2.35) it should be evident that the current distribution will not radiate if

$$\tilde{\mathbf{j}}(k\mathbf{s}) = 0 \quad \text{for all directions } \mathbf{s}. \tag{2.37}$$

This nonradiating condition is comparable to that of the scalar radiation source, Eq. (2.9), and theorems about nonradiating electromagnetic sources which are analogues to those of the scalar case may be derived [11].

The vectorial nature of the electromagnetic problem introduces another class of sources which do not generate power, as we now show. Let us suppose that the current density of the source has the form

$$\mathbf{j}(\mathbf{r}) = \mathbf{r}f(r), \tag{2.38}$$

where $f(r)$ ($r \equiv |\mathbf{r}|$) is a spherically symmetric, continuous function. The current is then purely radial, and the sphere might be described to be "pulsating".

On substituting from Eq. (2.38) into Eq. (2.36), and noting that

$$\mathbf{r}e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}} = \mathrm{i}\nabla_K e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}}, \tag{2.39}$$

we may express the spatial Fourier transform of the current density as

$$\begin{aligned}\tilde{\mathbf{j}}(\mathbf{K}) &= \frac{\mathrm{i}}{(2\pi)^3}\nabla_K\int_D f(r')e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}'}\mathrm{d}^3r' \\ &= \mathrm{i}\nabla_K\tilde{f}(K),\end{aligned} \tag{2.40}$$

where $\tilde{f}(K)$ is the three-dimensional Fourier transform of $f(r)$. Therefore

$$\tilde{\mathbf{j}}(k\mathbf{s}) = \mathrm{i}\nabla_K\tilde{f}(K)|_{\mathbf{K}=k\mathbf{s}} = \mathrm{i}\left(\frac{\partial}{\partial K}\tilde{f}(K)|_{K=k}\right)\mathbf{s}. \tag{2.41}$$

It can be seen that

$$\mathbf{s} \times \tilde{\mathbf{j}}(k\mathbf{s}) = \mathbf{s} \times [\mathbf{s} \times \tilde{\mathbf{j}}(k\mathbf{s})] = 0; \tag{2.42}$$

from Eqs. (2.35) it is evident that the electric and magnetic fields vanish in the far zone of the current density. Such pulsating spheres do not radiate because they produce purely longitudinal fields; however, unlike the scalar nonradiating sources that we have described previously, pulsating spheres will produce fields that can be detected in the near zone of the source.

2.2 Partially coherent nonradiating sources

In the previous section the existence of monochromatic nonradiating sources was demonstrated; we now consider the existence of partially coherent[6] nonradiating sources, for which the source function $Q(\mathbf{r}, t)$ is a random function of time t. We assume these fluctuations to be stationary, at least in the wide sense ([28], section 2.2).

The *mutual coherence function* of the source distribution, defined by the expression

$$\Gamma_Q(\mathbf{r}_1, \mathbf{r}_2, \tau) = \langle Q^*(\mathbf{r}_1, t) Q(\mathbf{r}_2, t + \tau) \rangle \,, \tag{2.43}$$

the angular brackets denoting ensemble averaging, describes the correlation of the source fluctuations at pairs of points $\mathbf{r}_1$ and $\mathbf{r}_2$ within the source at times t and $t+\tau$. It will be more convenient, however, to work in the space-frequency representation. For this purpose we consider the cross-spectral density $W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega)$ of the source

[6] For a complete description of classical coherence theory and the coherence concepts discussed in this section, see [28]; for a shorter description, see [36], chapter 10.

distribution, defined as the Fourier transform of the mutual coherence function,

$$W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma_Q(\mathbf{r}_1, \mathbf{r}_2, \tau) e^{\mathrm{i}\omega\tau} \mathrm{d}\tau. \tag{2.44}$$

We may also define a cross-spectral density $W_U(\mathbf{r}_1, \mathbf{r}_2, \omega)$ of the field $U(\mathbf{r}, t)$ in a similar manner. In the far zone of the source domain, the cross-spectral density of the field and that of the source may then be shown to be related by the formula ([28], section 4.4.5)

$$W_U(R_1\mathbf{s}_1, R_2\mathbf{s}_2, \omega) = \frac{(2\pi)^6}{R_1 R_2} e^{\mathrm{i}k(R_2 - R_1)} \tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2, \omega), \tag{2.45}$$

where $\mathbf{s}_1^2 = \mathbf{s}_2^2 = 1$, and

$$\tilde{W}_Q(\mathbf{K}_1, \mathbf{K}_2, \omega) = \frac{1}{(2\pi)^6} \int_D \int_D W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) e^{-\mathrm{i}k(\mathbf{K}_1 + \mathbf{K}_2)} \mathrm{d}^3 r_1 \mathrm{d}^3 r_2 \tag{2.46}$$

is the six-dimensional spatial Fourier transform of the cross-spectral density.

From Eq. (2.45), it seems that the following condition is necessary and sufficient for a source to be nonradiating at a frequency ω:

$$\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2, \omega) = 0 \quad \text{for all real unit vectors } \mathbf{s}_1, \mathbf{s}_2. \tag{2.47}$$

A simpler condition, however, may be stated for partially coherent nonradiating sources, as shown in the following theorem.

Theorem 2.5 *A partially coherent source is nonradiating at a given frequency ω if its cross-spectral density, $W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega)$, satisfies the condition [15]*

$$\tilde{W}_Q(-k\mathbf{s}, k\mathbf{s}, \omega) = 0 \quad \textit{for all } \mathbf{s}. \tag{2.48}$$

This theorem can be readily proven if we use the non-negative definiteness property ([28], section 4.3.2) of the cross-spectral density, i.e. the condition that for any well-behaved complex function $f(\mathbf{r})$, the cross-spectral density satisfies the inequality

$$\int_D \int_D W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) f^*(\mathbf{r}_1) f(\mathbf{r}_2) \mathrm{d}^3 r_1 \mathrm{d}^3 r_2 \geq 0. \tag{2.49}$$

We consider the following particular choice of $f(\mathbf{r})$:

$$f(\mathbf{r}) = a e^{\mathrm{i}k\mathbf{s}_1 \cdot \mathbf{r}} + b e^{\mathrm{i}k\mathbf{s}_2 \cdot \mathbf{r}}, \tag{2.50}$$

where a and b are arbitrary complex constants. On substituting from Eq. (2.50) into the non-negative definiteness condition (2.49) and using Eq. (2.46), condition (2.49) takes on the form

$$|a|^2 \tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_1, \omega) + |b|^2 \tilde{W}_Q(-k\mathbf{s}_2, k\mathbf{s}_2, \omega) + a^* b \tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2, \omega) + b^* a \tilde{W}_Q(-k\mathbf{s}_2, k\mathbf{s}_1, \omega) \geq 0. \tag{2.51}$$

We may express this condition in a matrix form as follows,

$$\begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} \tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_1, \omega) & \tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2, \omega) \\ \tilde{W}_Q(-k\mathbf{s}_2, k\mathbf{s}_1, \omega) & \tilde{W}_Q(-k\mathbf{s}_2, k\mathbf{s}_2, \omega) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \geq 0. \tag{2.52}$$

A *necessary* condition for this inequality to be satisfied is that the determinant of the matrix be non-negative. Furthermore, by using the fact that the cross-spectral density is a Hermitian function ([28], section 4.3.2), i.e. that

$$W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) = W_Q^*(\mathbf{r}_2, \mathbf{r}_1, \omega), \tag{2.53}$$

it is not difficult to show that

$$\tilde{W}_Q(-k\mathbf{s}_2, k\mathbf{s}_1, \omega) = \tilde{W}_Q^*(-k\mathbf{s}_1, k\mathbf{s}_2, \omega). \tag{2.54}$$

By use of Eq. (2.54), the requirement that inequality (2.52) be satisfied takes on the form

$$\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_1, \omega)\tilde{W}_Q(-k\mathbf{s}_2, k\mathbf{s}_2, \omega) \geq |\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2, \omega)|^2. \tag{2.55}$$

It is clear from inequality (2.55) that if Eq. (2.48) is satisfied, then Eq. (2.47) follows, thus establishing Theorem 2.5.

This meaning of this theorem may be readily understood by noting that the radiant intensity $J(\mathbf{s}, \omega)$, given by the formula

$$J(\mathbf{s}, \omega) = (2\pi)^6 \tilde{W}_Q(-k\mathbf{s}, k\mathbf{s}, \omega), \tag{2.56}$$

represents the rate at which the source radiates energy at frequency ω, per unit solid angle, into the far zone ([28], section 5.2.1). Thus if Theorem 2.5 is satisfied, the partially coherent nonradiating source does not radiate any power into the far zone at frequency ω.

Relatively few papers have been written about partially coherent nonradiating sources, no doubt because such sources are significantly more complicated than non-radiating monochromatic sources. Several papers have been written on the inverse source problem for partially coherent sources [16, 17, 18]. Nonradiating partially coherent electromagnetic sources have also been investigated [37, 38].

Other results regarding nonradiating partially coherent sources, which are analogous to those established for monochromatic nonradiating sources, may be proven by use of Theorem 4.1 of section 4.1 of this thesis.

2.3 The inverse source problem and nonradiating sources

In the most general sense of the term, an *inverse problem* may be said to be the determination of the "cause" of a phenomenon from measurements of the phenomenon itself (the "effect")[7]. Every inverse problem is based upon a *direct problem* whose solution represents the usual "cause – effect" sequence of events. Examples of inverse problems include the determination of atomic structure from measurements of atomic spectra (spectroscopy), the determination of crystal structure from X-ray scattering experiments (X-ray crystallography), and the determination of the characteristics of earthquake faults from measurements of seismic waves (seismology). One might even describe criminal investigations as inverse problems – the determination of a "cause" (the criminal and his motive) from measurements of the "effect" (evidence at the crime scene).

A large number of inverse problems that have been investigated are said to be *ill-posed* (in the sense of Hadamard, who originally introduced the concept [40]). For our purposes, it suffices to say that ill-posedness refers to two difficulties: nonuniqueness of the solution of the problem and large errors in the solution of the problem produced by small errors in the data (in which case the problem is said to be *ill-conditioned*). To explain these concepts more precisely, let us suppose that the direct problem may be represented by an operator A acting on an "object" f_o. Then the "image" of the object, f_i, may be represented by the equation $f_i = Af_o$. The inverse problem is characterized by the inverse operator A^{-1} acting upon the image f_i. Nonuniqueness of the inverse problem and the amplification of errors are illustrated

[7] An excellent and very readable introduction to inverse problems is given in [39].

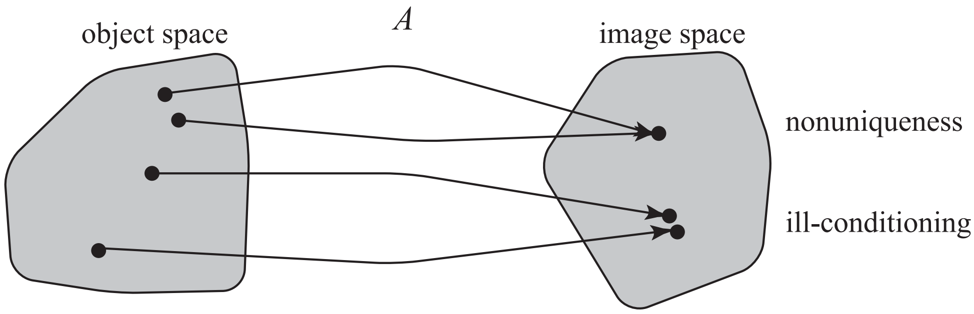


Figure 2.5: Demonstration of a direct problem whose inverse problem is ill-posed. Nonuniqueness corresponds to two different objects which have the same image. Ill-conditioning corresponds to two significantly different objects which have nearly the same image.

in figure 2.5. If the problem is ill-conditioned, a slight change in the image (due to noise, for instance) can result in a drastic change in the reconstructed object. If the problem is nonunique, even a noise-free image cannot be used to determine the object, because multiple objects can produce the same image. This difficulty is intimately related to nonradiating sources, as we shall see.

The general method used to deal with ill-conditioning and nonuniqueness of inverse problems is the application of *prior knowledge* in determining a solution. Prior knowledge is additional information, determined independently of the image, used to restrict the range of possible solutions. Such prior knowledge usually follows from knowledge of the physics of the problem, such as the size of the object being imaged. Its application is illustrated in figure 2.6. Nonuniqueness is, in principle, solved by isolating a single solution that satisfies the additional physical constraints. Ill-conditioning is, in principle, overcome by finding an acceptable physical solution which produces an image sufficiently close to the measured image. It should be

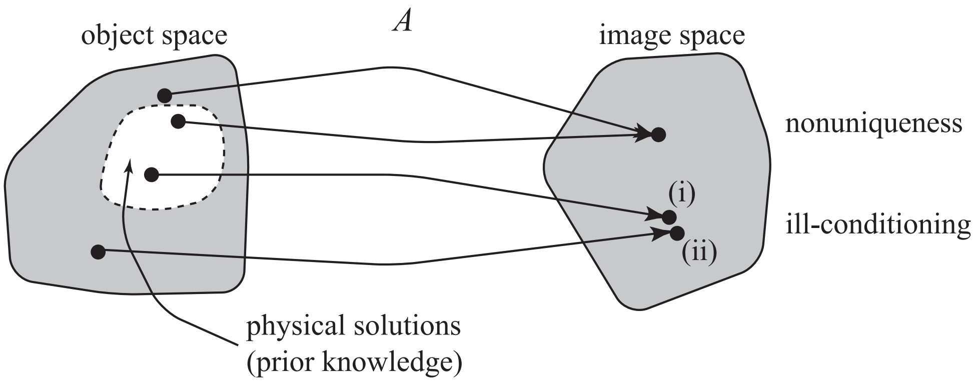


Figure 2.6: Demonstration of the use of prior knowledge. To deal with nonuniqueness, the solution is chosen that fits the additional physical constraint on the problem. To deal with ill-conditioning, a physical solution is chosen whose image (i) is sufficiently close to the measured image (ii).

noted that a particular piece of prior knowledge may be insufficient to construct uniquely the solution; the usefulness of specific prior knowledge must be analyzed within the context of the particular problem.

In optics, inverse problems are usually divided into two broad classes: (1) the determination of the scattering properties of an object from measurements of the field scattered by that object, and (2) the determination of the properties of a radiation source from measurements of the field radiated by that source. These problems are generally referred to as the *inverse scattering problem* and the *inverse source problem*, respectively.[8]

Inverse scattering problems have met with great success in recent years, most significantly in medical imaging. Foremost among these methods is computed tomog-

[8]It should be mentioned that the term *inverse source problem* is used to describe a variety of problems; see [41]. We will speak exclusively about three-dimensional, primary radiation sources.

raphy (CT), in which the attenuation of X-rays through an object is measured for a number of directions of incidence. By a non-trivial manipulation of the attenuation data, an image of a two-dimensional slice of the object may then be constructed.[9] Computed tomography is based upon a geometrical model of the propagation of radiation. When diffraction effects of scattering must be taken into account, the method known as diffraction tomography may be used, in which measurements of the scattered field for multiple directions of incidence may be used to reconstruct the object of interest.[10]

The inverse source problem, in contrast, has received little attention, no doubt because of its nonuniqueness. It would seem reasonable, then, to simply probe the structural information of an object by scattering experiments and forget about the source properties entirely, but there are several circumstances in which this is not useful or possible: (1) The object may not be accessible to scattering experiments, as in astronomy or seismology. In this case the only information available for study is the field radiated by the source. (2) The object may only be accessible for scattering experiments for a limited range of directions of incidence, as in geophysical inverse problems, for which the objects of interest are buried. The equivalence of the inverse source problem and the inverse scattering problem for a single direction of incidence is discussed in section 2.4. (3) The source of interest may be a subset of some larger scattering medium. In single photon emission computed tomography (SPECT)[11]

[9]The first CT machine was constructed by G.N. Hounsfield [42] in the early 1970s, who received the 1979 Nobel prize in physiology and medicine jointly with A.M. Cormack. For more information on computed tomography, see, for example, [43].

[10]The theoretical foundations of diffraction tomography were first developed by Wolf [44, 45]. A recent review of the field is given in [46].

[11]SPECT is one of the few three-dimensional inverse source problems to have been successfully

used in medicine, a radioactive isotope is injected into a patient's body and its position is determined by solving a particular inverse source problem. For these general reasons, it is worthwhile to understand the inverse source problem.

The nonuniqueness of the inverse source problem is easily understood as follows. Consider a monochromatic source $q(\mathbf{r})e^{-i\omega t}$ confined to a domain D. As we have seen in section 2.1, by measurement of the field in the far zone of the source we may determine the spatial Fourier transform of this distribution, $\tilde{q}(\mathbf{K})$, for all values $|\mathbf{K}| = k$. Now consider a source $q'(\mathbf{r}) = q(\mathbf{r}) + q_{NR}(\mathbf{r})$, where $q_{NR}(\mathbf{r})$ is a nonradiating source. The spatial Fourier transform of the source $q'(\mathbf{r})$ will be

$$\tilde{q}'(k\mathbf{s}) = \tilde{q}(k\mathbf{s}) + \tilde{q}_{NR}(k\mathbf{s}) = \tilde{q}(k\mathbf{s}), \tag{2.57}$$

which is the same as that of the source $q(\mathbf{r})$. Therefore the sources q and q' produce the same field in the far zone of the source; because of Theorem 2.3, their fields are identical everywhere outside the source domain. Therefore no measurements of the field outside the source can distinguish between the sources q and q'; consequently the inverse source problem for monochromatic sources is nonunique.

A similar argument may be made for partially coherent sources. If $W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega)$ is the cross-spectral density of a source Q, the cross-spectral density of its field will be identical to that of a source whose cross-spectral density is given by $W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) + W_{NR}(\mathbf{r}_1, \mathbf{r}_2, \omega)$, where W_{NR} is a nonradiating partially coherent source.

The uniqueness or nonuniqueness of an inverse problem can often be determined by a simple dimensional argument. Referring again to an "object" and an "image",

implemented. Its success is based on the assumption that the source is spatially uncorrelated and that the field propagates according to geometrical optics. One of the goals of this thesis is to determine whether such a problem is solvable under weaker assumptions. For details of SPECT, and a similar method, positron emission tomography (PET), see [47], section 4.2.

if the dimensionality of the image is equal to or greater than that of the object, the inverse problem is most likely unique. If the dimensionality of the image is less than that of the object, the inverse problem is certainly nonunique. The latter case corresponds to a loss of information in the direct problem – the image does not contain enough information to reconstruct the object. For instance, in the monochromatic inverse source problem, the "object" (the source) is a three-dimensional function, while the "image" (field measurements) is projected over a two-dimensional solid angle. For the inverse source problem for partially coherent sources, the "object" (the source cross-spectral density) is a six-dimensional function, while the "image" (field cross-spectral density measurements) is a four-dimensional function.

Because the inverse source problem is nonunique, we must use some sort of prior knowledge to perform a unique inversion. More precisely, we must limit ourselves to solving the inverse problem for some class of sources which have distinguishable radiation patterns. Let us consider the class of partially coherent sources which may be expressed in the form[12]

$$W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) = I_Q(\mathbf{r}_1)\delta(\mathbf{r}_2 - \mathbf{r}_1), \tag{2.58}$$

where $\delta(\mathbf{r}_2 - \mathbf{r}_1)$ is the three-dimensional Dirac delta function, and $I_Q(\mathbf{r})$ is a measure of the intensity of the source. Such sources are said to be *spatially incoherent*, as there is no correlation within the source distribution at different points in space. By use of Eq. (2.45), the cross-spectral density of the field at points $R\mathbf{s}_1$ and $R\mathbf{s}_2$ in the far zone is found to be

$$W_U(R\mathbf{s}_1, R\mathbf{s}_2, \omega) = \frac{(2\pi)^6}{R^2}\tilde{I}_Q[k(\mathbf{s}_2 - \mathbf{s}_1)], \tag{2.59}$$

where $\tilde{I}_Q$ is the three-dimensional Fourier transform of the source intensity.

[12]This problem was first considered in [48].

Let us consider the possibility of the existence of a nonradiating source of the form (2.58). To be nonradiating, the function $\tilde{I}_Q[k(\mathbf{s}_2 - \mathbf{s}_1)]$ must vanish for all directions $\mathbf{s}_1$, $\mathbf{s}_2$; in particular $\tilde{I}_Q(\mathbf{K})$ must vanish for all $|\mathbf{K}| \leq 2k$. However, it is known that because $\tilde{I}_Q(\mathbf{K})$ is the Fourier transform of a function of finite support, it is the boundary value of an entire analytic function in three complex variables (see [49], p. 352). It follows then that $\tilde{I}_Q(\mathbf{K})$ cannot vanish over any three-dimensional region of $\mathbf{K}$-space unless it vanishes identically – in which case $I_Q(\mathbf{r}, \omega)$ is itself identically zero. Therefore *nonradiating incoherent sources do not exist.* The inverse source problem can be readily seen to be solvable, for by use of the measurements of $\tilde{I}_Q(\mathbf{K})$ for all $|\mathbf{K}| \leq 2k$, we may perform a Fourier inversion of this function to reconstruct a low-pass filtered version of the source intensity, i.e.

$$I_Q^{l.p.}(\mathbf{r}) = \int_{|\mathbf{K}| \leq 2k} \tilde{I}_Q(\mathbf{K}) e^{i\mathbf{K}\cdot\mathbf{r}} \mathrm{d}^3 r. \tag{2.60}$$

Our dimensional argument for the uniqueness of inverse problems holds in this case, for the "object" (source cross-spectral density) is effectively a three-dimensional function, while the "image" (field cross-spectral density) is a four-dimensional function.

In retrospect, the uniqueness of the inverse source problem for incoherent sources may seem obvious because, as we have seen, nonradiating sources arise from destructive interference of the field generated by different points in the source. Incoherent sources cannot produce such destructive interference. Such sources are an extreme, and perhaps unphysical, approximation to realistic sources, however.[13] In chapter 5 we will determine a broader class of sources for which a unique solution to the inverse source problem may be found, the so-called quasi-homogeneous sources.

[13] A delta-correlated source would have an infinite source intensity. See [50], section 4.4.

2.4 Nonscattering scatterers and the localization of light

Up to now we have been considering only "invisible" objects of the second type mentioned at the beginning of this chapter: time-fluctuating sources which do not radiate. In this section we will consider the first type of invisible objects: objects which do not scatter fields incident upon them. We will see that the existence of such "nonscattering scatterers" is related to the existence of nonradiating sources.

We consider a monochromatic electromagnetic field with time dependence $e^{-i\omega t}$ incident upon a linear, isotropic, nonmagnetic medium occupying a finite domain D [see Fig. (2.7)]. If the index of refraction $n(\mathbf{r}, \omega)$ varies sufficiently slowly over space (see [36], section 13.1), the Cartesian components of the electromagnetic field are uncoupled to a good approximation and each component satisfies the scalar equation,

$$\nabla^2 U(\mathbf{r}, \omega) + k^2 n^2(\mathbf{r}, \omega) U(\mathbf{r}, \omega) = 0, \tag{2.61}$$

where $U(\mathbf{r}, \omega)$ is the spatial dependence of the *total* field, incident and scattered. This equation can be rewritten in the form

$$\nabla^2 U(\mathbf{r}, \omega) + k^2 U(\mathbf{r}, \omega) = -4\pi F(\mathbf{r}, \omega) U(\mathbf{r}, \omega), \tag{2.62}$$

where

$$F(\mathbf{r}, \omega) = \frac{1}{4\pi} k^2 [n^2(\mathbf{r}, \omega) - 1] \tag{2.63}$$

is known as the *scattering potential* of the medium. Equation (2.62) may be simplified further; let us write the total field in the form

$$U(\mathbf{r}, \omega) = U_i(\mathbf{r}, \omega) + U_s(\mathbf{r}, \omega), \tag{2.64}$$

where U_i is the field incident on the scatterer and U_s is the scattered field. We will

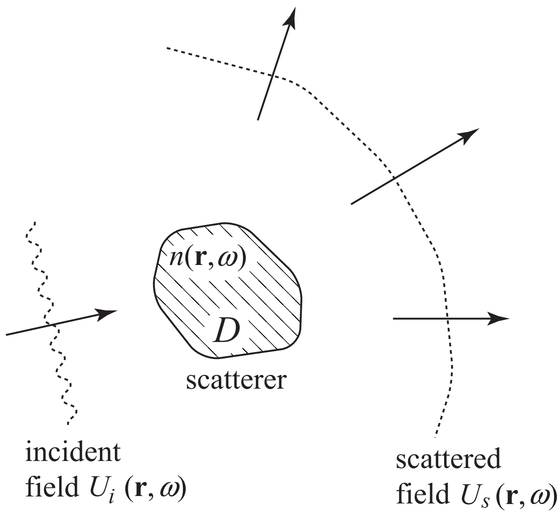


Figure 2.7: Illustrating the setup relating to the scattering of a monochromatic field from a scattering object.

consider only incident fields which satisfy the homogeneous Helmholtz equation, i.e.

$$\nabla^2 U_i(\mathbf{r},\omega) + k^2 U_i(\mathbf{r},\omega) = 0. \tag{2.65}$$

On substitution of Eq. (2.64) into Eq. (2.62), and using Eq. (2.65), we find the following partial differential equation for the scattered field,

$$\nabla^2 U_s(\mathbf{r},\omega) + k^2 U_s(\mathbf{r},\omega) = -4\pi F(\mathbf{r},\omega) U(\mathbf{r},\omega). \tag{2.66}$$

This equation should be compared with Eq. (2.3) of section 2.1. It is seen that the scattered field satisfies the inhomogeneous Helmholtz equation, with the source term given by

$$q(\mathbf{r},\omega) = F(\mathbf{r},\omega) U(\mathbf{r},\omega). \tag{2.67}$$

The scattered field may therefore be expressed in integral form as

$$U_s(\mathbf{r},\omega) = \int_D F(\mathbf{r}',\omega) U(\mathbf{r}',\omega) \frac{e^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \mathrm{d}^3 r'. \tag{2.68}$$

It is important to note that this equation does not readily give the scattered field, because U_s appears on *both* sides of the equation. However, if the scattering potential is sufficiently weak, the scattered field will be small compared to the incident field; we may then approximate $U(\mathbf{r}, \omega)$ in Eq. (2.68) by $U_i(\mathbf{r}, \omega)$, and arrive at the expression

$$U_s(\mathbf{r}, \omega) = \int_D F(\mathbf{r}', \omega) U_i(\mathbf{r}', \omega) \frac{e^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \mathrm{d}^3 r'. \tag{2.69}$$

This approximation for the scattered field is known as the *first Born approximation*, or often simply as the *Born approximation* ([36], section 13.1.2).

We are now in a position to investigate nonscattering scatterers. On comparison of Eq. (2.69) with the equation for the field of a radiation source, Eq. (2.5), it is clear that, for a given incident field U_i, we may treat the weak scattering problem as a radiation problem with a source given by the expression

$$q(\mathbf{r}, \omega) = F(\mathbf{r}, \omega) U_i(\mathbf{r}, \omega). \tag{2.70}$$

In particular, let us examine the scattered field in the far zone of the scatterer. In this case, the approximation given by Eq. (2.6) may be used, and the scattered field in the far zone (in a direction $\mathbf{s}$ and at a distance R) may be expressed in the form

$$U_s(R\mathbf{s}, \omega) \approx \frac{e^{\mathrm{i}kR}}{R} \int_D F(\mathbf{r}', \omega) U_i(\mathbf{r}', \omega) e^{-\mathrm{i}k\mathbf{s}\cdot\mathbf{r}'} \mathrm{d}^3 r'. \tag{2.71}$$

From this equation the following theorem regarding nonscattering scatterers follows immediately:

Theorem 2.6 *A weakly-scattering object with a scattering potential $F(\mathbf{r})$ will be nonscattering for a given incident field $U_i(\mathbf{r})$ if*

$$\int_D F(\mathbf{r}', \omega) U_i(\mathbf{r}', \omega) e^{-\mathrm{i}k\mathbf{s}\cdot\mathbf{r}'} \mathrm{d}^3 r' = 0 \quad \textit{for all } \mathbf{s}. \tag{2.72}$$

Because of this equivalence between these nonscattering scatterers and nonradiating sources, all results that apply to nonradiating sources also apply to scatterers which are nonscattering for one direction of incidence. For instance, the scattered field of a nonscattering scatterer will vanish everywhere outside the domain of the scattering object, as follows from the use of Theorem 2.3. Furthermore, the solution of the inverse source problem is equivalent to solving the inverse scattering problem for a single direction of incidence.

Such invisible scatterers were apparently first described by Kerker [51], who demonstrated that certain compound dielectric ellipsoids will not scatter a field incident from certain directions. Later work by Devaney [52] demonstrated that there exist weak scatterers which do not scatter incident plane waves for any *finite* number of directions of incidence. More recently, Wolf and Habashy [53] demonstrated that there do *not* exist weak scatterers which are nonscattering for *all* directions of incidence.[14] Because of the importance of this latter theorem for inverse scattering theory, and the relation of its derivation to later work in this thesis, we will prove it explicitly.

Theorem 2.7 *Within the accuracy of the first Born approximation, there is no medium occupying a finite region of space which is a nonscattering scatterer for all directions of incidence.*

Let us consider the incident fields to be plane waves,

$$U_i(\mathbf{r},\omega) = A_0 e^{ik\mathbf{s}_0\cdot\mathbf{r}}. \tag{2.73}$$

On substituting this formula into Eq. (2.71) for the scattered field in the far zone,

[14]This same paper refers to a theorem by Nachman [54] which demonstrates that this result is true for all scatterers, not just weak scatterers.

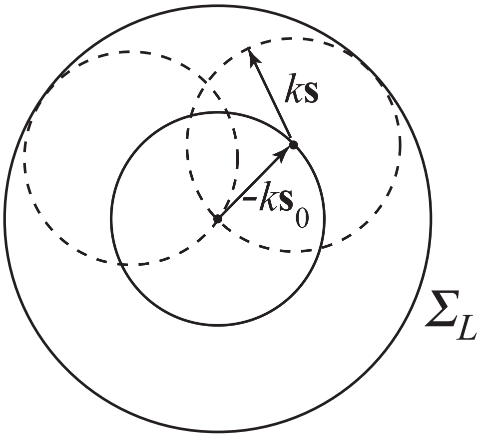


Figure 2.8: Demonstration of the Ewald spheres of reflection (dashed spheres) and the Ewald limiting sphere, Σ_L.

we find that

$$U_s(R\mathbf{s},\omega) = (2\pi)^3 \frac{e^{\mathrm{i}kR}}{R} \tilde{F}[k(\mathbf{s}-\mathbf{s}_0),\omega], \tag{2.74}$$

where

$$\tilde{F}(\mathbf{K},\omega) = \frac{1}{(2\pi)^3} \int_D F(\mathbf{r}',\omega) e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}'} \mathrm{d}^3 r' \tag{2.75}$$

is the three-dimensional Fourier transform of the scattering potential. For a fixed direction of incidence $\mathbf{s}_0$, measurements of the field in the far zone of the scatterer for all possible directions of scattering $\mathbf{s}$ can be used to determine those Fourier components of $\tilde{F}(\mathbf{K},\omega)$ which lie on a sphere of radius k, centered on $\mathbf{K} = k\mathbf{s}_0$ (see Fig. 2.8). This sphere, first described in the theory of X-ray scattering by crystals [55], is known as the Ewald sphere of reflection. For the scatterer to be nonscattering for all directions of incidence, the Fourier transform of F must therefore vanish on Ewald spheres of reflection for every direction of incidence and consequently it must vanish within a sphere of radius $|\mathbf{K}| \leq 2k$ (known as the Ewald limiting sphere). However, because $\tilde{F}(\mathbf{K},\omega)$ is the Fourier transform of a function of finite support, it is the boundary value of an entire analytic function in three complex variables (see [49], p. 352). It follows then that $\tilde{F}(\mathbf{K},\omega)$ cannot vanish over any three-dimensional region of $\mathbf{K}$-space unless it vanishes identically – in which case $F(\mathbf{r},\omega)$ is itself identically zero. Therefore Theorem 2.7 is proven, and a scatterer can only be nonscattering for a finite number of directions of incidence.

This result is important for the solution of the inverse scattering problem. Theorem 2.7 demonstrates that there are no truly invisible weak scatterers; measurements of the scattered field for enough directions of incidence will provide some information about the scattering object. Recall the dimensional argument given in section 2.3: if the dimensionality of the object and image are the same, the inverse problem is likely to be unique. In this case, the object (the scattering potential) is three-dimensional, and the image (the data obtained from field measurements for all directions of incidence and scattering) is three-dimensional, so the inverse problem is unique.

As mentioned earlier, when a nonscattering scatterer is illuminated by an incident field from a direction for which it is nonscattering, the scattered field is entirely contained within the region of the scattering object. It may be said that the scattered field is *localized* to the domain of the scatterer. In recent years, much attention has been given to the subject of light localization, and we discuss briefly the similarities and differences between this phenomenon and the properties of the nonscattering scatterers already mentioned.

The localization of light[15] has been investigated as an analogue of so-called Anderson localization [59] of electrons in disordered material. When the material

[15] A discussion of such light localization possiblities is given in [56]. Experimental observation of such localization has been done both for microwaves [57] as well as for visible light [58].

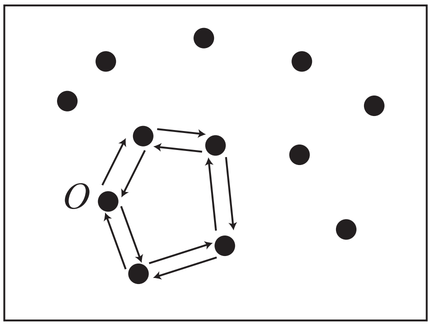


Figure 2.9: Schematic illustration of the localization of light. If the atom at the origin O is a radiating source, the waves radiated by the source along the two opposing paths will have the same phase and will interfere constructively. If the mean free path of the light is sufficiently short (of the order of a wavelength), the contribution of such closed loop paths will dominate and the light will tend to remain about the source.

through which the light (or electron) propagates is highly scattering and weakly absorbing, the propagation of the light may be described as a diffusion process. It has been demonstrated, both theoretically and experimentally, that if the mean free path of the light is of the order of a wavelength (i.e. the scattering is sufficiently strong in the material), diffusion in the system is impossible and the field is localized within the scattering material. This localization may be understood to arise from the *constructive* interference of the field arising from closed paths within the scatterer (see Fig. 2.9).

Such localization of light should not be confused with the nonscattering scatterers described in this section and the nonradiating sources described earlier. The most striking distinction is that light localization of the Anderson type only exists in materials which are strongly scattering. As we have seen above, scatterers may be nonscattering (for a given direction of incidence) even for scatterered fields which satisfy the first Born approximation. Furthermore, we have seen that non-scattering scatterers (and nonradiating sources) arise from a complete *destructive* interference of the outgoing radiation, whilst the localization of light arises from the *constructive* interference of fields returning to their point of origin. Despite these differences, some authors have confused the two types of localization (see, for instance, [60], section III, particularly Eq. (21)).

For completeness, we mention one more class of supposedly invisible objects that has been discussed in the literature. We consider spherical scatterers of uniform *complex* refractive index, for which the imaginary part of the index is negative (such active objects may be considered to be a good model for a gain medium pumped to saturation). Alexopoulos and Uzunoglu [61] have shown that, under certain circumstances, the extinction cross section for scattering of incident plane waves upon such objects vanishes. Such objects may be considered invisible, as the incident field appears to pass undisturbed through them. However, as pointed out by Kerker [62], the scattered field may in fact be quite large – the loss of the incident field due to scattering is counteracted by an amplification of the unscattered incident field by the active object. Such objects are therefore only invisible in a limited sense and not truly invisible as are the nonscattering scatterers described earlier in this section.

2.5 Moving charge distributions and radiation reaction

Up to this point we have considered “unmoving” charge-current distributions, that is, distributions whose radiation is due to some sort of internal oscillation of charges.

However, the original results in the theory of nonradiating sources were concerned with radiationless periodic translations of rigid charge distributions. In this section we demonstrate that radiationless motions of rigid charge distributions exist[16], and we relate this result to the theory of nonradiating sources that we have seen so far.

Let us consider a rigid three-dimensional charge distribution of total charge e and translational motion characterized by the position vector $\boldsymbol{\xi}(t)$. We assume that the motion of the charge distribution is nonrelativistic ($\dot{\boldsymbol{\xi}}(t) \ll c$). The time-dependent charge density of this object may be written as

$$\rho(\mathbf{r}, t) = ef(\mathbf{r} - \boldsymbol{\xi}(t)), \tag{2.76}$$

where $f(\mathbf{r})$ is the distribution of charge within the object, satisfying the constraint that

$$\int_V f(\mathbf{r}) \mathrm{d}^3 r = 1, \tag{2.77}$$

and the distribution is assumed to be localized within a volume V. The current density of such a moving charge distribution is given by the expression

$$\mathbf{j}(\mathbf{r}, t) = e\dot{\boldsymbol{\xi}}(t) f(\mathbf{r} - \boldsymbol{\xi}(t)). \tag{2.78}$$

Let us assume that the distribution is moving periodically with period T. Because the motion of the distribution is periodic, and because the distribution is of finite volume V, we may expand the current and charge densities in the following mixed Fourier integral/series representation:

$$\mathbf{j}(\mathbf{r}, t) = \sum_{n=-\infty}^{\infty} \int \mathbf{j}_n(\mathbf{K}) e^{\mathrm{i}(\mathbf{K}\cdot\mathbf{r} - \omega_n t)} \mathrm{d}^3 K, \tag{2.79a}$$

$$\rho(\mathbf{r}, t) = \sum_{n=-\infty}^{\infty} \int \frac{1}{\omega_n} \mathbf{K} \cdot \mathbf{j}_n(\mathbf{K}) e^{\mathrm{i}(\mathbf{K}\cdot\mathbf{r} - \omega_n t)} \mathrm{d}^3 K, \tag{2.79b}$$

[16]The method of derivation closely follows that discussed in [9].

where $\omega_n = 2\pi n/T$. The Fourier coefficients of $\mathbf{j}$ and ρ are related here by the use of the continuity equation. It is to be noted that the $n = 0$ term of Eq. (2.79b) represents the net charge of the source distribution.

We now define a scalar potential $\phi(\mathbf{r}, t)$ and a vector potential $\mathbf{A}(\mathbf{r}, t)$ in the usual way. In the Lorentz gauge, these potentials have the form ([26], chapter 6)

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{c} \int \frac{\mathbf{j}(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}^3 r', \tag{2.80a}$$

$$\phi(\mathbf{r}, t) = \int \frac{\rho(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}^3 r'. \tag{2.80b}$$

Far from the region within which the source is moving, at a distance r and in a direction specified by a unit vector $\hat{\mathbf{r}}$, these potentials take on the forms

$$\mathbf{A}(r\hat{\mathbf{r}}, t) \sim \frac{1}{c}\frac{1}{r} \int \mathbf{j}\left(\mathbf{r}', t - \frac{r - \hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right) \mathrm{d}^3 r', \tag{2.81a}$$

$$\phi(r\hat{\mathbf{r}}, t) \sim \frac{1}{r} \int \rho\left(\mathbf{r}', t - \frac{r - \hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right) \mathrm{d}^3 r'. \tag{2.81b}$$

We may substitute from Eqs. (2.79) into Eqs. (2.81) to express the potentials in terms of periodic oscillations as

$$\mathbf{A}(r\hat{\mathbf{r}}, t) \sim \frac{1}{c}\frac{1}{r} \sum_{n=-\infty}^{\infty} \int\int \mathrm{d}^3 K \mathrm{d}^3 r' \mathbf{j}_n(\mathbf{K}) e^{\mathrm{i}\left[\mathbf{K}\cdot\mathbf{r}' - \omega_n (t - \frac{r - \hat{\mathbf{r}}\cdot\mathbf{r}'}{c})\right]}, \tag{2.82a}$$

$$\phi(r\hat{\mathbf{r}}, t) \sim \frac{1}{r} \sum_{n=-\infty}^{\infty} \frac{1}{\omega_n} \int\int \mathrm{d}^3 K \mathrm{d}^3 r' \mathbf{K} \cdot \mathbf{j}_n(\mathbf{K}) e^{\mathrm{i}\left[\mathbf{K}\cdot\mathbf{r}' - \omega_n (t - \frac{r - \hat{\mathbf{r}}\cdot\mathbf{r}'}{c})\right]}. \tag{2.82b}$$

These expressions may be simplified by the use of the Fourier representation of the delta function, i.e.

$$\delta^{(3)}(\mathbf{K}) = \frac{1}{(2\pi)^3} \int e^{\mathrm{i}\mathbf{K}\cdot\mathbf{r}'} \mathrm{d}^3 r', \tag{2.83}$$

and Eqs. (2.82) become

$$\mathbf{A}(r\hat{\mathbf{r}}, t) \sim \frac{(2\pi)^3}{c} \sum_{n=-\infty}^{\infty} \frac{e^{\mathrm{i}k_n r}}{r} \mathbf{j}_n\left(k_n \hat{\mathbf{r}}\right) e^{-\mathrm{i}\omega_n t}, \tag{2.84a}$$

$$\phi(r\hat{\mathbf{r}}, t) \sim \frac{(2\pi)^3}{c} \sum_{n=-\infty}^{\infty} \frac{e^{\mathrm{i}k_n r}}{r} \hat{\mathbf{r}} \cdot \mathbf{j}_n \left(k_n \hat{\mathbf{r}}\right) e^{-\mathrm{i}\omega_n t}, \tag{2.84b}$$

where $k_n \equiv \omega_n / c$.

The electric field is given by the expression

$$\mathbf{E}(r\hat{\mathbf{r}}, t) = -\nabla\phi(r\hat{\mathbf{r}}, t) - \frac{1}{c}\frac{\partial \mathbf{A}(r\hat{\mathbf{r}}, t)}{\partial t}. \tag{2.85}$$

On substituting from Eqs. (2.84) into Eq. (2.85), neglecting all terms which decrease more rapidly than $1/r$, and using the elementary vector identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$, we find that the electric field is given by the expression

$$\mathbf{E}(r\hat{\mathbf{r}}, t) = -\frac{(2\pi)^3}{c} \sum_{n=-\infty}^{\infty} \mathrm{i}k_n \left[\hat{\mathbf{r}} \times \left[\hat{\mathbf{r}} \times \mathbf{j}_n(k_n \hat{\mathbf{r}})\right]\right] \frac{e^{\mathrm{i}k_n r}}{r} e^{-\mathrm{i}\omega_n t}. \tag{2.86}$$

Expression (2.86) indicates that the distribution will not radiate if the individual current contributions $\mathbf{j}_n(k_n\hat{\mathbf{r}}) = 0$ for all n, and all directions of observation $\hat{\mathbf{r}}$. This condition should be compared with the nonradiating condition (2.35a) for a monochromatic current distribution. We would like to express the electric field in terms of the structure of the rigid charge distribution, $f(\mathbf{r})$, and its motion, described by $\boldsymbol{\xi}(t)$. To express it in terms of these quantities, we first note that the coefficients $\mathbf{j}_n(\mathbf{K})$ of Eq. (2.86) are related to the current density $\mathbf{j}(\mathbf{r}, t)$ by the formula

$$\mathbf{j}_n(\mathbf{K}) = \frac{1}{(2\pi)^3 T} \int_V \int_0^T e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}} \mathbf{j}(\mathbf{r}, t) e^{\mathrm{i}\omega_n t} \mathrm{d}^3 r \mathrm{d}t. \tag{2.87}$$

On substitution from Eq. (2.78) into (2.87), it is not difficult to show that

$$\mathbf{j}_n(\mathbf{K}) = \frac{e}{T} \tilde{f}(\mathbf{K}) \int_0^T e^{-\mathrm{i}\mathbf{K}\cdot\boldsymbol{\xi}(t)} \dot{\boldsymbol{\xi}}(t) e^{\mathrm{i}\omega_n t} \mathrm{d}t, \tag{2.88}$$

where

$$\tilde{f}(\mathbf{K}) = \frac{1}{(2\pi)^3} \int_V f(\mathbf{r}) e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}} \mathrm{d}^3 r. \tag{2.89}$$

We may use this result to express the electric field far from the source in the form

$$\mathbf{E}(r\hat{\mathbf{r}},t) = -\frac{(2\pi)^3 e}{cT}\sum_{n=-\infty}^{\infty} k_n \tilde{f}(k_n\hat{\mathbf{r}})\frac{e^{\mathrm{i}k_n r}}{r}e^{-\mathrm{i}\omega_n t}\int_0^T e^{-\mathrm{i}k_n\hat{\mathbf{r}}\cdot\boldsymbol{\xi}(t')}(\hat{\mathbf{r}}\times[\hat{\mathbf{r}}\times\dot{\boldsymbol{\xi}}(t')])e^{\mathrm{i}\omega_n t'}\mathrm{d}t'. \tag{2.90}$$

From equation (2.90), the following remarkable theorem follows:

Theorem 2.8 *A rigid charge distribution* $ef(\mathbf{r})$ *undergoing periodic motion will not produce any radiation if the condition*

$$\tilde{f}(k_n\hat{\mathbf{r}}) = 0 \tag{2.91}$$

is satisfied for all positive integers n *and for all directions of observation* $\hat{\mathbf{r}}$*, independent of the precise path of the particle.*

This result was evidently first noted by Schott [6], who demonstrated it for the particular example of a charged spherical shell of radius $a = mcT/2$, where T is the period of oscillation and m is any positive integer. Later Bohm and Weinstein [8] determined other radiationless, spherically symmetric charge distributions, and Goedecke [9] demonstrated that radiationless motions exist even for some asymmetric, spinning charge distributions.

Several comments about these radiationless modes are in order. First, it is to be noted that, in general, the field will not be identically zero far from the moving charge; there will exist *static* electric and magnetic fields throughout space [produced by the $n = 0$ terms of Eqs. (2.79)]. Second, it is to be noted that, for the case of the spherical shell, the diameter of the shell is always equal to or greater than the distance that light may travel in one period T of oscillation. This suggests that the radius of the "orbit" of the charge distribution is always less than the spatial extent of the distribution – the motion of the distribution may be described more as a "wobble" than an "orbit". It has been speculated, though not rigorously proven, that this is a general feature of such radiationless motions [9].

If we focus our attention on a single frequency ω_n of the radiated field the moving charge will not radiate at that particular frequency if

$$\tilde{f}(k_n\hat{\mathbf{r}}) = 0 \quad \text{for all} \quad \hat{\mathbf{r}}. \tag{2.92}$$

This condition is formally identical to the nonradiating condition for a monochromatic distribution, Eq. (2.9). If it is satisfied, the spectrum of the radiated field will not contain the frequency ω_n, even though the source is moving at that frequency. This is an example of a *spectral change*, to be discussed further in sections 3.4 and 4.2.

The existence of such spectral changes implies that it is not possible in certain circumstances to determine the motion of a charge distribution by measurements of the radiation it produces. For instance, if $\tilde{f}(\omega_1\hat{\mathbf{r}}) = 0$ for all $\hat{\mathbf{r}}$, the lowest frequency measured in the field will be $\omega_2 = 2\omega_1$. This would lead one to suspect that the distribution is moving with period $T/2$, instead of the true period T. Again we see that the existence of nonradiating objects implies the nonuniqueness of an inverse problem.

The existence of radiationless motions has also played a significant role in the study of radiation reaction. Numerous authors ([8],[63],[64]) have pointed out that certain rigid charge distributions can oscillate not only without radiation, but without an external force acting upon them – the motion is maintained by the action of the particle's electromagnetic field upon itself. The condition for nonradiation described in Theorem 2.8 plays an important role in such oscillations.

One might wonder if the aforementioned results hold when the particle is moving at relativistic speeds. Very little work has been done in generalizing radiationless motions to the relativistic domain, at least in part due to the inconsistency of the concept of rigid charge distributions with relativity theory. However, Prigogine and Henin [65] have provided one generalization of extended electron theory in which self-oscillating modes are possible. Pearle [66] has shown that Schott's rigid, uniformly charged spherical shell moving in a relativistically invariant manner does not have any radiationless motions.

Chapter 3

One-dimensional localized excitations

The previous chapter introduced the phenomenon of nonradiating sources and noted that such sources are a general feature of linear, wavelike systems. Their origin lies in a complicated interference phenomenon involving the field emitted by every source point. As yet, no one has experimentally produced a nonradiating source and, at least in part, this is due to the complexity of the interference phenomenon.

Because nonradiating sources are a general feature of linear, wavelike systems, it would seem that they must also arise in one-dimensional systems, such as a vibrating string. Such *nonpropagating excitations* presumably would be easier to produce in a laboratory setting than the three-dimensional sources already discussed.

Furthermore, in the one-dimensional problem it is relatively easy to take into account effects such as external boundary conditions, damping, and driving forces which are quasi-monochromatic. By examining the influence such effects have on the existence of nonpropagating excitations, we may gain insight into the influence of similar effects in the analogous three-dimensional problem.

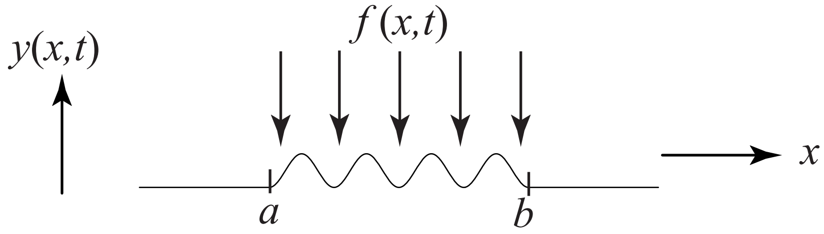


Figure 3.1: Illustrating the notation for a vibrating string. The string is shown oscillating with the nonpropagating excitation given by Eq. (3.14), with $n = 4$.

In this chapter, we investigate nonpropagating excitations for waves on a vibrating string. We begin by considering the ideal case of an infinite, undamped string, and then generalize the results to more complicated systems.

3.1 Nonpropagating string excitations

We consider an infinitely long flexible string under tension T and with mass per unit length μ, undergoing small displacements $y(x,t)$ from the equilibrium position, driven by a force density $f(x,t)$ (force per unit length) and localized in the region $a \leq x \leq b$ (see figure 3.1). The displacement obeys the wave equation (see, for instance, [67], chapter 4)

$$\mu \frac{\partial^2 y(x,t)}{\partial t^2} - T \frac{\partial^2 y(x,t)}{\partial x^2} = f(x,t). \tag{3.1}$$

Restricting ourselves to simple harmonic driving forces,

$$f(x,t) \equiv Re \left\{ f(x) e^{-\mathrm{i}\omega t} \right\}, \tag{3.2}$$

where Re denotes the real part, the steady-state solution $y(x,t)$ of Eq. (3.1) will have the same time dependence,

$$y(x,t) \equiv Re\left\{y(x)e^{-\mathrm{i}\omega t}\right\}, \tag{3.3}$$

and Eq. (3.1) then reduces to the one-dimensional inhomogeneous Helmholtz equation,

$$\frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} + k^2 y(x) = q(x), \tag{3.4}$$

where k is the wave number,

$$k = \frac{\omega}{v}, \quad v = \sqrt{T/\mu}, \tag{3.5}$$

and

$$q(x) = -f(x)/T. \tag{3.6}$$

We will call $q(x)$ the *effective force density*, or simply the *force density*.

The outgoing solution of Eq. (3.4) is well known to be (see, for example, [68], sections 16.5, 16.6)

$$y(x) = \frac{1}{2\mathrm{i}k}\int_a^b q(x')e^{\mathrm{i}k|x-x'|}\mathrm{d}x'. \tag{3.7}$$

For displacements to the right ($x > b$) and left ($x < a$) of the region of the applied force, $y(x)$ reduces to

$$y(x)|_R = \frac{e^{\mathrm{i}kx}}{2\mathrm{i}k}\int_a^b q(x')e^{-\mathrm{i}kx'}\mathrm{d}x' \tag{3.8}$$

and

$$y(x)|_L = \frac{e^{-\mathrm{i}kx}}{2\mathrm{i}k}\int_a^b q(x')e^{\mathrm{i}kx'}\mathrm{d}x'. \tag{3.9}$$

It is apparent from Eqs. (3.8) and (3.9) that the excitations will vanish everywhere outside the force region $a \leq x \leq b$ if

$$\tilde{q}(k) = 0, \quad \tilde{q}(-k) = 0, \tag{3.10}$$

with k given by Eq. (3.5), and $\tilde{q}(k)$ is the Fourier transform of the force density, i.e.

$$\tilde{q}(K) = \frac{1}{2\pi} \int_a^b q(x) e^{-\mathrm{i}Kx} \mathrm{d}x. \tag{3.11}$$

Nontrivial force densities that satisfy Eq. (3.10) will generate displacements of the string only within the region of the applied force, and will not produce any displacement outside it. As mentioned in the introduction to this chapter, we will refer to such a situation as a *nonpropagating excitation.*

As a simple example of a nonpropagating excitation, let $a = -l$, $b = l$, $l > 0$, and let the force be constant throughout this domain, i.e.

$$q(x) = \begin{cases} Q_0 & \text{when} \quad |x| \leq l, \\ 0 & \text{when} \quad |x| > l. \end{cases} \tag{3.12}$$

Upon substituting from Eq. (3.12) into Eq. (3.11) and requiring that the two conditions (3.10) be fulfilled, we find that there will be nontrivial solutions if and only if

$$kl = n\pi, \tag{3.13}$$

where $n = 1, 2, \ldots$. This result shows that a *constant* localized force distribution within the region $-l \leq x \leq l$ produces a nonpropagating excitation only for certain special values of kl. Using this result in the general expression (3.7) for the displacement, one readily finds that

$$y(x) = \begin{cases} \frac{Q_0}{(n\pi/l)^2} \left[1 - (-1)^n \cos \frac{n\pi x}{l}\right] & \text{when} \quad |x| \leq l, \\ 0 & \text{when} \quad |x| > l. \end{cases} \tag{3.14}$$

This displacement, along with the associated force density, is shown in Fig. 3.2 for the cases $n = 1$ and $n = 2$. We note that the displacement $y(x)$ given by Eq. (3.14) is continuous everywhere on the string, in particular at the boundary of the

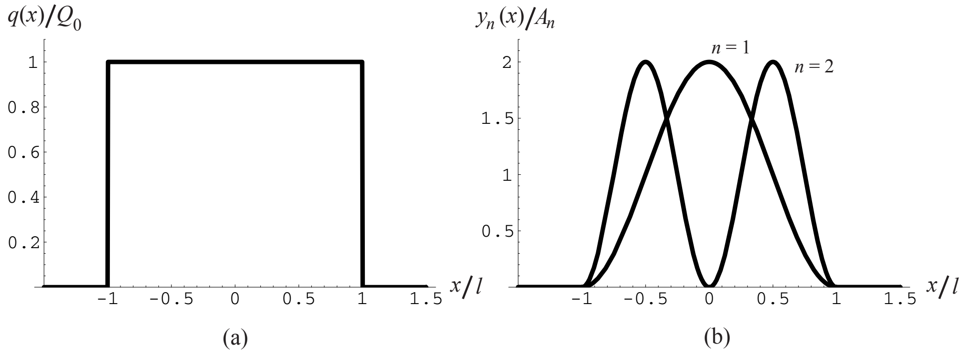


Figure 3.2: Force density distribution $q(x)$, shown in (a), which produces nonpropagating excitations (b), with $A_n = Q_0/(n\pi/l)^2$, $k = n\pi/l$, for $n = 1$ and $n = 2$.

region of applied force. One can readily verify that the first derivative $\mathrm{d}y(x)/\mathrm{d}x$ is also continuous everywhere on the string. In Appendix I, we demonstrate that this behavior is a general property of excitations due to piecewise continuous, localized force distributions. This fact leads to the following theorem.

Theorem 3.1 *A nonpropagating excitation $y(x)$ on an infinitely long string and the piecewise continuous force distribution $f(x)$, assumed to be confined to a finite region $a \leq x \leq b$, which generates it are related by the inhomogeneous Helmholtz equation (3.4), subject to the boundary conditions*

$$y(a) = y(b) = 0, \quad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a} = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=b} = 0. \tag{3.15}$$

This theorem, which is a one-dimensional analogue of Theorem 2.4, implies that the nonpropagating excitations are solutions to an *overspecified* Sturm-Liouville boundary-value problem, because only one of the two sets of boundary conditions (3.15) is required for a unique solution of the equation. Using Theorem 3.1, one can construct numerous examples of nonpropagating excitations. The following theorem is also a one-dimensional analogue of a theorem in radiation theory [19].

Theorem 3.2 *A force distribution, confined to a region $a \leq x \leq b$, which generates a nonpropagating excitation on an infinitely long string, related by Eq. (3.4), is orthogonal to every solution of the homogeneous Helmholtz equation with wavenumber k.*

To establish this theorem, let $y(x)$ be a nonpropagating excitation and $q(x)$ the force distribution which generates it, and let $u(x)$ be any solution of the homogeneous Helmholtz equation,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + k^2 u = 0. \tag{3.16}$$

We first multiply Eq. (3.4) by $u(x)$ and Eq. (3.16) by $y(x)$ and subtract the equations from each other. We then obtain the formula

$$u\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - y\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = q(x)u(x). \tag{3.17}$$

On integrating both sides of Eq. (3.17) with respect to x over the range $a \leq x \leq b$ and then integrating by parts on the left we obtain the relation

$$\left[u\frac{\mathrm{d}y}{\mathrm{d}x} - y\frac{\mathrm{d}u}{\mathrm{d}x}\right]_a^b = \int_a^b q(x)u(x)\mathrm{d}x. \tag{3.18}$$

According to Theorem 3.1, the left-hand side of Eq. (3.18) vanishes and consequently

$$\int_a^b q(x)u(x)\mathrm{d}x = 0, \tag{3.19}$$

as asserted by Theorem 3.2.

We have therefore shown in this section that nonpropagating excitations exist for one-dimensional wave problems and we have demonstrated some of their properties.

We conclude by noting that in the example that we have presented, the force density which produced the nonpropagating excitation is quite simple [see Eqs. (3.12) and (3.13)]. This force density is, of course, not the only one with these properties. Any localized force density which obeys the conditions (3.10) will produce nonpropagating excitations on the string.

3.2 Nonpropagating excitations on a string of finite length

In our investigation of nonpropagating excitations in the previous section, we dealt with a string of infinite length. Of course, any real string is of finite length and any realistic treatment of the vibrating string problem must therefore take into account the boundaries of the string. In this section we consider how the theory of nonpropagating excitations must be modified when the string under consideration is of finite length.

We consider a string of finite length $2L$, fixed at the endpoints $x = -L$, $x = L$. The displacement again satisfies the wave equation (3.1), subject now to the boundary conditions

$$y(-L,t) = y(L,t) = 0. \tag{3.20}$$

We again restrict ourselves to simple harmonic driving forces, so that

$$f(x,t) \equiv Re\left\{f(x)e^{-\mathrm{i}\omega t}\right\}, \tag{3.21}$$

$$y(x,t) \equiv Re\left\{y(x)e^{-\mathrm{i}\omega t}\right\}. \tag{3.22}$$

Equation (3.1) then reduces to the one-dimensional inhomogeneous Helmholtz equation,

$$\frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} + k^2 y(x) = q(x), \tag{3.23}$$

but now subject to the boundary conditions

$$y(-L) = y(L) = 0. \tag{3.24}$$

The quantities k and v are defined again as in Eq. (3.5), and $q(x)$ as in Eq. (3.6).

The steady-state solution of Eq. (3.23), for a given force density localized in the region $a \leq x \leq b$, is given by ([67], chapter 4)

$$y(x) = \int_a^b G(x, x')q(x')\mathrm{d}x', \tag{3.25}$$

where $G(x, x')$ is the Green's function of the system, given by the formula

$$G(x, x') = \frac{1}{k \sin 2kL} \times \begin{cases} \sin k(x+L) \sin k(x'-L), & -L < x < x' < L, \\ \sin k(x-L) \sin k(x+L), & -L < x' < x < L. \end{cases} \tag{3.26}$$

To begin with, we assume that the string is driven off-resonance, i.e. $2kL \neq n\pi$, with n being any positive integer. In such a case, the Green's function is well-behaved, and the displacement to the right (R) and left (L) of the applied force is given by the expressions

$$y(x)|_R = \frac{1}{k \sin 2kL} \sin k(x-L) \left[\int_a^b \sin k(x'+L) q(x')\mathrm{d}x' \right], \tag{3.27}$$

$$y(x)|_L = \frac{1}{k \sin 2kL} \sin k(x+L) \left[\int_a^b \sin k(x'-L) q(x')\mathrm{d}x' \right]. \tag{3.28}$$

Using an elementary trigonometric identity in Eqs. (3.27) and (3.28), we find that the solutions will be nonpropagating, i.e. $y(x) = 0$ for $x < a$ and $x > b$, if

$$\int_a^b \sin kx' q(x')\mathrm{d}x' = \int_a^b \cos kx' q(x')\mathrm{d}x' = 0, \tag{3.29}$$

or equivalently if

$$\int_a^b e^{-\mathrm{i}kx'}q(x')\mathrm{d}x' = \int_a^b e^{\mathrm{i}kx'}q(x')\mathrm{d}x' = 0. \tag{3.30}$$

On comparing these relations with Eqs. (3.10) and (3.11) of section 3.1, we see that the conditions for nonpropagation on a finite string with end points fixed are the same as the conditions for an infinite string. This result is perhaps not surprising, because the nonpropagating solutions vanish identically outside a finite domain and it does not matter where the ends of the string are, or what boundary conditions are placed upon them. The constraint that the end points of the string are fixed does not therefore influence the existence of nonpropagating excitations. We might expect, then, that nonpropagating solutions will be independent of the length $2L$ of the string, and be well-behaved even for values of kL associated with resonance, i.e. when $2kL = n\pi$, where n is an integer.

Using Eq. (3.25) to determine the string displacement in the interior of the region of the applied force (labeled by the subscript IN), we find that

$$\begin{aligned} y(x)|_{IN} = \frac{1}{k\sin 2kL}\Bigg\{ & \sin kx\cos^2 kL\int_a^b q(x')\sin kx'\mathrm{d}x' \\ & - \cos kx\sin^2 kL\int_a^b q(x')\cos kx'\mathrm{d}x' \\ & + \frac{1}{2}\cos kx\sin 2kL\int_a^b \mathrm{sgn}(x-x')q(x')\sin kx'\mathrm{d}x' \\ & - \frac{1}{2}\sin kx\sin 2kL\int_a^b \mathrm{sgn}(x'-x)q(x')\cos kx'\mathrm{d}x' \Bigg\}, \end{aligned} \tag{3.31}$$

where

$$\mathrm{sgn}(x'-x) = \begin{cases} 1 & x' > x, \\ -1 & x' < x. \end{cases} \tag{3.32}$$

Since a nonpropagating excitation satisfies Eq. (3.29), the first two terms of Eq. (3.31) vanish and the displacement within the region of the applied force may be expressed as

$$y(x)|_{IN} = \frac{1}{2k}\int_a^b \sin(k|x-x'|)q(x')\mathrm{d}x', \tag{3.33}$$

which, as expected, is independent of the length of the string.

In dealing with nonpropagating excitations, then, we may ignore the boundary conditions on the string and work with the simpler mathematical formalism appropriate to an infinite string. This result suggests that such excitations are not affected or perturbed by external constraints upon the system.

3.3 Nonpropagating excitations on a damped string of infinite length

Let us next consider the effect of a damping force per unit length, $R\partial y/\partial t$ (R being a constant), upon the existence and behavior of nonpropagating excitations. We return to the case of an infinite string and now use, instead of Eq. (3.1), the more general wave equation ([67], chapter 4)

$$\mu\frac{\partial^2 y(x,t)}{\partial t^2} = T\frac{\partial^2 y(x,t)}{\partial x^2} - R\frac{\partial y(x,t)}{\partial t} + f(x,t). \tag{3.34}$$

Restricting ourselves to simple harmonic driving forces and the corresponding steady-state solutions (3.2) and (3.3), Eq. (3.34) reduces to a one-dimensional inhomogeneous Helmholtz equation with a complex wave number,

$$\frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} + \left[\frac{\omega^2}{T/\mu} + \mathrm{i}\frac{\omega R}{T}\right]y(x) = -\frac{f(x)}{T} = q(x). \tag{3.35}$$

The solution to Eq. (3.35) can be shown to be[1]

$$y(x) = \frac{1}{2\mathrm{i}\beta}\int_a^b e^{\mathrm{i}\beta|x-x'|}q(x')\mathrm{d}x', \tag{3.36}$$

[1] Eq. (3.36) is an obvious modification of the solution (3.7) for the undamped infinite string.

where β is given by

$$\beta \equiv \sqrt{\frac{\omega^2}{T/\mu} + \mathrm{i}\frac{\omega R}{T}}$$
$$= k\sqrt{\frac{1}{2}\left(\left[1+\left(\frac{R}{\mu\omega}\right)^2\right]^{1/2}+1\right)} + \mathrm{i}k\sqrt{\frac{1}{2}\left(\left[1+\left(\frac{R}{\mu\omega}\right)^2\right]^{1/2}-1\right)}. \quad (3.37)$$

Here k, as before, is given by Eq. (3.5). The solution (3.36) represents exponentially damped waves propagating away from the region of the applied force. In the limit of weak damping, i.e. when

$$R \ll \mu\omega, \quad (3.38)$$

β may be approximated as

$$\beta \approx \frac{\omega}{v} + \frac{1}{2}\frac{\mathrm{i}R}{\mu v} = k + \mathrm{i}\alpha, \quad \alpha \equiv \frac{R}{2\mu v}. \quad (3.39)$$

This approximation is likely to hold for many situations of practical interest.

To the left (L) and right (R) of the applied force, the excitation is given by the expressions

$$y(x)|_L = \frac{1}{2\mathrm{i}\beta}e^{-\mathrm{i}\beta x}\int_a^b e^{\mathrm{i}\beta x'}q(x')\mathrm{d}x', \quad (3.40)$$
$$y(x)|_R = \frac{1}{2\mathrm{i}\beta}e^{\mathrm{i}\beta x}\int_a^b e^{-\mathrm{i}\beta x'}q(x')\mathrm{d}x'. \quad (3.41)$$

These excitations will vanish outside the region of applied force if and only if

$$\int_a^b e^{\mathrm{i}kx'-\alpha x'}q(x')\mathrm{d}x' = 0, \quad (3.42)$$
$$\int_a^b e^{-\mathrm{i}kx'+\alpha x'}q(x')\mathrm{d}x' = 0, \quad (3.43)$$

where the expression (3.39) for β was used.

The appearance of the parameter α in the exponentials in Eqs. (3.42) and (3.43) results in a non-trivial departure from the theory of the undamped string. To see the difference, we consider again the step function force density

$$q(x') = \begin{cases} Q_0 & |x'| \leq l \\ 0 & |x'| > l. \end{cases} \tag{3.44}$$

We have seen that for the undamped string, the force density represented by Eq. (3.44) gives nonpropagating solutions only for values of kl such that

$$kl = n\pi, \quad (n = 1, 2, 3, ...). \tag{3.45}$$

Let us now determine if nonpropagating solutions exist for this force density on a damped string. Substituting from Eq. (3.44) into Eqs. (3.42) and (3.43), we find that the step function force will generate nonpropagating excitations if and only if

$$\tan kl + \mathrm{i} \tanh \alpha l = 0. \tag{3.46}$$

This equation can be satisfied only for $\alpha = 0$, i.e. for the undamped case. On a string with damped oscillations, the step function force therefore never generates nonpropagating excitations. The amplitude of vibration of the string to the right of the region of applied force on the boundary of that region, i.e. at $x = l$, is given by the formula

$$|y(l)|_R| = \frac{1}{\sqrt{2}} \frac{Q_0}{[k^2 + \alpha^2]} e^{-\alpha l} [\cosh 2\alpha l - \cos 2kl]^{1/2}. \tag{3.47}$$

If we vary the wavenumber k (or, equivalently, the frequency of vibration ω), we see that, for $2\alpha l \ll 1$, the minima of intensity occur roughly at the values of kl given by Eq. (3.45) (see figure 3.3).

This example raises the question of whether or not nonpropagating excitations can exist on a damped string. It is clear, though, that the displacement $y(x)$ and the

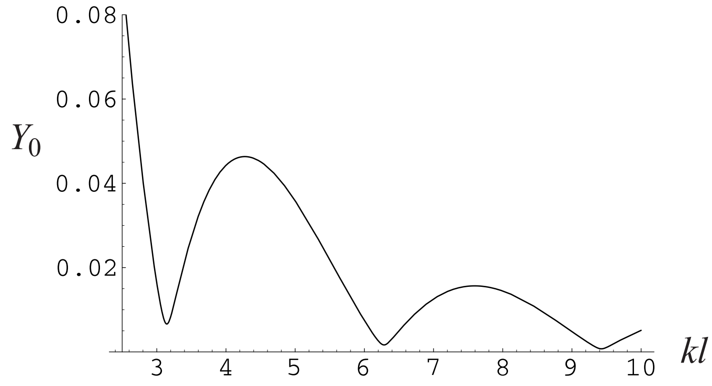


Figure 3.3: The (normalized) amplitude $Y_0 = \left| y(l)|_R \right| / Q_0 l^2$ of waves propagating to the right of the region of the applied force, evaluated at the boundary $x = l$. Though never strictly zero, the wave amplitude has minimums at frequencies approximately given by Eq. (3.45). Here $\alpha l = 0.07$.

force distribution that generates it are still related by an inhomogeneous Helmholtz equation. Hence, by the same arguments as those given in Appendix I, one finds that the field and its derivative must be continuous at all points. Theorem 3.1 may therefore be modified for the case of a damped string as follows.

Theorem 3.3 *A nonpropagating excitation on an infinitely long damped string and the piecewise continuous force distribution, assumed to be confined to a finite region* $a \le x \le b$*, which generates it are related by the inhomogeneous Helmholtz equation (3.35), with complex coefficients, subject to the boundary conditions*

$$y(a) = y(b) = 0, \quad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a} = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=b} = 0. \tag{3.48}$$

In general, the relation between the displacement of a localized excitation and the force distribution that generates it is more complicated for solutions on the damped string, due to the complex coefficient in the Helmholtz equation (3.35). One can, however, create nonpropagating excitations with relatively simple force distributions, as the following example shows. Let

$$q(x) = \begin{cases} Q_0 & |x| \leq x_0, \\ Q_1 & x_0 \leq |x| \leq x_1, \\ 0 & |x| \geq x_1. \end{cases} \tag{3.49}$$

Since this distribution is symmetric about the point $x = 0$, we need only satisfy one of the Eqs. (3.42) and (3.43). Substitution of Eq. (3.49) into Eq. (3.42) leads to the relation

$$\begin{aligned} \int_{-x_1}^{x_1} e^{\mathrm{i}kx' - \alpha x'} q(x') \mathrm{d}x' = Q_0 \int_{-x_0}^{x_0} e^{\mathrm{i}kx' - \alpha x'} \mathrm{d}x' \\ + Q_1 \left\{ \int_{-x_1}^{-x_0} e^{\mathrm{i}kx' - \alpha x'} \mathrm{d}x' + \int_{x_0}^{x_1} e^{\mathrm{i}kx' - \alpha x'} \mathrm{d}x' \right\}. \end{aligned} \tag{3.50}$$

This relation leads, after some calculation, to the formula

$$\begin{aligned} \int_{-x_1}^{x_1} e^{\mathrm{i}kx' - \alpha x'} q(x') \mathrm{d}x' = \frac{2\mathrm{i}}{\mathrm{i}k - \alpha} \{ Q_1 [\sin kx_1 \cosh \alpha x_1 + \mathrm{i} \sinh \alpha x_1 \cos kx_1] \\ + [Q_0 - Q_1][\sin kx_0 \cosh \alpha x_0 + \mathrm{i} \sinh \alpha x_0 \cos kx_0] \}. \end{aligned} \tag{3.51}$$

To satisfy Eq. (3.42), we therefore require that the real and imaginary parts of Eq. (3.51) vanish, i.e. that

$$Q_1 \sin kx_1 \cosh \alpha x_1 + [Q_0 - Q_1] \sin kx_0 \cosh \alpha x_0 = 0, \tag{3.52}$$

and

$$Q_1 \sinh \alpha x_1 \cos kx_1 + [Q_0 - Q_1] \sinh \alpha x_0 \cos kx_0 = 0. \tag{3.53}$$

It is clear that this pair of equations has many solutions – we will consider only one of them. First, note that if we choose

$$x_1 = \frac{n\pi}{k}, \quad x_0 = \frac{m\pi}{k}, \quad n > m > 0, \tag{3.54}$$

then Eq. (3.52) is automatically satisfied, and Eq. (3.53) reduces to

$$(-1)^{n-m}\frac{Q_1 - Q_0}{Q_1} = \frac{\sinh \alpha x_1}{\sinh \alpha x_0}. \tag{3.55}$$

Let us choose both n and m to be even. The right-hand side of Eq. (3.55) exceeds unity and it is clear that there exist choices $Q_0 < 0$, $Q_1 > 0$ such that Eq. (3.55), and consequently Eqs. (3.42) and (3.43), are satisfied. We have therefore found a force distribution which is nonpropagating on the damped string. An example of this force distribution is shown in figure 3.4; unlike the step function force given by Eq. (3.44), this distribution generates true nonpropagating excitations on the damped string. The real and imaginary parts of the amplitude of vibration are shown in Fig. 3.5. It is to be noted that the amplitude of the string vibrations decreases smoothly to zero at the boundary of the region of applied force, in agreement with Theorem 3.3.

We may also investigate nonpropagating excitations on a damped string of finite length with fixed end points. It is clear, though, from the arguments given in this section and in section 3.2, that the results are unchanged for the case of a string of finite length. The nonpropagating excitations, damped or undamped, are not influenced by the length of the string.

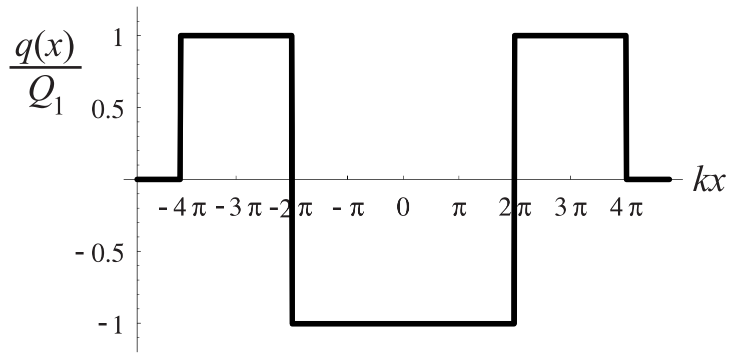


Figure 3.4: An example of a force distribution which generates nonpropagating excitations on a damped string of infinite length, as given by Eq. (3.49). Here $kx_0 = 2\pi$, $kx_1 = 4\pi$, and $\alpha/k = 0.01$.

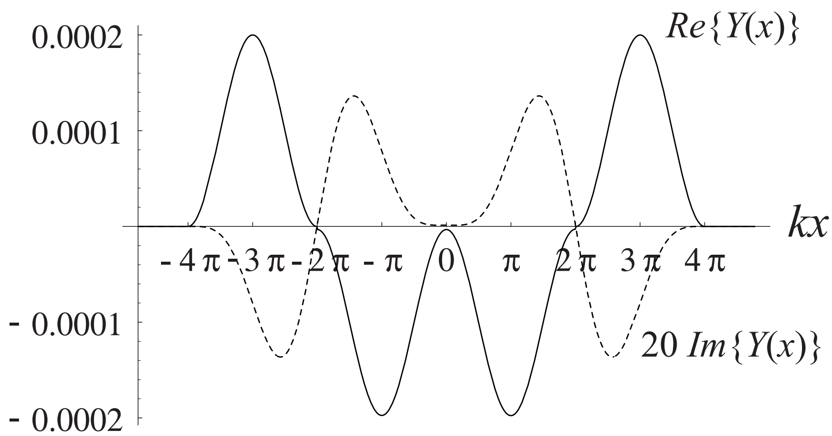


Figure 3.5: The real and imaginary parts of the string amplitude $y(x)$ for the force density shown in Fig. 3.4. Here $Y(x) = y(x)\alpha^2/Q_1$.

3.4 Nonpropagating excitations of quasi-monochromatic driving forces

In section 3.1, we demonstrated that a monochromatic step function driving force of the form of Eq. (3.12) generates nonpropagating excitations only for certain frequencies, given by Eq. (3.13). In order to learn how nonpropagating solutions affect the behavior of waves on the string, we will now consider a similar force distribution on an infinite undamped string which has oscillations over a range of frequencies. More specifically, we consider a simple quasi-monochromatic force distribution with random phase fluctuations $\phi(t)$ and center frequency ω_0, given by the expression

$$f(x,t) = 2F(x)\cos[\phi(t) - \omega_0 t], \tag{3.56}$$

where

$$F(x) = \begin{cases} F_0 & |x| \leq x_0 \\ 0 & |x| > x_0, \end{cases} \tag{3.57}$$

and $\phi(t)$ is a stationary random function, assumed to be slowly varying over times of the order of $2\pi/\omega_0$. The second-order correlation properties of the force density are characterized by the cross-correlation function

$$\Gamma_f(x_1, x_2, \tau) = \langle f_A^*(x_1,t) f_A(x_2, t+\tau)\rangle\,, \tag{3.58}$$

where $f_A(x,t)$ is the complex analytic signal representation of $f(x,t)$ ([28], chapter 4) and the sharp brackets denote an ensemble average. It can be shown that, for a quasi-monochromatic signal of the form (3.56), one has to a good approximation ([28], section 3.1.2)

$$f_A(x,t) = F(x)e^{\mathrm{i}[\phi(t)-\omega_0 t]}. \tag{3.59}$$

On substituting this expression into Eq. (3.58), we find that

$$\Gamma_f(x_1, x_2, \tau) = F(x_1)F(x_2)\left\langle e^{-\mathrm{i}\phi(t)}e^{\mathrm{i}\phi(t+\tau)}\right\rangle e^{-\mathrm{i}\omega_0\tau}. \tag{3.60}$$

Let us assume that the ensemble average in Eq. (3.60) has the form of a Gaussian distribution, i.e. that

$$\left\langle e^{-\mathrm{i}\phi(t)}e^{\mathrm{i}\phi(t+\tau)}\right\rangle = e^{-\tau^2\sigma^2/2}. \tag{3.61}$$

The cross-spectral density of the source is then given by the expression

$$W_f(x_1, x_2, \omega) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma_f(x_1, x_2, \tau)e^{\mathrm{i}\omega\tau}\mathrm{d}\tau = F(x_1)F(x_2)\frac{1}{\sqrt{2\pi}\sigma}e^{-(\omega-\omega_0)^2/2\sigma^2}. \tag{3.62}$$

We show in Appendix I that the cross-spectral density of the displacement at points to the right of the source is given by the formula

$$W_y(x_1, x_2, \omega) = \frac{(2\pi)^2}{4k^2T^2}\tilde{W}_f(-k, k, \omega)e^{-\mathrm{i}k(x_1-x_2)}, \tag{3.63}$$

where $\tilde{W}_f$ is the two-dimensional Fourier transform of the force density, i.e.

$$\tilde{W}_f(-k_1, k_2, \omega) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}W_f(x_1, x_2, \omega)e^{\mathrm{i}k_1x_1-\mathrm{i}k_2x_2}\mathrm{d}x_1\mathrm{d}x_2. \tag{3.64}$$

The total “wave intensity”, here denoted by $I(x)$, is given by the formula

$$I(x) = \langle y_A^*(x,t)y_A(x,t)\rangle \equiv \Gamma_y(x, x, 0) = \int_0^{\infty}W_y(x, x, \omega)\mathrm{d}\omega, \tag{3.65}$$

where $y_A(x, t)$ is again the complex analytic signal representation of $y(x, t)$. Substituting from Eqs. (3.58), (3.63) and (3.64) into Eq. (3.65), the wave intensity is found to be given by the expression

$$I(x) = \frac{F_0^2}{T^2}\int_0^{\infty}\frac{\left[\sin\frac{\omega x_0}{v}\right]^2}{(\omega/v)^4}\frac{1}{\sqrt{2\pi}\sigma}e^{-(\omega-\omega_0)^2/2\sigma^2}\mathrm{d}\omega. \tag{3.66}$$

This expression is, of course, only approximate, because of the approximation leading to Eq. (3.59). In particular, any realistic force spectrum would not include a constant ($\omega = 0$) component; such a constant force represents a continuous "pushing" on the string which would significantly distort its shape. In the calculations which follow, we will neglect all components of the force spectrum beyond a distance 3σ from the center frequency.

It is clear from Eq. (3.66) that for this force density, nonpropagating solutions do not exist. Nevertheless, the nonpropagating phenomenon still affects the behavior of outgoing waves. To see this, we consider a situation as described above for which the center frequency ω_0 of the driving force may be adjusted. If the bandwidth of the driving force is sufficiently narrow, one would expect the intensity $I(x)$ of the waves propagating away from the region of applied force to approach local minima as the center frequency approaches values for which $\sin(\omega_0 x_0/v) = 0$, i.e. values for which $\omega_0/v \to n\pi/x_0$. This effect can be seen in figure 3.6. These special frequencies correspond to the frequencies for which a constant monochromatic force distribution produces nonpropagating solutions.

The example which we have just considered is a good illustration of the phenomenon of correlation-induced spectral changes, which has attracted a good deal of attention in recent years [69]. In the present case, the spectrum of the force density is independent of position throughout the region of the applied force and is given by the expression

$$S_f(x,\omega) \equiv W_f(x,x,\omega) = |F_0|^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-(\omega-\omega_0)^2/2\sigma^2}, \quad |x| \leq x_0. \tag{3.67}$$

The spectrum of the waves propagating away from the region of the applied force,

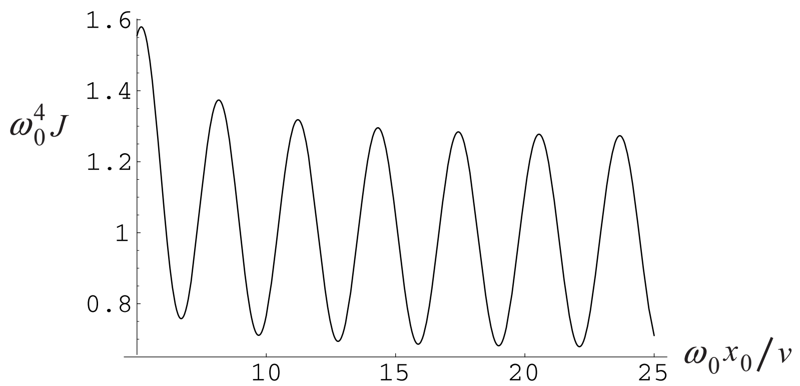


Figure 3.6: The normalized intensity $J = \sqrt{2\pi}\sigma T^2 I/(F_0^2 x_0 v^3)$ of waves propagating away from the region of applied force, scaled by the fourth power of the center frequency ω_0, with the choice $\sigma x_0/v = \pi/4$. I is given by Eq. (3.66).

however, is given by the expression

$$S_y(x,\omega) \equiv W_y(x,x,\omega) = \frac{F_0^2}{T^2}\frac{[\sin kx_0]^2}{k^4}\frac{1}{\sqrt{2\pi}\sigma}e^{-(\omega-\omega_0)^2/2\sigma^2}, \tag{3.68}$$

which is clearly different from the spectrum S_f of the applied force. The normalized spectra $s_f(\omega)$ and $s_y(\omega)$ are plotted in figure 3.7 for the case when $\omega_0 x_0/v = 4\pi$, $\sigma x_0/v = \pi/4$. We see that whilst the spectrum of the force density has a single peak, the spectrum of the propagated wave has two peaks because the frequency component $\omega = 4\pi v/x_0$ is nonpropagating. This example shows that nonpropagating excitations, and nonradiating sources, are a special case of a correlation-induced spectral change, for which the wave (or field) spectrum vanishes at a particular frequency. We will discuss correlation-induced spectral changes further in the following chapters.

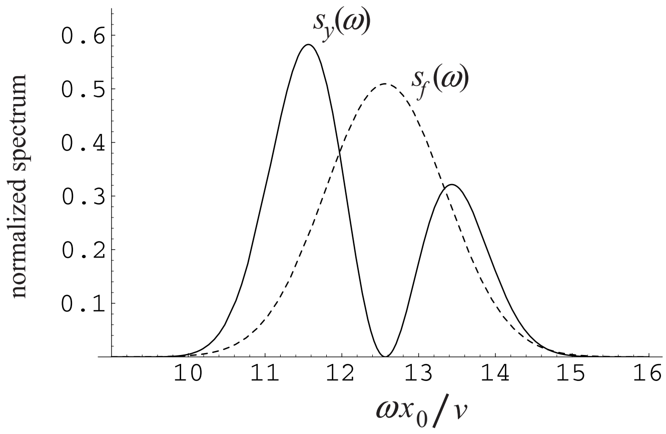


Figure 3.7: The normalized spectrum of the applied force density and of the string displacement, with the choice $\omega_0 x_0/v = 4\pi$, $\sigma x_0/v = \pi/4$. The normalized spectrum of the force density is defined by the expression $s_f(\omega) \equiv S_f(x,\omega)/(\int_0^\infty S_f(x,\omega')\mathrm{d}\omega')$, with a similar expression for the normalized string displacement spectrum, $s_y(\omega)$.

3.5 Proposal of a simple experiment to observe nonpropagating excitations

We are now in a position to suggest a simple experiment which might be used to detect nonpropagating excitations on a string.

The preceding sections have demonstrated that an important characteristic of such excitations is their *stability*; by stability we mean that the behavior of nonpropagating solutions remains essentially the same when the system is perturbed slightly. For instance, we have seen that the step function driving force produces nonpropagating excitations when $kl = n\pi$. If the system is perturbed by a damping force, these special frequencies become frequencies at which the outgoing wave amplitude is a minimum (recall figure 3.3). Likewise, if the driving force is quasi-monochromatic instead of monochromatic, the wave amplitude is minimum at the special frequencies.

This stability suggests the following experiment: Apply a step function driving force, with a tunable frequency, to a flexible string. Arrange the system to measure the amplitude of waves leaving the region of applied force. If the driving force is highly quasi-monochromatic, and the damping sufficiently weak, one should see minimum wave amplitude at the frequencies $kl = n\pi$, for integer n. These frequencies are those for which the outgoing waves are nearly cancelled by destructive interference.

Perhaps the most challenging obstacle in setting up this experiment will be the production of the driving force. It is to be noted that, as can be seen from the solution to the Helmholtz equation (3.7), waves propagate freely in the region of applied force, which suggests that the string must be free to vibrate in that region. The force density must be applied to the string *without touching or otherwise restraining* the string. It should be possible to generate such vibrations by electromagnetic or acoustical methods.

3.6 Partially propagating excitations

In this chapter, we have been investigating localized excitations in one dimension in part to gain insight into the behavior and properties of three-dimensional nonradiating sources. While it has been productive to do so and many results in the one-dimensional theory have three-dimensional analogues, the relationship between the two cases is not exact. In this section we illustrate one such difference by considering an unusual class of force distributions which produce propagating excitations in only one direction.

Let us consider again monochromatic excitations on an undamped string of infinite length. We have seen that the displacement $y(x)$ of the string to the right (R) and left (L) of the region of applied force is given by the equations (Eqs. (3.8) and (3.9) of section 3.1)

$$y(x)|_R = \frac{e^{\mathrm{i}kx}}{2\mathrm{i}k}\int_a^b q(x')e^{-\mathrm{i}kx'}\mathrm{d}x' = \frac{\pi e^{\mathrm{i}kx}}{\mathrm{i}k}\tilde{q}(k) \tag{3.69}$$

and

$$y(x)|_L = \frac{e^{-\mathrm{i}kx}}{2\mathrm{i}k}\int_a^b q(x')e^{\mathrm{i}kx'}\mathrm{d}x' = \frac{\pi e^{-\mathrm{i}kx}}{\mathrm{i}k}\tilde{q}(-k). \tag{3.70}$$

From these relations, it is clear that the displacement vanishes on the right of the region of applied force if and only if

$$\tilde{q}(k) = 0, \tag{3.71}$$

and the displacement vanishes on the left of the region of applied force if and only if

$$\tilde{q}(-k) = 0. \tag{3.72}$$

Let us first consider a force density for which the spatial part $q(x)$ is real; in such a case, the two equations (3.71) and (3.72) are equivalent, as can be seen by considering the complex conjugate of Eq. (3.71), viz.

$$[\tilde{q}(k)]^* = \left[\frac{1}{2\pi}\int_a^b q(x')e^{-\mathrm{i}kx'}\mathrm{d}x'\right]^* = \frac{1}{2\pi}\int_a^b q(x')e^{\mathrm{i}kx'}\mathrm{d}x' = \tilde{q}(-k). \tag{3.73}$$

If $q(x)$ is a complex function, however, then Eqs. (3.71) and (3.72) are not equivalent, and the possibility exists of one equation but not the other being satisfied. Such systems may be said to have *partially propagating excitations*, as waves will propagate away from the region of the applied force only on the side for which

$$\tilde{q}(k) \neq 0 \quad \text{or} \quad \tilde{q}(-k) \neq 0. \tag{3.74}$$

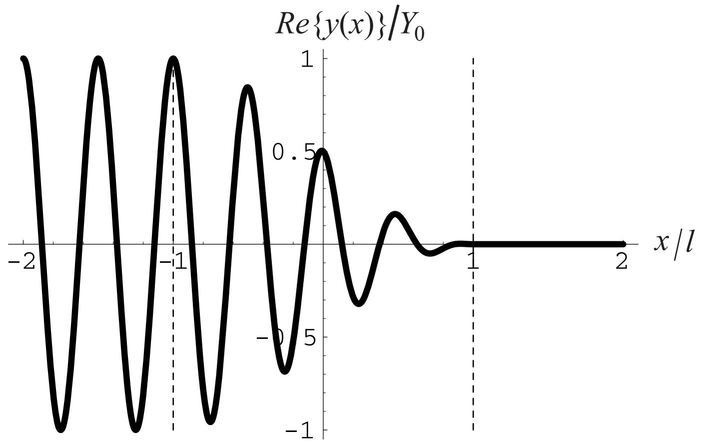


Figure 3.8: The real part of the displacement $y(x)$ of a damped string of infinite length when acted upon by a monochromatic force density given by Eq. (3.75). The dashed lines indicate the boundaries of the region of applied force. Here $kl = 1$.

As an example of such a force density, consider the distribution given by the formula

$$q(x) = \begin{cases} -\frac{3Y_0}{2d}\left[\mathrm{i}k - \frac{\mathrm{i}kx^2}{d^2} + \frac{x}{d^2}\right]e^{-\mathrm{i}kx}, & |x| \le l, \\ 0, & |x| > l. \end{cases} \tag{3.75}$$

Using Eq. (3.7), the displacement generated by this force distribution can be shown to be

$$y(x) = \begin{cases} Y_0 e^{-\mathrm{i}kx}, & x < -l, \\ Y_0\left[\frac{1}{2} - \frac{3}{4d}x + \frac{1}{4d^3}x^3\right]e^{-\mathrm{i}kx}, & |x| \le l, \\ 0, & x > l. \end{cases} \tag{3.76}$$

The real part of the displacement $y(x)$ is plotted in figure 3.8.

In the three-dimensional radiation problem, the radiation pattern outside the source either vanishes identically or vanishes only for a set of directions of observation of zero measure. In the one-dimensional string problem, the excitations may vanish over one half of the domain outside the region of applied force. This difference can be traced to the different analytic properties of the fields outside the sources in the two cases, and seems to indicate a significant difference between the one-dimensional and three-dimensional radiation problems.

It may be argued, however, that this difference has arisen from an improper comparison of the two problems. We have compared the uncountably infinite number of directions into which a three-dimensional source may radiate to the two directions into which a one-dimensional source may radiate. Let us consider instead the field of a three-dimensional source $q(\mathbf{r})$ in *two* directions, $k\mathbf{s}$ and $-k\mathbf{s}$, and ask the following question: Is it possible for the source $q(\mathbf{r})$ to produce a field in the direction $k\mathbf{s}$ but not the direction $-k\mathbf{s}$? It should be clear that the answer to this question is "yes," and in this sense the one-dimensional theory is in complete agreement with the theory of three-dimensional sources. Nevertheless, the existence of partially propagating excitations illustrates that the comparison between the two problems must be treated with some caution.

3.7 Two-dimensional nonpropagating excitations

We have demonstrated that the wave equation has localized solutions in one dimension (nonpropagating excitations) and in three dimensions (nonradiating sources). One may wonder if such excitations are also possible in a two-dimensional space, and in this section we show that this is so.

We consider waves on a two-dimensional surface, such as waves on a vibrating drumhead or waves on the surface of a pond (see, for instance, [67], section 5.2). The amplitude $V(\boldsymbol{\rho}, t)$ of waves produced by an applied force $F(\boldsymbol{\rho}, t)$ (localized to the domain D) satisfies the two-dimensional wave equation

$$\left[\nabla^2 - \frac{\partial^2}{\partial t^2}\right] V(\boldsymbol{\rho}, t) = F(\boldsymbol{\rho}, t), \tag{3.77}$$

where ∇^2 is the Laplacian with respect to the two-dimensional position vector $\boldsymbol{\rho}$. We consider again only monochromatic applied forces and monochromatic solutions, so that

$$V(\boldsymbol{\rho}, t) = Re\left\{v(\boldsymbol{\rho})e^{-\mathrm{i}\omega t}\right\}, \tag{3.78}$$

$$F(\boldsymbol{\rho}, t) = Re\left\{f(\boldsymbol{\rho})e^{-\mathrm{i}\omega t}\right\}. \tag{3.79}$$

For such solutions, Eq. (3.77) reduces to the inhomogeneous Helmholtz equation in two dimensions, i.e.

$$\left[\nabla^2 + k^2\right] v(\boldsymbol{\rho}) = f(\boldsymbol{\rho}). \tag{3.80}$$

The solution to this problem can be shown to be ([68], section 16.6)

$$v(\boldsymbol{\rho}) = -\frac{\mathrm{i}}{4}\int_D f(\boldsymbol{\rho}')H_0^{(1)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|)\mathrm{d}^2\rho', \tag{3.81}$$

where $H_0^{(1)}(x)$ is the zeroth order Hankel function of the first kind[2].

We first consider solutions to this equation far from the region of the applied force, for which $k|\boldsymbol{\rho} - \boldsymbol{\rho}'| \gg 1$. We may then examine the asymptotic form of the Hankel function of the first kind, which is given by ([68], section 11.6)

$$H_0^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}}e^{-\mathrm{i}\pi/4}e^{\mathrm{i}x}, \quad x \gg 1. \tag{3.82}$$

Furthermore, for $|\boldsymbol{\rho}| \gg |\boldsymbol{\rho}'|$, we may write

$$|\boldsymbol{\rho} - \boldsymbol{\rho}'| \approx \rho - \mathbf{s}\cdot\boldsymbol{\rho}', \tag{3.83}$$

[2]Hankel functions of the first kind are defined in [68], section 11.4. Another description of such functions can be found in [70], chapter 5.

where $\boldsymbol{\rho} = \rho\mathbf{s}$. On substituting from Eqs. (3.82) and (3.83) into Eq. (3.81), one finds that the wave amplitude far from the region of the applied force is

$$v(\boldsymbol{\rho}) \approx -\sqrt{\frac{2}{\pi\rho}} e^{-\mathrm{i}\pi/4} e^{\mathrm{i}k\rho} \frac{(2\pi)^2 \mathrm{i}}{4} \tilde{f}(k\mathbf{s}), \tag{3.84}$$

where

$$\tilde{f}(\mathbf{K}) = \frac{1}{(2\pi)^2} \int_D f(\boldsymbol{\rho}') e^{-\mathrm{i}\mathbf{K}\cdot\boldsymbol{\rho}'} \mathrm{d}^2\rho' \tag{3.85}$$

is the *two*-dimensional Fourier transform of the applied force. Evidently this two-dimensional source will not radiate if

$$\tilde{f}(k\mathbf{s}) = 0 \quad \text{for all} \quad \mathbf{s}. \tag{3.86}$$

This condition is comparable to the nonradiating condition (2.9) derived for three-dimensional sources and the nonpropagating condition (3.10) for one-dimensional sources. We have seen that in three dimensions, a source is nonradiating if its Fourier transform vanishes on a sphere of radius k; in one dimension, a source is nonpropagating if its Fourier transform vanishes at distances $\pm k$. A two-dimensional source is nonpropagating if its Fourier transform vanishes on a circle of radius k.

As an example of such a two-dimensional source, we consider a force density applied uniformly within a circle of radius a, i.e.

$$f(\boldsymbol{\rho}) = \begin{cases} f_0 & |\boldsymbol{\rho}| \leq a, \\ 0 & |\boldsymbol{\rho}| > a. \end{cases} \tag{3.87}$$

This is analogous to the homogeneous spherical source defined in Eq. (2.10) and the step function driving force defined in Eq. (3.12). On substitution of this force density into the nonpropagating condition (3.86), it is not difficult to show that Eq. (3.86) reduces to

$$J_1(ka) = 0, \tag{3.88}$$

where J_1 is a Bessel function of order 1. The uniform circular force density is therefore nonpropagating if ka is a zero of J_1. This condition is comparable to that derived for a uniform spherical nonradiating source, Eq. (2.11), and that derived for a step function nonpropagating source, Eq. (3.13).

Evidently nonpropagating sources exist in two dimensions, and have comparable behaviors to their one and three-dimensional counterparts. To investigate them further, and in particular to study further the uniform circular force density, it would be worthwhile to obtain a multipole expansion of the two-dimensional Green's function $H_0^{(1)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|)$, similar to the expansion (2.32) for the three-dimensional Green's function. Fortunately, it is not difficult to derive such an expansion, as we now show.[3]

Let us assume that the two-dimensional Green's function $G(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ is expandable in terms of a series of angular orthonormal functions, i.e.

$$G(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \sum_{m=-\infty}^{\infty} g_m(\rho_1, \rho_2)\Theta_m^*(\theta_1)\Theta_m(\theta_2), \tag{3.89}$$

where

$$\Theta_m(\theta) = \frac{1}{\sqrt{2\pi}}e^{im\theta}. \tag{3.90}$$

By definition the Green's function must satisfy the relation

$$\nabla_1^2 G(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) + k^2 G(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \delta^{(2)}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1), \tag{3.91}$$

where $\delta^{(2)}$ is the two-dimensional Dirac delta function, which may be expressed in terms of ρ and θ as

$$\delta^{(2)}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1) = \frac{1}{\rho_1}\delta(\rho_2 - \rho_1)\delta(\theta_2 - \theta_1). \tag{3.92}$$

[3]This calculation has undoubtedly been done elsewhere, but explicit derivations are difficult to find. We perform the derivation here for completeness.

The delta function with respect to θ may be expanded in terms of the angular functions Θ_m as

$$\delta(\theta_2 - \theta_1) = \sum_{m=-\infty}^{\infty} \Theta_m^*(\theta_1)\Theta_m(\theta_2). \tag{3.93}$$

On substituting from Eqs. (3.89), (3.92), and (3.93) into Eq. (3.91), and noting that, because of the orthogonality of the Θ_m, the resulting equation must be satisfied independently for each m, we arrive at the set of equations

$$\frac{\partial}{\partial\rho_1}\left(\rho_1\frac{\partial g_m}{\partial\rho_1}\right) - \frac{m^2}{\rho_1}g_m + k^2\rho_1 g_m = \delta(\rho_2 - \rho_1). \tag{3.94}$$

We have therefore reduced the two-dimensional Green's function problem to the problem of determining the Green's function for a one-dimensional Sturm-Liouville equation. It can be found by use of standard methods in Sturm-Liouville theory[4] that

$$g_m(\rho_1, \rho_2) = \begin{cases} \frac{\pi}{2\mathrm{i}} J_m(k\rho_1)H_m^{(1)}(k\rho_2), & \rho_2 > \rho_1, \\ \frac{\pi}{2\mathrm{i}} H_m^{(1)}(k\rho_1)J_m(k\rho_2), & \rho_2 < \rho_1, \end{cases} \tag{3.95}$$

where J_m and $H_m^{(1)}$ are the Bessel and Hankel functions of the first kind of order m, respectively. This solution is found by requiring that the Green's function is finite at the origin and has the form of an outgoing wave ($\sim e^{\mathrm{i}k\rho}/\sqrt{\rho}$) far away from the origin. On substituting from Eq. (3.95) into Eq. (3.89), we may expand the two-dimensional Green's function as

$$-\frac{\mathrm{i}}{4}H_0^{(1)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|) = \frac{\pi}{2\mathrm{i}} \sum_{m=-\infty}^{\infty} J_m(k\rho_<)H_m^{(1)}(k\rho_>)\Theta_m^*(\theta_1)\Theta_m(\theta_2), \tag{3.96}$$

where $\rho_<$ and $\rho_>$ are respectively the lesser and greater of ρ_1, ρ_2. This expression is analogous to that of the three-dimensional Green's function, Eq. (2.32), with

[4]For a discussion of the solution of inhomogeneous Sturm-Liouville problems, see [68], section 16.5, or [71], chapter 8.

the replacement of the spherical Bessel functions j_m, $h_m^{(1)}$ by the ordinary Bessel functions J_m, $H_m^{(1)}$.

Let us use Eq. (3.96) to determine the field within the uniform circular force density. On substituting from Eq. (3.96) into Eq. (3.81), one finds that

$$v(\boldsymbol{\rho}) = \frac{\pi f_0}{2\mathrm{i}} \left\{ H_0^{(1)}(k\rho) \int_0^{\rho} J_0(k\rho')\rho' \mathrm{d}\rho' + J_0(k\rho) \int_{\rho}^{a} H_0^{(1)}(k\rho')\rho' \mathrm{d}\rho' \right\}. \tag{3.97}$$

The integrations may be carried out explicitly by using the following recurrence relation ([68], Eq. (11.15))

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^n W_n(x)] = x^n W_{n-1}(x), \tag{3.98}$$

where W_n may be either a Bessel function or Hankel function of order n. Equation (3.97) then reduces to

$$v(\boldsymbol{\rho}) = \frac{\pi f_0}{2k^2\mathrm{i}} \left\{ H_0^{(1)}(k\rho)J_1(k\rho)k\rho + J_0(k\rho)H_1^{(1)}(ka)ka - J_0(k\rho)H_1^{(1)}(k\rho)k\rho \right\}. \tag{3.99}$$

This expression may be simplified by using another Bessel function relation ([68], exercise 11.4.1)

$$J_{m-1}(x)H_m^{(1)}(x) - J_m(x)H_{m-1}^{(1)}(x) = \frac{2}{\mathrm{i}\pi x}. \tag{3.100}$$

By use of this relation, the wave amplitude within the domain of the source is found to be given by the expression

$$v(\boldsymbol{\rho}) = \frac{\pi f_0}{2k^2\mathrm{i}} \left\{ -\frac{2}{\pi\mathrm{i}} + J_0(k\rho)H_1^{(1)}(ka)ka \right\} \tag{3.101}$$

The real part of the wave amplitude is plotted for the first three nonpropagating excitations in figure 3.9. It is clear from the figure that the amplitude goes smoothly to zero at the boundary of the region of the applied force; this suggests that there is a theorem for two-dimensional nonpropagating excitations analogous to Theorem 2.4

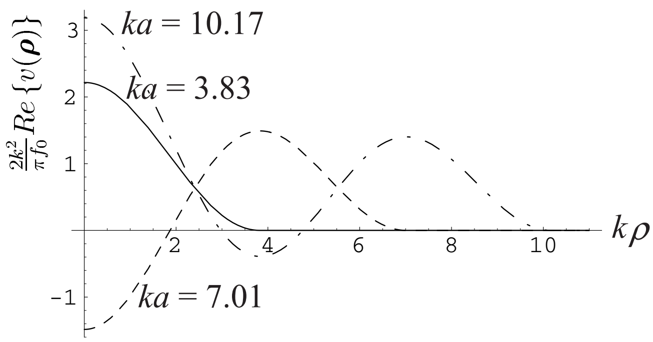


Figure 3.9: The (normalized) wave amplitude $v(\boldsymbol{\rho})$ as a function of $k\rho$ for the first three uniform circular force densities which produce nonpropagating excitations. The (scaled) radii of these force densities are $ka = 3.83$, $ka = 7.01$, and $ka = 10.17$.

for nonradiating sources and Theorem 3.1 for nonpropagating excitations on a string. For a uniform circular force density, this can be seen explicitly by differentiating Eq. (3.101). The wave amplitude for the force density for $ka = 10.17$ is plotted in three dimensions in figure 3.10.

This section was intended to serve as a brief introduction to two-dimensional nonpropagating excitations. We have seen that they exhibit many of the properties of their one-dimensional and three-dimensional counterparts. Presumably most of the results of the one and three dimensional theories could be proven in the two-dimensional case as well. As we are primarily interested in the three-dimensional radiation problem, we now leave nonpropagating excitations behind and return to nonradiating sources.

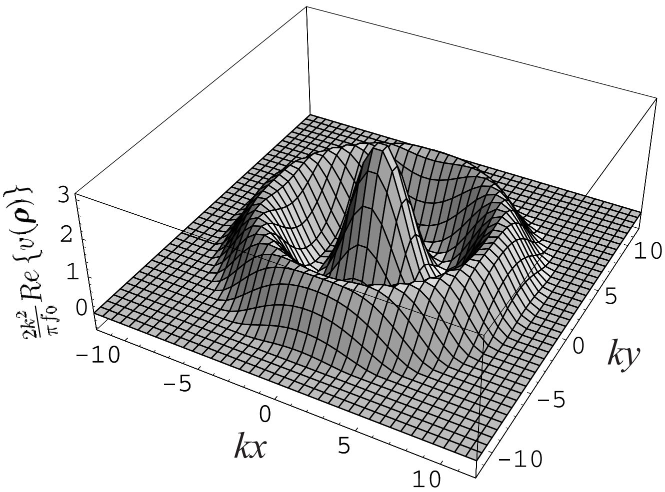


Figure 3.10: The (normalized) wave amplitude $v(\boldsymbol{\rho})$ as a function of kx, ky for the nonpropagating excitation with $ka = 10.17$.

Chapter 4

Effects associated with nonradiating sources

4.1 Sources of arbitrary states of coherence that generate completely coherent fields outside the source domain

We have seen that, for a fully coherent source, it is not possible to determine the structural features of the source from measurements of its external field. This difficulty is directly related to the existence of nonradiating sources. As nonradiating partially coherent sources also exist, one might expect that the structural features of a partially coherent source, such as the spectral degree of coherence, cannot be determined from measurements of the cross-spectral density of the field. A consequence of this observation is that sources with many different states of coherence may produce fully coherent fields, or no field at all, outside the source domain. In this section we demonstrate that there exist such sources. We begin by deriving a theorem about partially coherent nonradiating sources which we will need for our analysis.

Consider a fluctuating scalar source distribution $Q(\mathbf{r}, t)$ that occupies, for all time, a finite domain D. Here $\mathbf{r}$ denotes a position vector and t denotes the time. We assume that the source fluctuations are statistically stationary, at least in the wide sense ([28], p. 47).

Let $W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega)$ be the cross-spectral density function, at frequency ω, of the source distribution. It is known that under very general conditions W_Q may be expanded in a Mercer-type series, viz. ([28], section 4.7.1 and [72])

$$W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) = \sum_n \lambda_n(\omega)\phi_n^*(\mathbf{r}_1, \omega)\phi_n(\mathbf{r}_2, \omega), \tag{4.1}$$

where the λ_n's and ϕ_n's (called the coherent modes of the source) are the eigenvalues and the eigenfunctions, respectively, of the integral equation

$$\int_D W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega)\phi_n(\mathbf{r}_1, \omega)\mathrm{d}^3 r_1 = \lambda_n(\omega)\phi_n(\mathbf{r}_2, \omega), \tag{4.2}$$

and

$$\lambda_n(\omega) > 0 \quad \text{for all } n. \tag{4.3}$$

For a three-dimensional source, n generally represents a triad of indexes.

The radiant intensity $J(\mathbf{s}, \omega)$, i.e. the rate at which the source radiates energy at frequency ω per unit solid angle around a direction specified by a unit vector $\mathbf{s}$ is given by the expression ([28], p. 232, with a slightly different definition of the Fourier transform)

$$J(\mathbf{s}, \omega) = \tilde{W}_Q(-k\mathbf{s}, k\mathbf{s}, \omega), \tag{4.4}$$

where

$$\tilde{W}_Q(\mathbf{K}_1, \mathbf{K}_2, \omega) = \int_D \int_D W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) e^{-\mathrm{i}(\mathbf{K}_1\cdot\mathbf{r}_1 + \mathbf{K}_2\cdot\mathbf{r}_2)}\mathrm{d}^3 r_1 \mathrm{d}^3 r_2 \tag{4.5}$$

is the six-dimensional spatial Fourier transform of the cross-spectral density function and

$$k = \omega/c, \tag{4.6}$$

c being the speed of light in vacuum.

On substituting from Eq. (4.1) into Eq. (4.4), it readily follows that, in terms of the source modes ϕ_n, the radiant intensity may be expressed in the form

$$J(\mathbf{s}, \omega) = \sum_n \lambda_n(\omega) |\tilde{\phi}_n(k\mathbf{s}, \omega)|^2, \tag{4.7}$$

where

$$\tilde{\phi}_n(\mathbf{K}, \omega) = \int_D \phi_n(\mathbf{r}, \omega) e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}} \mathrm{d}^3 r \tag{4.8}$$

is the three-dimensional spatial Fourier transform of the source mode $\phi_n(\mathbf{r}, \omega)$.

Because the λ_n's are positive, it is clear from Eq. (4.7) that if the source does not radiate at frequency ω, i.e. if

$$J(\mathbf{s}, \omega) = 0 \tag{4.9}$$

for all directions $\mathbf{s}$, then

$$\tilde{\phi}_n(\mathbf{K}, \omega) = 0 \tag{4.10}$$

for all real vectors $\mathbf{K}$ of magnitude $|\mathbf{K}| = k = \omega/c$, and for all n. This requirement implies that the monochromatic source

$$Q_n(\mathbf{r}, t) = \phi_n(\mathbf{r}, \omega) e^{-\mathrm{i}\omega t} \tag{4.11}$$

itself does not radiate [12, 19]. We have therefore established the following theorem:

Theorem 4.1 *If a statistically stationary stochastic source does not radiate at frequency ω, all of its coherent modes $\phi_n(\mathbf{r}, \omega)$ are nonradiating modes.*

It is also known that the field $\psi(\mathbf{r},\omega)$ generated by a nonradiating monochromatic source of frequency ω (i.e. by a monochromatic source that generates a field whose radiant intensity $J(\mathbf{s},\omega)=0$ for all directions $\mathbf{s}$) vanishes at every point outside the source. Expressed differently, if Eq. (4.10) holds for all $|\mathbf{K}|$ of magnitude ω/c, then [10]

$$\psi(\mathbf{r},\omega)=0 \quad \text{for all } \mathbf{r}\notin D. \tag{4.12}$$

In view of Theorem 4.1, we may now describe sources of arbitrary states of spatial coherence that produce fields that are spatially completely coherent outside the source domain.

Let us consider a stochastic, statistically stationary, source occupying a finite domain D, whose cross-spectral density has the mode expansion

$$\begin{aligned} W_Q(\mathbf{r}_1,\mathbf{r}_2,\omega) &= \lambda_0(\omega)\phi_0^{R*}(\mathbf{r}_1,\omega)\phi_0^{R}(\mathbf{r}_2,\omega) \\ &\quad + \sum_{n=1}^{N}\lambda_n(\omega)\phi_n^{NR*}(\mathbf{r}_1,\omega)\phi_n^{NR}(\mathbf{r}_2,\omega), \end{aligned} \tag{4.13}$$

where $\mathbf{r}_1\in D$, $\mathbf{r}_2\in D$, and N is an arbitrary positive integer. In this expression $\phi_0^R(\mathbf{r},\omega)$ is a radiating mode and $\phi_n^{NR}(\mathbf{r},\omega)$ (with $n=1,2,...$) are nonradiating modes.[1] It is clear that only the mode $\phi_0^R(\mathbf{r},\omega)$ will generate a field outside D. Consequently the cross-spectral density of the field throughout the exterior of the source domain, i.e. for $\mathbf{r}_1\notin D$, $\mathbf{r}_2\notin D$, will be given by the expression (see [28], section 4.7.3, and [73])

$$W_\psi(\mathbf{r}_1,\mathbf{r}_2,\omega)=\lambda_0(\omega)\psi_0^{R*}(\mathbf{r}_1,\omega)\psi_0^{R}(\mathbf{r}_2,\omega), \tag{4.14}$$

[1] A general method for specifying nonradiating monochromatic sources localized in a given domain was described in [20].

where

$$\psi_0^R(\mathbf{r},\omega) = \int_D \phi_0^R(\mathbf{r}',\omega)\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3r'. \tag{4.15}$$

Because, according to Eq. (4.14), W_ψ is a product of a function of $\mathbf{r}_1$ and a function of $\mathbf{r}_2$, the field is necessarily spatially completely coherent at frequency ω throughout the exterior of the source domain D (see [28], section 4.5.3, and [74]). However, Eq. (4.13) does not factorize in this way and, therefore, the source distribution is necessarily only partially coherent. One can show that as the number N of the nonradiating source modes is increased, i.e. as larger and larger numbers of such modes contribute to the field within the source region and if the coefficients λ_n are of the same order of magnitude, the source will become spatially highly incoherent. Yet all such sources will produce a completely spatially coherent field outside the source domain.

This fact is illustrated in figure 4.1. The spectral degree of coherence is plotted for three source distributions, with the position vectors chosen to be collinear; each source distribution contains a different number of nonradiating modes. The nonradiating modes (written below in an unnormalized form) were taken to be given by the expression

$$\phi_{lmN}^{NR}(\mathbf{r}) = \sum_{n=0}^{N-1}\alpha_{ln}\Lambda_{lmn}(\mathbf{r}) - \frac{1}{\alpha_{lN}}\left[\sum_{n=0}^{N-1}[\alpha_{ln}]^2\right]\Lambda_{lmN}(\mathbf{r}), \tag{4.16}$$

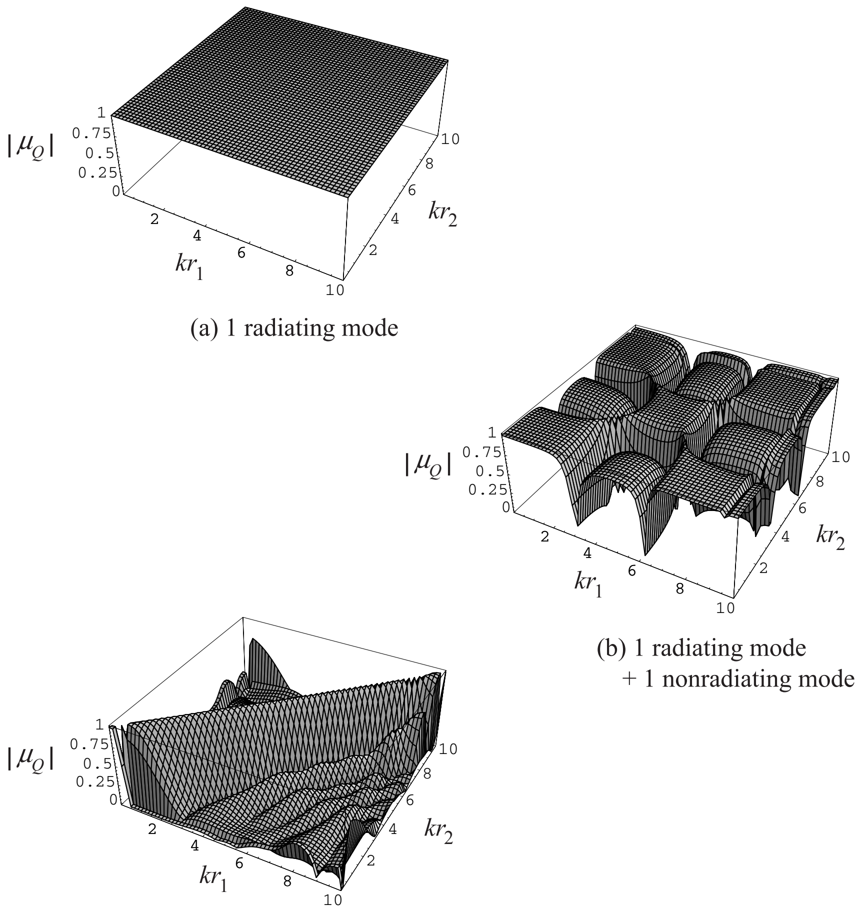
where

$$\Lambda_{lmn}(\mathbf{r}) = \left[\frac{2}{a^3}\right]^{1/2}\frac{1}{|j_{l+1}(k_{ln}a)|}j_l(k_{ln}r)Y_l^m(\theta,\phi), \tag{4.17}$$

$$\alpha_{ln} = -\frac{k_{ln}}{k^2-k_{ln}^2}\frac{j_l'(k_{ln}a)}{|j_{l+1}(k_{ln}a)|}, \tag{4.18}$$

and

$$j_\nu'(u) \equiv \frac{\mathrm{d}}{\mathrm{d}u}j_\nu(u). \tag{4.19}$$



(a) 1 radiating mode

(b) 1 radiating mode
+ 1 nonradiating mode

(c) 1 radiating mode
+ 845 nonradiating modes

Figure 4.1: Absolute value of the spectral degree of coherence $\mu_Q(\mathbf{r}_1, \mathbf{r}_2, \omega)$ of sources confined to spherical domains, with $ka = 10$, consisting of suitable linear combinations of a radiating mode and of nonradiating modes. The position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are collinear. It is to be noted that in (c) the source is highly spatially incoherent. Yet all these sources produce a completely spatially coherent field outside the domain containing the source.

In Eqs. (4.16) and (4.18), $k_{ln}a$ is the nth zero of the spherical Bessel function of order l, i.e. $j_l(k_{ln}a) = 0$, a is the radius of the spherical source, k is defined as in Eq. (4.6), and $Y_l^m(\theta, \phi)$ are the spherical harmonics. These modes, which are derived in Appendix II, are all nonradiating and mutually orthogonal, and together with the unnormalized "radiating" functions

$$\phi_{lmn}^{R}(\mathbf{r}, \omega) = j_n(kr) Y_l^m(\theta, \phi), \tag{4.20}$$

the modes $\phi_{lmn}^{NR}(\mathbf{r}, \omega)$ form a complete set of basis functions within the sphere $r < a$.

We have thus demonstrated that sources of quite arbitrary states of coherence can produce fields which are completely coherent outside the source domain. Similarly, by omitting the radiating mode from the expression (4.13), one can create sources of arbitrary states of coherence which produce *no* field at all outside the domain occupied by the source.

This example shows that even a quite incoherent source may be nonradiating, and it therefore might seem to suggest that the inverse source problem may not be uniquely solved for any class of partially coherent sources. However, this example may also be used to demonstrate that a certain class of highly incoherent sources must radiate, as we now show.

We consider a source whose cross-spectral density is of the form

$$W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) = \sum_{n=1}^{N} \lambda_n(\omega) \phi_n^{NR*}(\mathbf{r}_1, \omega) \phi_n^{NR}(\mathbf{r}_2, \omega), \tag{4.21}$$

where $\phi_n^{NR}(\mathbf{r}, \omega)$ are the orthonormal NR modes contained within the sphere $r < a$. This source will produce no field outside the domain of the sphere. As we have seen, as the number of equally-weighted modes increases (i.e. as $N \to \infty$), the source will become spatially highly incoherent and the cross-spectral density will approach the form of a delta function. However, as we have noted, the set of all nonradiating modes ϕ_n^{NR} does not form a complete set of basis functions for the sphere; the complete basis is the set of all nonradiating modes (ϕ_n^{NR}), together with the set of all radiating modes (ϕ_n^{R}). In this basis, a delta function can be expressed in the form[2]

$$\delta(\mathbf{r}_1 - \mathbf{r}_2) = \sum_{n=1}^{\infty} \phi_n^{R*}(\mathbf{r}_1, \omega)\phi_n^{R}(\mathbf{r}_2, \omega) + \sum_{n=1}^{\infty} \phi_n^{NR*}(\mathbf{r}_1, \omega)\phi_n^{NR}(\mathbf{r}_2, \omega). \tag{4.22}$$

The most incoherent nonradiating source that can be constructed is more coherent than a delta function source, because the nonradiating source does not include the radiating modes. It is evidently not even arbitrarily close to a delta function, because there are an infinite number of modes which are not included in the nonradiating source representation. This example suggests that there exists a class of very incoherent sources that will radiate. We will see in chapter 5 that quasi-homogeneous sources belong to this class.

4.2 Energy conservation and spectral changes for randomly fluctuating electromagnetic fields

In section 3.4, we demonstrated that a nonradiating field (or nonpropagating excitation) provides an example of a correlation-induced spectral change. When a source is nonradiating at frequency ω, that frequency does not appear at all in the spectrum of the field, even though it appears in the spectrum of the source.

It would seem at first sight that such spectral changes violate energy conser-

[2]Such a *closure relation* is well-known in the theory of orthogonal functions. See, for instance, [68], p. 528.

vation. That this is not so was demonstrated not long ago within the framework of scalar theory in [75]. It is expected that similar results will hold for randomly fluctuating electromagnetic sources and fields. Around 1960, after the rigorous laws of coherence theory of the electromagnetic field had been formulated, various conservation laws for such fields were derived in the space-time domain [76, 77, 78]. They turned out to be rather complicated and, probably because of this, little use has been made of them.

In this section we generalize the results of reference [75] and determine an energy conservation law in the space-frequency domain for all statistically stationary, fluctuating electromagnetic fields, and we demonstrate that correlation-induced spectral changes are in fact consistent with this conservation law.

We consider an electromagnetic field generated by a randomly fluctuating source polarization $\mathbf{P}(\mathbf{r}, \omega)$ occupying a domain D. We will assume that the fluctuations are stationary, at least in the wide sense([28], p. 47). Let $\langle \mathbf{F}(\mathbf{r}, \omega) \rangle$ represent the expectation value of the flux density vector (the Poynting vector) at frequency ω, at an arbitrary point $\mathbf{r}$ in the field. Using the space-frequency representation for the electromagnetic field [72, 73], the expectation value of the Poynting vector may be expressed in the form

$$\langle \mathbf{F}(\mathbf{r}, \omega) \rangle = \frac{c}{8\pi} Re \langle \mathbf{E}^*(\mathbf{r}, \omega) \times \mathbf{H}(\mathbf{r}, \omega) \rangle, \tag{4.23}$$

where the angular brackets denote the average taken over an ensemble of space-frequency realizations. On taking the divergence of this expression and on using the vector identity

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}), \tag{4.24}$$

it follows that

$$\nabla \cdot \langle \mathbf{F}(\mathbf{r},\omega)\rangle = \frac{c}{8\pi} Re\left\{\langle \mathbf{H}^*(\mathbf{r},\omega)\cdot[\nabla\times\mathbf{E}(\mathbf{r},\omega)]\rangle - \langle \mathbf{E}^*(\mathbf{r},\omega)\cdot[\nabla\times\mathbf{H}(\mathbf{r},\omega)]\rangle\right\}. \tag{4.25}$$

The right hand side of Eq. (4.25) may be simplified by the use of Maxwell's equations for monochromatic fields, viz.

$$\nabla\times\mathbf{E}(\mathbf{r},\omega) = \mathrm{i}k\mathbf{H}(\mathbf{r},\omega), \tag{4.26a}$$

$$\nabla\times\mathbf{H}(\mathbf{r},\omega) = -\mathrm{i}k\mathbf{E}(\mathbf{r},\omega) - 4\pi\mathrm{i}k\mathbf{P}(\mathbf{r},\omega). \tag{4.26b}$$

Substituting from these equations into Eq. (4.25), one finds that

$$\begin{aligned}\nabla \cdot \langle \mathbf{F}(\mathbf{r},\omega)\rangle = \frac{kc}{8\pi} Re\{&\mathrm{i}\langle \mathbf{H}^*(\mathbf{r},\omega)\cdot\mathbf{H}(\mathbf{r},\omega)\rangle + \mathrm{i}\langle \mathbf{E}^*(\mathbf{r},\omega)\cdot\mathbf{E}(\mathbf{r},\omega)\rangle \\ &+ 4\pi\mathrm{i}\langle \mathbf{E}^*(\mathbf{r},\omega)\cdot\mathbf{P}(\mathbf{r},\omega)\rangle\}.\end{aligned} \tag{4.27}$$

The first two terms on the right of Eq. (4.27) are purely imaginary, and do not contribute to the left-hand side. Equation (4.27) therefore reduces to

$$\nabla \cdot \langle \mathbf{F}(\mathbf{r},\omega)\rangle = -\frac{kc}{2} Im\langle \mathbf{E}^*(\mathbf{r},\omega)\cdot\mathbf{P}(\mathbf{r},\omega)\rangle. \tag{4.28}$$

Now it is known[3] that the electric field produced by a monochromatic source may be expressed in the form

$$\mathbf{E}(\mathbf{r},\omega) = [k^2 + \nabla(\nabla\cdot)]\int_D \mathbf{P}(\mathbf{r}',\omega)\frac{e^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3r'. \tag{4.29}$$

Substituting from this expression into Eq. (4.28), and introducing the cross-spectral density of the source polarization by the formula

$$W_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle P_i^*(\mathbf{r}_1,\omega)P_j(\mathbf{r}_2,\omega)\rangle, \tag{4.30}$$

[3]See [27], Eqs. (73) and (74) of chapter 2, with the change $\mathbf{J}(\mathbf{r},\omega) = \mathrm{i}\omega\mathbf{P}(\mathbf{r},\omega)$.

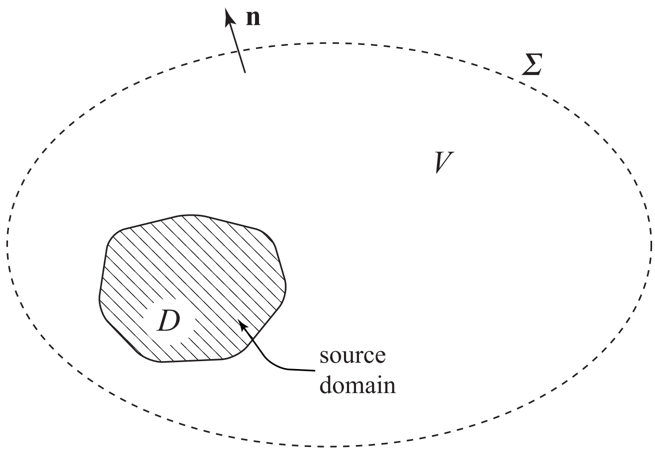


Figure 4.2: Illustrating notation relating to the integral form of the energy conservation law (4.34) for statistically stationary electromagnetic fields.

Eq. (4.28) may be expressed (in tensor notation) in the form

$$\nabla \cdot \langle \mathbf{F}(\mathbf{r}, \omega) \rangle = -\frac{kc}{2} Im \int_D W_{ij}^{(P)}(\mathbf{r}', \mathbf{r}, \omega)(k^2 \delta_{ij} + \partial_i \partial_j) \frac{e^{-\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \mathrm{d}^3 r', \qquad (4.31)$$

where summation over repeated indices is to be taken.

Equation (4.31) is the *differential form* of an energy conservation law for statistically stationary random electromagnetic fields. It is to be noted that when the point represented by the position vector $\mathbf{r}$ is located outside the source domain D, $W_{ij}^{(P)}(\mathbf{r}', \mathbf{r}, \omega) = 0$, and Eq. (4.31) reduces to the simple form

$$\nabla \cdot \langle \mathbf{F}(\mathbf{r}, \omega) \rangle = 0. \qquad (4.32)$$

The physical significance of formula (4.31) becomes more apparent if one converts it into integral form. Let us, therefore, integrate both sides of Eq. (4.31) over a volume V, bounded by a surface Σ, which completely encloses the source domain D (see Fig. 4.2). Making use of the divergence theorem and the fact that the polarization tensor vanishes everywhere outside the source domain D, it follows that

$$\int_{\Sigma}\langle\mathbf{F}(\mathbf{r},\omega)\rangle\cdot\mathbf{n}\mathrm{d}\Sigma=-\frac{kc}{2}Im\int_{D}\int_{D}W_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)(k^2\delta_{ij}+\partial_i\partial_j)\frac{e^{-\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3r\mathrm{d}^3r', \tag{4.33}$$

where $\mathbf{n}$ denotes the unit outward normal to Σ at the point $\mathbf{r}$. Noting that the cross-spectral density tensor, $W_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)$, summed over the subscripts i and j, is Hermitian, and that $e^{-\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}/|\mathbf{r}-\mathbf{r}'|$ is symmetric with respect to $\mathbf{r}$ and $\mathbf{r}'$, Eq. (4.33) may be rewritten in the form

$$\int_{\Sigma}\langle\mathbf{F}(\mathbf{r},\omega)\rangle\cdot\mathbf{n}\mathrm{d}\Sigma=\frac{k^2c}{2}\int_{D}\int_{D}W_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)(k^2\delta_{ij}+\partial_i\partial_j)\frac{\sin k|\mathbf{r}-\mathbf{r}'|}{k|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3r\mathrm{d}^3r'. \tag{4.34}$$

This formula is the *integral form* of the energy conservation law. It demonstrates that the rate at which the source radiates energy across any surface Σ which completely encloses the source domain D depends upon the second-order correlation properties of the source polarization, represented by the cross-spectral density tensor $W_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)$. The conservation laws (4.31) and (4.34) are a generalization to electromagnetic fields of energy conservation laws derived not long ago for fluctuating scalar fields([75], Eqs. (3.4) and (3.6)).

Let us now turn to the phenomenon of correlation-induced spectral changes. Consider the field in the far zone of the source at a point specified by the position vector $R\mathbf{u}$, ($\mathbf{u}^2=1$). The electric and magnetic fields are given by the expressions ([27], Eqs. (83) and (84) of chapter 2)

$$E_i(R\mathbf{u},\omega)\sim(2\pi)^3k^2\frac{e^{\mathrm{i}kR}}{R}(\delta_{ij}-u_iu_j)\tilde{P}_j(k\mathbf{u},\omega), \tag{4.35a}$$

$$H_i(R\mathbf{u},\omega)\sim(2\pi)^3k^2\frac{e^{\mathrm{i}kR}}{R}\epsilon_{ijk}u_j\tilde{P}_k(k\mathbf{u},\omega), \tag{4.35b}$$

where δ_{ij} is the Kroenecker delta symbol, ϵ_{ijk} is the completely antisymmetric unit tensor of Levi-Civita ([79], p. 685), and

$$\tilde{P}_i(\mathbf{K},\omega) = \frac{1}{(2\pi)^3}\int_D P_i(\mathbf{r},\omega)e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}}\mathrm{d}^3r \tag{4.36}$$

is the spatial Fourier transform of the source polarization.

Now the power spectrum of the field in the far zone at distance R from the source, in a direction specified by a unit vector $\mathbf{u}$, may be identified with the ensemble average of the energy density multiplied by the speed of light c, viz.

$$\begin{aligned} S^{(\infty)}(R\mathbf{u},\omega) &= c\langle U^{(\infty)}(R\mathbf{u},\omega)\rangle \\ &= \frac{c}{16\pi}\langle E_i^*(R\mathbf{u},\omega)E_i(R\mathbf{u},\omega)\rangle + \frac{c}{16\pi}\langle H_i^*(R\mathbf{u},\omega)H_i(R\mathbf{u},\omega)\rangle \\ &= \frac{c}{16\pi}\left(W_{ii}^{(E)}(R\mathbf{u},R\mathbf{u},\omega) + W_{ii}^{(H)}(R\mathbf{u},R\mathbf{u},\omega)\right). \end{aligned} \tag{4.37}$$

Here the cross-spectral density tensor of the electric field in the far zone is defined as

$$W_{ij}^{(E)}(R\mathbf{u}_1,R\mathbf{u}_2,\omega) = \langle E_i^*(R\mathbf{u}_1,\omega)E_j(R\mathbf{u}_2,\omega)\rangle, \tag{4.38}$$

with a similar definition for the cross-spectral density tensor of the magnetic field. Using Eqs. (4.35a) and (4.35b) in Eq. (4.37), we obtain for the spectrum of the field in the far zone the expression

$$S^{(\infty)}(R\mathbf{u},\omega) = \frac{8\pi^5k^4c}{R^2}\left[(\delta_{ij} - u_iu_j)\tilde{W}_{ij}^{(P)}(-k\mathbf{u},k\mathbf{u},\omega)\right], \tag{4.39}$$

where

$$\tilde{W}_{ij}^{(P)}(-k\mathbf{u},k\mathbf{u},\omega) = \frac{1}{(2\pi)^6}\int_D\int_D W_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega)e^{-\mathrm{i}k\mathbf{u}\cdot(\mathbf{r}_2-\mathbf{r}_1)}\mathrm{d}^3r_1\mathrm{d}^3r_2 \tag{4.40}$$

is the six-dimensional Fourier transform of the cross-spectral density tensor of the source. The spectrum of each Cartesian component of the source polarization may be defined by the expression

$$S_i^{(P)}(\mathbf{r},\omega) \equiv W_{ii}^{(P)}(\mathbf{r},\mathbf{r},\omega) \quad \text{(no summation)}, \tag{4.41}$$

and, consequently, the (tensorial) spectral degree of coherence for pairs of components of the source polarization may be defined by the formula

$$\mu_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \frac{W_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega)}{\sqrt{S_i^{(P)}(\mathbf{r}_1,\omega)}\sqrt{S_j^{(P)}(\mathbf{r}_2,\omega)}}. \tag{4.42}$$

The (tensorial) spectral degree of coherence represents the strength of correlation between Cartesian components of the polarization at different points $\mathbf{r}_1, \mathbf{r}_2$; it can be shown that

$$0 \le |\mu_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega)| \le 1 \tag{4.43}$$

for all values of $\mathbf{r}_1$, $\mathbf{r}_2$, ω, and all values of i,j. To see this, we note that

$$\left\langle |a_1 P_i(\mathbf{r}_1,\omega) + a_2 P_j(\mathbf{r}_2,\omega)|^2 \right\rangle \ge 0 \tag{4.44}$$

for all complex values of a_1, a_2. Upon expanding the left side of this inequality, and using Eqs. (4.30) and (4.41), it may be written as

$$|a_1|^2 S_i^{(P)}(\mathbf{r}_1,\omega) + a_1^* a_2 W_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega) + a_2^* a_1 W_{ji}^{(P)}(\mathbf{r}_2,\mathbf{r}_1,\omega) + |a_2|^2 S_j^{(P)}(\mathbf{r}_2,\omega) \ge 0. \tag{4.45}$$

We note that it follows from the definition of $W_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ that $W_{ij}^{(P)*}(\mathbf{r}_1,\mathbf{r}_2,\omega) = W_{ji}^{(P)}(\mathbf{r}_2,\mathbf{r}_1,\omega)$. Since inequality (4.45) must hold for all values of a_1 and a_2, it follows from a well-known property of non-negative definite quadratic forms[4] that

$$S_i^{(P)}(\mathbf{r}_1,\omega) S_j^{(P)}(\mathbf{r}_2,\omega) \ge |W_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega)|^2. \tag{4.46}$$

From this inequality (4.43) follows.

[4] See [80], Theorem 20, p. 337.

If we substitute for $\tilde{W}_{ij}^{(P)}$ from Eq. (4.40) into the expression for the far zone power spectrum given by Eq. (4.39), it follows that

$$S^{(\infty)}(R\mathbf{u},\omega) = \frac{8\pi^5 k^4 c}{R^2}\left[(\delta_{ij} - u_i u_j)\int_D\int_D W_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)e^{-ik\mathbf{u}\cdot(\mathbf{r}-\mathbf{r}')}\mathrm{d}^3r\mathrm{d}^3r'\right]. \tag{4.47}$$

Using Eq. (4.42) in Eq. (4.47), we arrive at the expression

$$\begin{aligned} S^{(\infty)}(R\mathbf{u},\omega) &= \frac{8\pi^5 k^4 c}{R^2}(\delta_{ij} - u_i u_j) \\ &\times \int_D\int_D \sqrt{S_i^{(P)}(\mathbf{r}',\omega)}\sqrt{S_j^{(P)}(\mathbf{r},\omega)}\mu_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)e^{-ik\mathbf{u}\cdot(\mathbf{r}-\mathbf{r}')}\mathrm{d}^3r\mathrm{d}^3r'. \end{aligned} \tag{4.48}$$

This formula clearly demonstrates the existence of correlation-induced spectral changes. It is evident from Eq. (4.48) that the spectrum $S^{(\infty)}$ of the far field depends not only upon the spectrum of the source polarization, but also upon the correlations between Cartesian components of the polarization. Hence, except perhaps in some special cases, the spectrum of the far field will differ from the source spectrum and will also depend upon the direction of observation **u**.

It is not difficult to show that in spite of the fact that source correlations induce spectral changes in the far field, formula (4.48) is consistent with the energy conservation law (4.34). For this purpose, we return to Eq. (4.47) and integrate both sides over all directions **u**, and multiply both sides by R^2. We then obtain the formula

$$\begin{aligned} \int_{\Sigma^{(\infty)}} S^{(\infty)}(R\mathbf{u},\omega)\mathrm{d}\Sigma^{(\infty)} &= \frac{1}{8\pi}k^4 c\int_{(4\pi)}\mathrm{d}\Omega \\ &\times \int_D\int_D W_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)(\delta_{ij} - u_i u_j)e^{-ik\mathbf{u}\cdot(\mathbf{r}-\mathbf{r}')}\mathrm{d}^3r\mathrm{d}^3r', \end{aligned} \tag{4.49}$$

where we have used the fact that $R^2\mathrm{d}\Omega = \mathrm{d}\Sigma^{(\infty)}$ is the differential surface element of a large sphere $\Sigma^{(\infty)}$ centered in the source region (see Fig. 4.3). The product $u_i u_j$

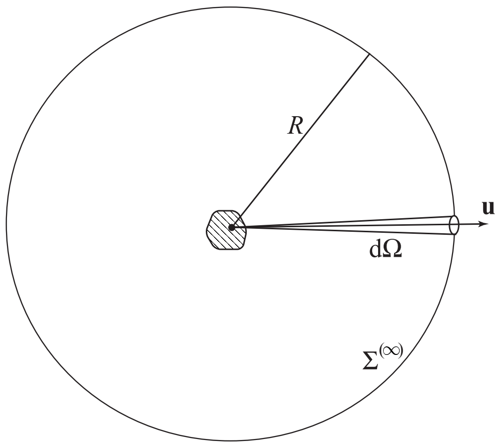


Figure 4.3: Illustrating notation relating to the spectrum of the radiated field in the far zone of a fluctuating source polarization.

on the right side of Eq. (4.49) may be expressed as a differential operator acting on the exponent. We may also use the well-known identity ([28], footnote on p. 123)

$$\frac{\sin k|\mathbf{r}-\mathbf{r}'|}{k|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi}\int_{(4\pi)} e^{-\mathrm{i}k\mathbf{u}\cdot(\mathbf{r}-\mathbf{r}')}\mathrm{d}\Omega, \tag{4.50}$$

to simplify Eq. (4.49). One finds that

$$\int_{\Sigma^{(\infty)}} S^{(\infty)}(R\mathbf{u},\omega)\mathrm{d}\Sigma^{(\infty)} = \frac{k^2 c}{2}\int_D\int_D W_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega) \times (k^2\delta_{ij}+\partial_i\partial_j)\frac{\sin k|\mathbf{r}-\mathbf{r}'|}{k|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3r\mathrm{d}^3r'. \tag{4.51}$$

The right hand side of this equation is identical to the right-hand side of the integral form of the energy conservation law (4.34). The left-hand sides are also equal to each other because of the well-known relation between the average flux vector $\langle\mathbf{F}^{(\infty)}\rangle$ and the spectral density $S^{(\infty)}$ in the far field, viz. $\langle\mathbf{F}^{(\infty)}(R\mathbf{u},\omega)\rangle = S^{(\infty)}(R\mathbf{u},\omega)\mathbf{u}$. Hence equations (4.34) and (4.48) are consistent with each other and, consequently, correlation-induced spectral changes are consistent with energy conservation.

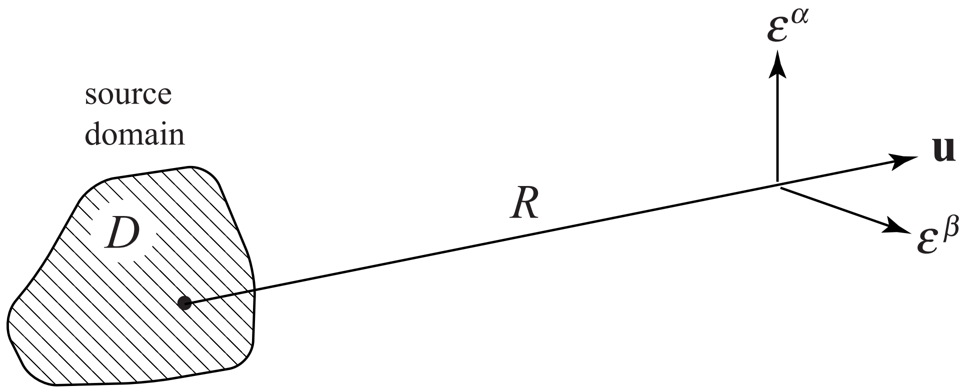


Figure 4.4: Illustrating notation relating to Eqs. (4.53) through (4.55).

4.3 Unpolarized sources which generate highly polarized fields

In section 4.1 it was shown that fluctuating scalar sources with quite different degrees of spatial coherence, even sources which are highly incoherent, can generate fields which are spatially completely coherent. In this section we consider similar effects with a source of electromagnetic radiation. In particular, we will show that certain unpolarized electromagnetic sources can produce fields outside the source domain which are almost completely polarized in nearly all directions.

We consider a fluctuating source polarization $\mathbf{P}(\mathbf{r},\omega)$, confined to a domain D (see Fig. 4.4), for which the fluctuations are quasi-homogeneous, i.e. such that the cross-spectral density of the source polarization may be approximated in the form

$$W_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega) \approx S^{(P)}\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2},\omega\right)\mu_{ij}^{(P)}\left(\mathbf{r}_2-\mathbf{r}_1,\omega\right), \tag{4.52}$$

where $S^{(P)}(\mathbf{r},\omega)$ is the spectral density of the source and $\mu_{ij}^{(P)}(\mathbf{r}',\omega)$ is its spectral degree of coherence. Moreover, $S^{(P)}(\mathbf{r},\omega)$ varies slowly over distances comparable to the width of $\mu_{ij}^{(P)}(\mathbf{r}',\omega)$. A more detailed description of the quasi-homogeneous approximation, at least for scalar radiation sources, will be given in chapter 5.

Far from the source domain D, the cross-spectral density tensor of the electric field may be expressed in the form[5]

$$W_{ij}^{(E)}(R\mathbf{u},R\mathbf{u},\omega) = \frac{k^4(2\pi)^6}{R^2}\left(\delta_{im} - u_i u_m\right)\left(\delta_{jn} - u_j u_n\right)\tilde{W}_{mn}^{(P)}(-k\mathbf{u},k\mathbf{u},\omega), \quad (4.53)$$

where R is the distance and $\mathbf{u}$ the direction from the source to the field point, u_i are the Cartesian components of $\mathbf{u}$, δ_{ij} is the Kronecker delta symbol and $\tilde{W}_{ij}^{(P)}$ is given by Eq. (4.40). In the far zone, the electric field will only have components transverse to $\mathbf{u}$. The *coherence matrix* $M_{\alpha\beta}$ ([81]; see also [28]) of the far field is defined as the projection of the far zone field tensor onto these transverse components,

$$M_{\alpha\beta}(R\mathbf{u},\omega) \equiv \epsilon_i^{\alpha}\epsilon_j^{\beta}W_{ij}^{(E)}(R\mathbf{u},R\mathbf{u},\omega), \quad (4.54)$$

where ϵ_i^{α}, ϵ_j^{β} are the Cartesian components of unit vectors $\boldsymbol{\epsilon}^{\alpha}$,$\boldsymbol{\epsilon}^{\beta}$ perpendicular to $\mathbf{u}$. Substituting from Eq. (4.53) into Eq. (4.54), and using the property that $\boldsymbol{\epsilon}^{\alpha}\cdot\mathbf{u} = \boldsymbol{\epsilon}^{\beta}\cdot\mathbf{u} = 0$, the coherence matrix takes the form

$$M_{\alpha\beta}(R\mathbf{u},\omega) = \frac{k^4(2\pi)^6}{R^2}\epsilon_i^{\alpha}\epsilon_j^{\beta}\tilde{W}_{ij}^{(P)}(-k\mathbf{u},k\mathbf{u},\omega). \quad (4.55)$$

The matrix $M_{\alpha\beta}$ describes the correlations which exist between components of the transverse electric field in the far zone. The *degree of polarization* of the field ([28], p. 354) is then defined as

$$P(R\mathbf{u},\omega) = \left[1 - \frac{4Det\left\{M_{\alpha\beta}(R\mathbf{u},\omega)\right\}}{\left(Tr\left\{M_{\alpha\beta}(R\mathbf{u},\omega)\right\}\right)^2}\right]^{1/2}, \quad (4.56)$$

where *Det* and *Tr* denote the determinant and trace of the coherence matrix, respectively.

[5] This relation may be derived using Eq. (4.35a).

We will now demonstrate that it is possible for a completely unpolarized source to generate a field that is almost completely polarized in the far zone, i.e. an unpolarized source may generate a field for which $P(R\mathbf{u}, \omega) = 1$ for nearly all directions of observation $\mathbf{u}$. A necessary and sufficient condition for the field to be completely polarized is that the coherence matrix has the form ([28], section 6.3.2)

$$M_{\alpha\beta} = A q_\alpha q_\beta, \tag{4.57}$$

where A is a constant and q_α are the components of a real vector $\mathbf{q}$. Sufficiency may be proven by substitution of Eq. (4.57) into Eq. (4.56). A source may be regarded as unpolarized if the cross-spectral density tensor at each point $\mathbf{r}$ has the form

$$W_{ij}^{(P)}(\mathbf{r}, \mathbf{r}, \omega) = \delta_{ij} S^{(P)}(\mathbf{r}, \omega). \tag{4.58}$$

Now we consider a quasi-homogeneous source polarization whose cross-spectral density tensor has the form

$$W_{ij}^{(P)}(\mathbf{r}_1, \mathbf{r}_2, \omega) \approx S^{(P)}\left(\frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \omega\right) \left\{\delta_{ij} A(\mathbf{r}_2 - \mathbf{r}_1, \omega) + a_i a_j B(\mathbf{r}_2 - \mathbf{r}_1, \omega)\right\}. \tag{4.59}$$

We will choose the functions A and B in such a way as to satisfy Eqs. (4.57) and (4.58). Substituting from Eq. (4.59) into Eq. (4.55), the coherence matrix for such a source polarization is found to be given by the expression

$$M_{\alpha\beta}(R\mathbf{u}, \omega) = \frac{k^4 (2\pi)^6}{R^2} \tilde{S}^{(P)}(0, \omega) \left\{\epsilon_i^\alpha \epsilon_i^\beta \tilde{A}(k\mathbf{u}, \omega) + (\epsilon_i^\alpha a_i)(\epsilon_j^\beta a_j) \tilde{B}(k\mathbf{u}, \omega)\right\}. \tag{4.60}$$

Comparing Eqs. (4.57) and (4.58) with Eqs. (4.60) and (4.59), respectively, it is clear that Eqs. (4.57) and (4.58) will be satisfied if

$$B(0, \omega) = 0, \tag{4.61}$$

$$\tilde{A}(k\mathbf{u}, \omega) = 0. \tag{4.62}$$

A source distribution of the form of Eq. (4.59) which satisfies Eqs. (4.61) and (4.62) will be unpolarized, yet it will produce a completely polarized far field.

The origin of this phenomenon is closely related to the theory of nonradiating sources. Equation (4.62) indicates that the unpolarized part of the source does not manifest itself in any way in the far field. However, we will show in chapter 5 that Eq. (4.62) cannot be satisfied exactly, i.e. that nonradiating quasi-homogeneous sources do not exist. Nevertheless, a source of the form of Eq. (4.59) that satisfies Eq. (4.61) will produce an almost completely polarized field if

$$\left|\tilde{A}(k\mathbf{u},\omega)\right| \ll \left|\tilde{B}(k\mathbf{u},\omega)\right| \qquad \text{for all } \mathbf{u}. \tag{4.63}$$

One might ask if a function which satisfies Eqs. (4.59), (4.61) and (4.63) can possess all the properties of a valid correlation function. In Appendix III we demonstrate that this is so.

For a source which satisfies the inequality (4.63), the field in the far zone will not be completely polarized in all directions. To see this, let us substitute Eq. (4.60) into Eq. (4.56). The degree of polarization is then found to be given by the expression

$$P(R\mathbf{u},\omega) = \frac{\left[(\boldsymbol{\epsilon}^1\cdot\mathbf{a})^2 + (\boldsymbol{\epsilon}^2\cdot\mathbf{a})^2\right]\tilde{B}}{2\tilde{A} + \left[(\boldsymbol{\epsilon}^1\cdot\mathbf{a})^2 + (\boldsymbol{\epsilon}^2\cdot\mathbf{a})^2\right]\tilde{B}}, \tag{4.64}$$

where the functional dependence of $\tilde{A}$ and $\tilde{B}$ has been suppressed for brevity. In Eq. (4.64), $\boldsymbol{\epsilon}^1$, $\boldsymbol{\epsilon}^2$ are vectors perpendicular to $\mathbf{u}$ and therefore depend upon the direction $\mathbf{u}$. If θ denotes the angle between $\mathbf{a}$ and $\mathbf{u}$, Eq. (4.64) may be expressed in the simpler form

$$P(\theta,\omega) = \frac{\tilde{B}\sin^2\theta}{2\tilde{A} + \tilde{B}\sin^2\theta}. \tag{4.65}$$

If the Fourier transform of A vanishes for all directions $\mathbf{u}$, the degree of polarization will evidently be unity for all directions. If instead $|\tilde{A}(k\mathbf{u},\omega)|$ is small compared to the magnitude of the Fourier transform of B but is nonzero, the degree of polarization will vanish in directions parallel and antiparallel to $\mathbf{a}$.

As an example, consider the case when

$$A(\mathbf{r},\omega) = \frac{\sin qr}{qr}\exp[-r^2/2\sigma^2], \tag{4.66a}$$

$$B(\mathbf{r},\omega) = \exp[-r^2/2\sigma^2] - \frac{\sin qr}{qr}\exp[-r^2/2\sigma^2]. \tag{4.66b}$$

The radial dependence of these functions is displayed in figure 4.5. $A(\mathbf{r},\omega)$ has been chosen as the product of two non-negative definite Hermitian functions; it follows then that $A(\mathbf{r},\omega)$ itself will be non-negative definite and Hermitian.[6] The Fourier transforms of these functions are readily found to be

$$\tilde{A}(k\mathbf{u},\omega) = \frac{\sigma^3}{(2\pi)^{3/2}} e^{-k^2\sigma^2/2} e^{-q^2\sigma^2/2} \frac{\sinh\left[kq\sigma^2\right]}{kq\sigma^2}, \tag{4.67a}$$

$$\tilde{B}(k\mathbf{u},\omega) = \frac{\sigma^3}{(2\pi)^{3/2}} e^{-k^2\sigma^2/2}\left[1 - e^{-q^2\sigma^2/2}\frac{\sinh\left[kq\sigma^2\right]}{kq\sigma^2}\right]. \tag{4.67b}$$

Both these functions are independent of the direction $\mathbf{u}$. Substituting these expressions into the inequality (4.63), it follows that such a source will produce an almost completely polarized field if

$$e^{-q^2\sigma^2/2}\frac{\sinh\left[kq\sigma^2\right]}{kq\sigma^2} \ll 1. \tag{4.68}$$

There are two independent parameters in this inequality, namely $q\sigma$ and $k\sigma$. Because of the rapid rate of decay of the Gaussian as compared to the growth of the sinh function, it is possible to choose these two parameters to satisfy the inequality (4.68). As discussed above, such a field will produce an almost completely polarized field for nearly all directions of observation. Figure 4.6 shows the dependence of $P(\theta,\omega)$ upon θ for several values of $q\sigma$ and for a fixed value of $k\sigma$.

[6] A correlation function has the same properties as a characteristic function. See corollary 1 to theorem 3.3.1, p. 38 of [82].

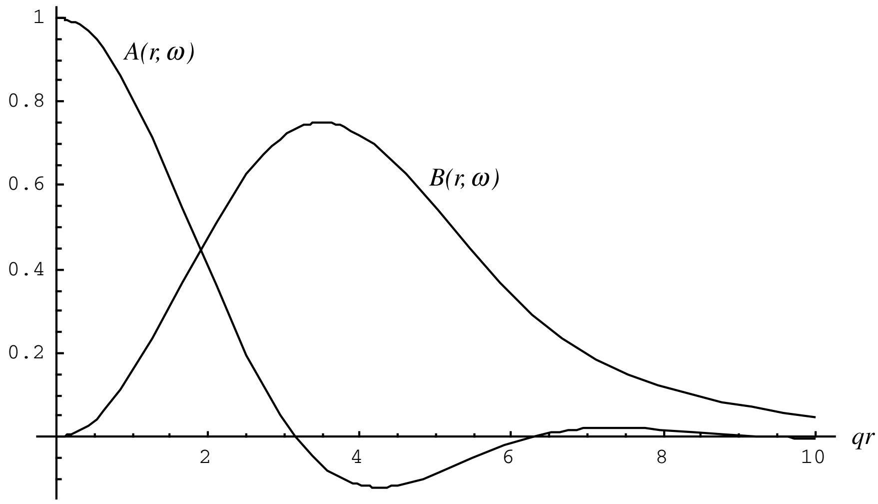


Figure 4.5: The radial dependence of the functions $A(\mathbf{r}, \omega)$ and $B(\mathbf{r}, \omega)$, given by Eqs. (4.66a) and (4.66b), with $q\sigma = 4$.

This example demonstrates that the polarization properties of a source distribution do not necessarily reflect themselves in the field generated by the source. This effect, like the correlation-induced spectral changes discussed earlier, is a consequence of the spatial coherence of the source, described in this case by the functions $A(\mathbf{r}, \omega)$ and $B(\mathbf{r}, \omega)$, and some degree of anisotropy of the polarization of the source, as described by the vector a_i. Although, in general, any polarization changes will not be as extreme as those described here, any research involving sources with appreciable spatial coherence should take into account the possibility of such effects.[7]

[7]It has also been shown that polarization changes can occur in the propagation of partially coherent beams. See, for instance, [83], [84] and [85].

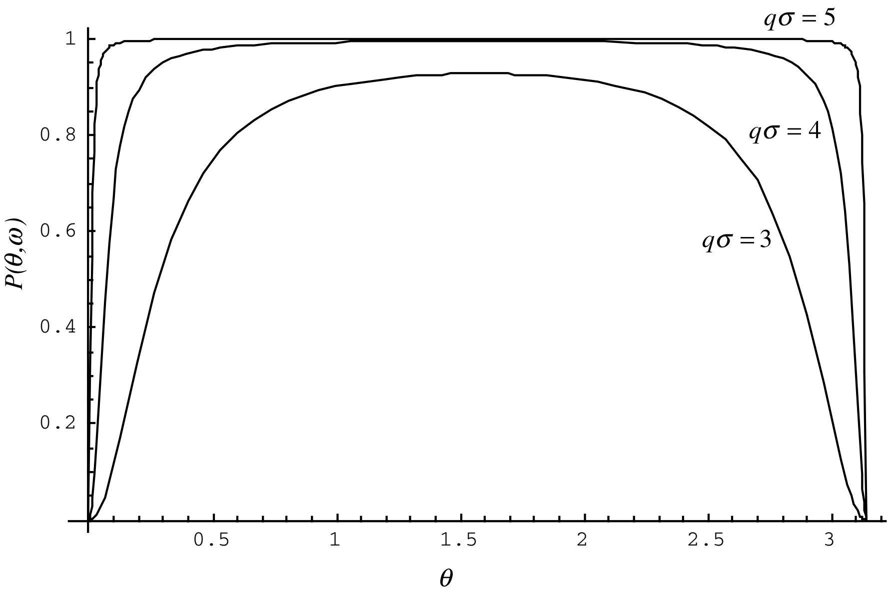


Figure 4.6: The degree of polarization $P(\theta, \omega)$ of the electric field in the far zone of the source, for various values of the parameter $q\sigma$. The angle θ is the angle between the direction of observation $\mathbf{u}$ and the unit vector $\mathbf{a}$. For larger values of the parameter $q\sigma$, the degree of polarization can be made arbitrarily close to unity for nearly all directions θ.

Chapter 5

Radiation from globally incoherent sources

In the previous chapters we investigated nonradiating sources and some of the consequences of their existence. By examining one-dimensional problems (chapter 3), we demonstrated that one-dimensional analogues to nonradiating sources (known as nonpropagating excitations) exist and that they produce noticeable effects even when the medium is damping or the force density applied to the string is quasi-monochromatic. In chapter 4, we considered some of the unusual consequences of the existence of nonradiating sources, such as the suppression of spectral lines or substantial differences existing between the field polarization and the source polarization.

Of course, the results of chapter 4 seem unusual because they conflict with physical intuition – intuition that comes from observation and experiment. Evidently nonradiating sources are not a general feature of most radiation problems, and this realization leads one to ask why they are not – after all, we have seen that nonradiating sources are extremely robust mathematical objects whose existence is not threatened by statistical fluctuations, damping forces, or external boundary conditions. If effects such as those described in chapter 4 frequently occurred, it would not be possible, even in principle, to determine the three-dimensional structure of objects such as fluorescent bulbs and neon signs. This apparent disagreement between classical radiation theory and observation might lead one to suspect that classical radiation theory is a poor model for most, if not all, radiation problems.

However, it is known that for spatially incoherent sources, whose spatial correlation properties may be represented by a delta function, the inverse problem is unique [48, 86]. This was shown explicitly in section 2.3. For such sources, a band-limited version of the source intensity can be reconstructed from measurements of the cross-spectral density of the field.

It has also been suggested in several papers that the inverse source problem for so-called *quasi-homogeneous sources* is unique, allowing reconstruction of the source intensity or the spectral degree of coherence from measurements of the cross-spectral density of the field, if one has sufficient prior knowledge of the source [17, 18]. A quasi-homogeneous source is one whose cross-spectral density is well-approximated by an expression of the form

$$\begin{aligned} W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) &\approx W_Q^{qh}(\mathbf{r}_1, \mathbf{r}_2, \omega) \\ &= I_Q\left(\frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \omega\right) \mu_Q(\mathbf{r}_2 - \mathbf{r}_1, \omega), \end{aligned} \tag{5.1}$$

where I_Q, the source intensity, varies slowly over distances comparable to the width of the spectral degree of coherence μ_Q (see figure 5.1). Alternatively, it is often said that I_Q is a 'slow' function of position and μ_Q is a 'fast' function of position. Sources with delta-correlations, as mentioned above, are a subclass of the set of

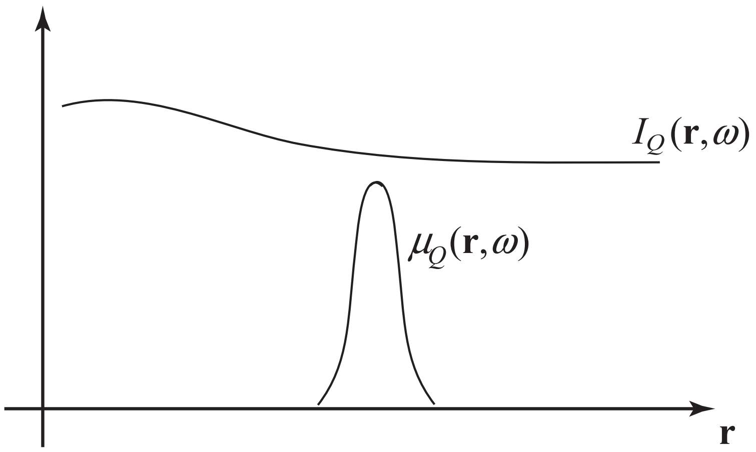


Figure 5.1: Illustrating the conventional requirement for the validity of the quasi-homogeneous approximation. At a given frequency ω, the spectral density $I_Q(\mathbf{r}, \omega)$ must be a 'slowly varying' function of position over distances comparable to the width of the spectral degree of coherence, $\mu_Q(\mathbf{r}', \omega)$, where $\mathbf{r}' = \mathbf{r}_2 - \mathbf{r}_1$.

quasi-homogeneous sources, and quasi-homogeneous sources may be considered to be a subclass of the set of so-called globally incoherent sources, i.e. sources for which the source intensity varies slowly over distances comparable to the width of the spectral degree of coherence [but not necessarily with a factorized cross-spectral density of the form of Eq. (5.1)].

The quasi-homogeneous approximation has been used quite often since its introduction, both in modelling scatterers [87, 88, 89] as well as modelling sources [90]. It has also been used to elucidate the foundations of radiometry [91, 92, 93].

Furthermore, quasi-homogeneous sources seem to be a good model for the class of highly incoherent radiating sources discussed at the end of section 4.1.

As prevalent as it has been in statistical optics, however, this approximation has not as yet been put on firm mathematical ground. Probably because of this, the question of uniqueness in the quasi-homogeneous inverse source problem is still open[1], and few attempts have been made to investigate methods of inversion with sources of this kind.

We begin this chapter by investigating the validity of the quasi-homogeneous approximation for the class of three-dimensional primary Gaussian Schell-model sources. We then investigate the quasi-homogeneous approximation for all three-dimensional, statistically stationary Schell-model sources in the space-frequency domain. From this analysis we indeed find, as suggested in section 4.1, that nonradiating quasi-homogeneous sources do not exist. This result suggests that, for the class of quasi-homogeneous sources, the inverse source problem is uniquely solvable, and the consequences of this uniqueness are discussed in section 5.3. Methods of determining the source structure from field measurements for quasi-homogeneous sources are discussed and undertaken in sections 5.4 and 5.5. Finally, the general inverse problem for globally incoherent sources is briefly considered in section 5.6.

5.1 The quasi-homogeneous approximation for three-dimensional Gaussian Schell-model sources

We consider a statistically stationary random radiation source $Q(\mathbf{r}, t)$ confined to a domain D for which the cross-spectral density is of the form

$$W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) = \sqrt{I_Q(\mathbf{r}_1, \omega)}\sqrt{I_Q(\mathbf{r}_2, \omega)}\mu_Q(\mathbf{r}_2 - \mathbf{r}_1, \omega), \tag{5.2}$$

[1]Though one earlier paper [37] hinted that a certain class of quasi-homogeneous sources must radiate.

where, as mentioned earlier, I_Q is the intensity, and μ_Q is the spectral degree of coherence, of the source at frequency ω. The spectral degree of coherence is *defined* by the relation

$$\mu_Q(\mathbf{r}_2 - \mathbf{r}_1) \equiv \frac{W_Q(\mathbf{r}_1, \mathbf{r}_2)}{\sqrt{I_Q(\mathbf{r}_1)}\sqrt{I_Q(\mathbf{r}_2)}}. \tag{5.3}$$

It is to be noted that $\mu_Q(\mathbf{r}_2 - \mathbf{r}_1)$ is undefined for values of $\mathbf{r}_1$, $\mathbf{r}_2$ such that $\mathbf{r}_1 \notin D$, $\mathbf{r}_2 \notin D$. The absolute value of the spectral degree of coherence can be shown to be restricted to the range (see [28], section 4.3.2)

$$0 \leq |\mu_Q(\mathbf{r}_2 - \mathbf{r}_1)| \leq 1. \tag{5.4}$$

The extreme value zero represents spatial incoherence and the value unity represents complete spatial coherence, at frequency ω. When μ_Q has non-negligible values only for very small values of the magnitude of the difference variable $|\mathbf{r}_2 - \mathbf{r}_1|$ of the order of a wavelength, it is usually said that the source is *incoherent*. In Eq. (5.2), the spectral degree of coherence depends only upon the difference between the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$; sources of this kind are known as Schell-model sources ([94], section 7.5; see also [28], sections 5.2.2 and 5.3.2).

We are interested in determining under what conditions a Schell-model source may be well-approximated by the quasi-homogeneous model, Eq. (5.1). More precisely, we are interested in determining under what conditions the field produced by a Schell-model source is well-approximated by the field of the corresponding quasi-homogeneous source.

In the far zone of a three-dimensional, statistically stationary source, the cross-spectral density of the field is given by the expression (Eq. (5.2-5) from [28])

$$W_U^{(\infty)}(R_1\mathbf{s}_1, R_2\mathbf{s}_2, \omega) = (2\pi)^6 \frac{e^{ik(R_2-R_1)}}{R_1 R_2} \tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2, \omega), \tag{5.5}$$

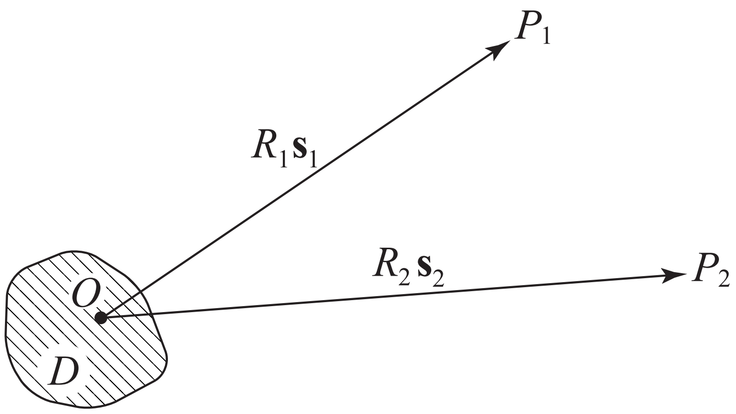


Figure 5.2: Notation used in analysis of the quasi-homogeneous approximation.

where $\mathbf{s}_1$, $\mathbf{s}_2$ are unit vectors pointing from the origin in the source region to points P_1 and P_2 in the far zone, R_1 and R_2 are the distances from the origin to the points P_1 and P_2 (see Fig. 5.2), and

$$\tilde{W}_Q(\mathbf{K}_1, \mathbf{K}_2, \omega) = \frac{1}{(2\pi)^6} \iint W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) e^{-\mathrm{i}(\mathbf{K}_1\cdot\mathbf{r}_1 + \mathbf{K}_2\cdot\mathbf{r}_2)} \mathrm{d}^3 r_1 \mathrm{d}^3 r_2 \tag{5.6}$$

is the six-dimensional spatial Fourier transform of the cross-spectral density of the source. In Eq. (5.5), $k = \omega/c$ is the free-space wavenumber of the radiation. As we will consider for the most part a single frequency ω, we will no longer display the dependence of the various quantities on ω.

For a quasi-homogeneous source, the Fourier transform (5.6) of the cross-spectral density of the source (5.1) has the simple form

$$\tilde{W}_Q^{qh}(-k\mathbf{s}_1, k\mathbf{s}_2) = \tilde{I}_Q\left[k(\mathbf{s}_2 - \mathbf{s}_1)\right] \tilde{\mu}_Q \left(k\frac{\mathbf{s}_1 + \mathbf{s}_2}{2}\right), \tag{5.7}$$

where $\tilde{I}_Q$ and $\tilde{\mu}_Q$ are the three-dimensional Fourier transforms of I_Q and μ_Q respectively, defined by the formulas

$$\tilde{I}_Q(\mathbf{K}) = \frac{1}{(2\pi)^3} \int I_Q(\mathbf{r}') e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}'} \mathrm{d}^3 r', \tag{5.8}$$

$$\tilde{\mu}_Q(\mathbf{K}) = \frac{1}{(2\pi)^3}\int \mu_Q(\mathbf{r}')e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}'}\mathrm{d}^3r'. \tag{5.9}$$

Let us compare Eq. (5.7) with the Fourier transform of the cross-spectral density (5.2) of the Schell-model source. On substituting from Eq. (5.2) into Eq. (5.6), and introducing the variables

$$\mathbf{R} \equiv (\mathbf{r}_1 + \mathbf{r}_2)/2, \quad \mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1, \tag{5.10}$$

we may express the Fourier transform of the cross-spectral density of a Schell-model source in the form

$$\begin{aligned}\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \frac{1}{(2\pi)^6}\iint &\sqrt{I_Q(\mathbf{R}+\mathbf{r}/2)}\sqrt{I_Q(\mathbf{R}-\mathbf{r}/2)} \\ &\times \mu_Q(\mathbf{r})e^{-\mathrm{i}k(\mathbf{s}_2-\mathbf{s}_1)\cdot\mathbf{R}}e^{-\mathrm{i}k\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)\cdot\mathbf{r}}\mathrm{d}^3R\mathrm{d}^3r. \end{aligned} \tag{5.11}$$

In general, the $\mathbf{R}$-integration in Eq. (5.11) is difficult to perform and usually cannot be evaluated analytically. Let us consider first the class of sources for which the intensity profile is Gaussian, i.e.

$$I_Q(\mathbf{r}) = I_0 e^{-r^2/2\sigma_I^2}. \tag{5.12}$$

For such sources, the $\mathbf{R}$-integral in Eq. (5.11) can be evaluated, and that formula reduces to

$$\begin{aligned}\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = &\frac{I_0(2\pi\sigma_I^2)^{3/2}}{(2\pi)^6}e^{-\sigma_I^2k^2(\mathbf{s}_2-\mathbf{s}_1)^2/2} \\ &\times \int e^{-\mathrm{i}k[(\mathbf{s}_1+\mathbf{s}_2)/2]\cdot\mathbf{r}}e^{-r^2/8\sigma_I^2}\mu_Q(\mathbf{r})\mathrm{d}^3r. \end{aligned} \tag{5.13}$$

Noting that the Fourier transform of the intensity (5.12) of the source is

$$\tilde{I}_Q(\mathbf{K}) = \frac{(2\pi)^{3/2}}{(2\pi)^3}I_0\sigma_I^3 e^{-\sigma_I^2K^2/2}, \tag{5.14}$$

we may rewrite Eq. (5.13) in the form

$$\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \tilde{I}_Q\left[k(\mathbf{s}_2 - \mathbf{s}_1)\right] \frac{1}{(2\pi)^3} \int e^{-ik\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)\cdot\mathbf{r}} e^{-r^2/8\sigma_I^2} \mu_Q(\mathbf{r}) \mathrm{d}^3 r. \quad (5.15)$$

This expression is similar to the Fourier transform of the cross-spectral density of a quasi-homogeneous source, given by Eq. (5.7), save for the appearance of the Gaussian term $\exp(-r^2/8\sigma_I^2)$ within the integral. This term represents the influence of the overlapped intensity integral from Eq. (5.11). On comparison of the Schell-model result (5.15) with Eq. (5.7), it seems clear that Eq. (5.7) will well-approximate the Schell-model result only if the exponential $\exp(-r^2/8\sigma_I^2)$ can be replaced by unity for all values of $\mathbf{r}$ for which $\mu_Q(\mathbf{r})$ is appreciable. This observation is in agreement with the usual statement of the quasi-homogeneous approximation, which suggests that $\mu_Q(\mathbf{r})$ must be a narrow function compared to any distance characterizing the rate of variation of $I_Q(\mathbf{r})$, in this case σ_I.

Let us simplify Eq. (5.15) further by also choosing the spectral degree of coherence to have a Gaussian form, i.e.

$$\mu_Q(\mathbf{r}) = e^{-r^2/2\sigma_\mu^2}. \quad (5.16)$$

With this spectral degree of coherence, the Fourier transform of the cross-spectral density is readily found to be given by the expression

$$\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \tilde{I}_Q\left[k(\mathbf{s}_2 - \mathbf{s}_1)\right] \frac{\sigma_T^3}{(2\pi)^{3/2}} e^{-k^2(\mathbf{s}_1+\mathbf{s}_2)^2\sigma_T^2/8}, \quad (5.17)$$

where

$$\sigma_T^2 = \frac{\sigma_I^2\sigma_\mu^2}{\sigma_\mu^2/4 + \sigma_I^2}. \quad (5.18)$$

By using Eqs. (5.12) and (5.16) in the quasi-homogeneous approximation, Eq. (5.1), we find that the quasi-homogeneous approximate form of Eq. (5.17) is given by the function

$$\tilde{W}_Q^{qh}(-k\mathbf{s}_1, k\mathbf{s}_2) = \tilde{I}_Q\left[k(\mathbf{s}_2 - \mathbf{s}_1)\right] \frac{\sigma_\mu^3}{(2\pi)^{3/2}} e^{-k^2(\mathbf{s}_1+\mathbf{s}_2)^2\sigma_\mu^2/8}. \tag{5.19}$$

The difference between the exact Fourier transform (5.17) and the transform which results from using the quasi-homogeneous approximation, Eq. (5.19), is given by

$$\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) - \tilde{W}_Q^{qh}(-k\mathbf{s}_1, k\mathbf{s}_2) = \frac{\tilde{I}_Q\left[k(\mathbf{s}_2 - \mathbf{s}_1)\right]}{(2\pi)^{3/2}} \times \left[\sigma_T^3 e^{-k^2(\mathbf{s}_1+\mathbf{s}_2)^2\sigma_T^2/8} - \sigma_\mu^3 e^{-k^2(\mathbf{s}_1+\mathbf{s}_2)^2\sigma_\mu^2/8}\right]. \tag{5.20}$$

This difference represents the complete correction, in closed form, to the quasi-homogeneous approximation for Gaussian Schell-model sources. When the magnitude of this term is small compared to the magnitude of the individual transforms, it is clear that the quasi-homogeneous approximation will be a good approximation. For a given pair of directions $\mathbf{s}_1$ and $\mathbf{s}_2$, we may characterize the relative magnitude of the correction term by the ratio $\Delta(\mathbf{s}_1, \mathbf{s}_2)$, defined as

$$\Delta(\mathbf{s}_1, \mathbf{s}_2) \equiv \left| \frac{\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) - \tilde{W}_Q^{qh}(-k\mathbf{s}_1, k\mathbf{s}_2)}{\tilde{W}_Q^{qh}(-k\mathbf{s}_1, k\mathbf{s}_2)} \right|. \tag{5.21}$$

For a Gaussian Schell-model source, this ratio may be shown to have the value [using Eqs. (5.19) and (5.20)]

$$\Delta(\mathbf{s}_1, \mathbf{s}_2) = \left| 1 - \frac{\sigma_I^3}{[\sigma_I^2 + \sigma_\mu^2/4]^{3/2}} \exp\left[-\frac{K^2}{2} \left(\frac{\sigma_I^2 \sigma_\mu^2}{\sigma_I^2 + \sigma_\mu^2/4} - \sigma_\mu^2 \right) \right] \right|, \tag{5.22}$$

where $K = |\mathbf{K}|$, and $\mathbf{K} \equiv k(\mathbf{s}_1 + \mathbf{s}_2)/2$. If the quasi-homogeneous approximation is to be valid for all directions $\mathbf{s}_1$ and $\mathbf{s}_2$, Δ must be negligible for all possible directions. Since $\mathbf{s}_1$ and $\mathbf{s}_2$ are unit vectors, $|\mathbf{K}|$ may take on all values within the range

$$0 \leq |\mathbf{K}| \leq k. \tag{5.23}$$

For the ratio to be roughly constant over all these values, the exponential must vary slowly over all possible $|\mathbf{K}|$ values, implying that

$$\left|\frac{k^2}{2}\left[\frac{\sigma_I^2\sigma_\mu^2}{\sigma_I^2+\sigma_\mu^2/4}-\sigma_\mu^2\right]\right| \ll 1. \tag{5.24}$$

The expression on the left-hand side of inequality (5.24) may be simplified and one finds that

$$\frac{k^2\sigma_\mu^2}{1/4+\sigma_I^2/\sigma_\mu^2} \ll 1. \tag{5.25}$$

The condition (5.25) is *necessary* for the validity of the quasi-homogeneous approximation in our particular model. When this condition is satisfied, the exponential in Eq. (5.22) is approximately equal to unity for all $\mathbf{K}$ values given by Eq. (5.23). However, by setting the exponential equal to unity in that inequality, it is clear that the ratio Δ will only be small if

$$1-\frac{\sigma_I^3}{[\sigma_I^2+\sigma_\mu^2/4]^{3/2}} = 1-\frac{1}{[1+\sigma_\mu^2/(4\sigma_I^2)]^{3/2}} \approx 0. \tag{5.26}$$

This condition will clearly only be satisfied when

$$\sigma_\mu/\sigma_I \ll 1. \tag{5.27}$$

Inequality (5.27) represents the usual requirement for the quasi-homogeneous approximation, expressed in a form appropriate for Gaussian Schell-model sources – the width of the spectral degree of coherence (σ_μ) must be very small compared to the length characterizing the rate of variation of the source intensity with position (represented here by σ_I). Our analysis has also suggested an additional constraint, represented by inequality (5.25), which depends upon the wavenumber k of the radiation. Noting that the denominator of inequality (5.25) must be a very large quantity because of inequality (5.27), inequality (5.25) may be replaced by the simpler constraint

$$k\sigma_\mu^2 = (2\pi)\frac{\sigma_\mu^2}{\lambda} \ll \sigma_I. \tag{5.28}$$

This inequality appears to be unappreciated in applications of the quasi-homogeneous approximation. It implies that if the wavelength of the radiation is sufficiently small, then the quasi-homogeneous approximation will not be satisfied, regardless of the validity of inequality (5.27).

We may roughly understand the significance of this inequality as follows. The Fourier transform of a three-dimensional function at $|\mathbf{K}| = k = 2\pi/\lambda$ is sensitive to variations of that function over distances on the order of or greater than a few wavelengths. We have seen that, for Gaussian Schell-model sources, the difference between the quasi-homogeneous approximation and the exact Schell-model representation is that, for the quasi-homogeneous approximation, one takes the Fourier transform of a correlation function of width σ_μ, while in the exact result one takes the Fourier transform of a modified function of width σ_T. The difference between the two widths may be represented by a length $\Delta\sigma$, given by the formula

$$\Delta\sigma \equiv \sqrt{|\sigma_T^2 - \sigma_\mu^2|} \approx \sigma_\mu^2/2\sigma_I, \tag{5.29}$$

where the approximate form holds if inequality (5.27) is satisfied. For the Fourier transform of the correlation function to be insensitive to this difference, $\Delta\sigma$ must be very small relative to the wavelength, i.e.

$$\Delta\sigma/\lambda \ll 1. \tag{5.30}$$

Using the relation $k = 2\pi/\lambda$, this inequality is seen to be equivalent to inequality (5.28).

We have found, therefore, that the validity of the quasi-homogeneous approximation depends not only upon the variations of the source intensity and the spectral degree of coherence, but also upon the wavelength of the radiation. Inequalities (5.27) and (5.28) together comprise necessary and sufficient conditions for the validity of the quasi-homogeneous approximation for Gaussian Schell-model sources. Although we derived these results for a simple class of sources, it seems clear that the approximation will be influenced by the wavelength under broader circumstances. Although that dependence may, in general, be quite complicated, the arguments leading to inequality (5.30) indicate that the wavelength must be large compared to a distance $\Delta\sigma$ which will depend upon the width of the correlation function and the specific intensity profile.

At this point one may be left with the impression that our analysis of the quasi-homogeneous approximation is more mathematical pedantry than physics. However, as we will see in the following sections, a detailed analysis of the justification for the approximation leads to new and easier methods of solving the inverse source problem for quasi-homogeneous sources, a problem which has not as yet convincingly been shown to have a unique solution. Furthermore, the inequalities derived are of some interest for understanding the foundations of radiometry, in which quasi-homogeneous sources have played an important role [91, 92, 93], as we now briefly discuss.

It has been shown that one may construct a generalized radiance function from the cross-spectral density of a partially coherent source which, in the limit $\lambda \to 0$, will satisfy all the requirements of traditional radiometry, *provided the source is quasi-homogeneous*. However, we have shown that the quasi-homogeneous approximation depends upon wavelength, and the satisfaction of inequality (5.28) seems to be in conflict with the limit $\lambda \to 0$. Apparently, one must be more careful in using calculations with this radiometric limit. Inequality (5.28) can only be satisfied for $\lambda \to 0$ if one simultaneously takes the limit $\sigma_\mu \to 0$, i.e. if one considers the source to be completely incoherent. Realistic sources, with small but nonzero correlation length and wavelength, will apparently satisfy approximately the postulates of traditional radiometry provided they also satisfy the newly-derived condition (5.28). The true importance of this condition still remains to be fully clarified.

5.2 The quasi-homogeneous approximation for Schell-model sources

In the previous section we examined the quasi-homogeneous approximation for Gaussian Schell-model sources, and determined that the validity of the approximation for such sources depends on two inequalities, one of which includes the wavelength of the radiation and was previously unappreciated in the theory of such sources. In this section we investigate the quasi-homogeneous approximation for general Schell-model sources, and determine conditions under which a general Schell-model source may be considered to be quasi-homogeneous.

We consider again a three-dimensional, primary, random scalar Schell-model radiation source $Q(\mathbf{r}, t)$, confined to a domain D. The cross-spectral density of such a source may be expressed in the form

$$\begin{aligned} W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) &= \sqrt{I_Q(\mathbf{r}_1, \omega)}\sqrt{I_Q(\mathbf{r}_2, \omega)}\mu_Q(\mathbf{r}_2 - \mathbf{r}_1, \omega) \\ &= h_Q(\mathbf{r}_1, \omega) h_Q(\mathbf{r}_2, \omega)\mu_Q(\mathbf{r}_2 - \mathbf{r}_1, \omega), \end{aligned} \tag{5.31}$$

where $h_Q(\mathbf{r}, \omega) \equiv \sqrt{I_Q(\mathbf{r}, \omega)}$. As before, we will confine our analysis to a single frequency component ω, and will therefore not display the dependence of the various quantities on ω.

According to Eq. (5.5), the cross-spectral density of the radiated field in the far zone is

$$W_U^{(\infty)}(R_1\mathbf{s}_1, R_2\mathbf{s}_2) = (2\pi)^6 \frac{e^{\mathrm{i}k(R_2-R_1)}}{R_1R_2}\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2), \tag{5.32}$$

Equation (5.32) shows that all information about the source structure that is obtainable from the cross-spectral density of the far field is contained within the function $\tilde{W}_Q$, and we will therefore focus our investigation upon that function.

Substituting from Eq. (5.31) into Eq. (5.32), we may express the Fourier transform of a Schell-model source in the form

$$\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \frac{1}{(2\pi)^6}\int\int h_Q(\mathbf{r}_1)h_Q(\mathbf{r}_2)\mu_Q(\mathbf{r}_2-\mathbf{r}_1)e^{-\mathrm{i}k(\mathbf{s}_2\cdot\mathbf{r}_2-\mathbf{s}_1\cdot\mathbf{r}_1)}\mathrm{d}^3r_1\mathrm{d}^3r_2. \tag{5.33}$$

Changing the variables of integration to

$$\mathbf{R} \equiv \frac{\mathbf{r}_1+\mathbf{r}_2}{2} \quad , \quad \mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1, \tag{5.34}$$

we may express Eq. (5.33) in the form

$$\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \frac{1}{(2\pi)^6}\int M_Q\left[k(\mathbf{s}_2-\mathbf{s}_1), \mathbf{r}\right]\mu_Q(\mathbf{r})e^{-\mathrm{i}k\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)\cdot\mathbf{r}}\mathrm{d}^3r, \tag{5.35}$$

where

$$M_Q[\mathbf{K}, \mathbf{r}] \equiv \int h_Q\left(\mathbf{R}+\frac{\mathbf{r}}{2}\right)h_Q\left(\mathbf{R}-\frac{\mathbf{r}}{2}\right)e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{R}}\mathrm{d}^3R. \tag{5.36}$$

It is to be noted that M_Q is of finite extent with respect to the $\mathbf{r}$-variable, because the function h_Q is of finite extent. Also, because the function h_Q is non-negative (h_Q describing, as before, the square root of the source intensity), M_Q satisfies the inequality

$$|M_Q[\mathbf{K}, \mathbf{r}]| \leq M_Q[0, \mathbf{r}] \quad \text{for all } \mathbf{r}. \tag{5.37}$$

From these two properties it is clear that if we define a function $B(\mathbf{r})$ by the formula

$$\begin{aligned} B(\mathbf{r}) &= 0 \quad \{\mathbf{r} : M_Q(0,\mathbf{r}) = 0\} \\ &= 1 \quad \{\mathbf{r} : M_Q(0,\mathbf{r}) \neq 0\}, \end{aligned} \tag{5.38}$$

we may incorporate this function into the integrand of Eq. (5.35) without changing the value of that integral. This is possible because the domain of support of $M(\mathbf{K},\mathbf{r})$ is always contained within the domain of support of $B(\mathbf{r})$. On substituting $B(\mathbf{r})$ into Eq. (5.35), we obtain for $\tilde{W}_Q$ the expression

$$\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \frac{1}{(2\pi)^6}\int M_Q\left[k(\mathbf{s}_2 - \mathbf{s}_1), \mathbf{r}\right] \mu_Q^B(\mathbf{r}) e^{-\mathrm{i}k\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)\cdot\mathbf{r}} \mathrm{d}^3 r, \tag{5.39}$$

where

$$\mu_Q^B(\mathbf{r}) \equiv B(\mathbf{r})\mu_Q(\mathbf{r}). \tag{5.40}$$

In the usual description of the quasi-homogeneous approximation, h_Q must be "slowly varying" over distances comparable to the "width" of μ_Q^B. This is a global requirement, however, in that it must hold for all locations within the source domain. It would seem more appropriate, then, to convert Eq. (5.39) into an integral involving the Fourier transforms of h_Q and of μ_Q^B. We introduced the function $B(\mathbf{r})$ for this purpose; the function μ_Q is by itself undefined for points (represented by $\mathbf{r}_1$ and $\mathbf{r}_2$) not contained within the domain of support of $B(\mathbf{r})$ [see Eq. (5.3)].

As M_Q and μ_Q^B are both well-behaved functions, they each have a Fourier representation,

$$M_Q[\mathbf{K}_0, \mathbf{r}] = \int \tilde{M}_Q[\mathbf{K}_0, \mathbf{K}] e^{\mathrm{i}\mathbf{K}\cdot\mathbf{r}} \mathrm{d}^3 K, \tag{5.41}$$

and

$$\mu_Q^B(\mathbf{r}) = \int \tilde{\mu}_Q^B[\mathbf{K}] e^{\mathrm{i}\mathbf{K}\cdot\mathbf{r}} \mathrm{d}^3 K. \tag{5.42}$$

From Eq. (5.39), we see that $\tilde{W}_Q$ is the Fourier transform of a product of two functions. By the convolution theorem, $\tilde{W}_Q$ may therefore be written as the three-dimensional convolution of the Fourier transforms of these functions, so that

$$\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \frac{1}{(2\pi)^3}\int \tilde{M}_Q\left[k(\mathbf{s}_2 - \mathbf{s}_1), \mathbf{K}\right] \tilde{\mu}_Q^B \left[k\left(\frac{\mathbf{s}_1 + \mathbf{s}_2}{2}\right) - \mathbf{K}\right] \mathrm{d}^3K. \quad (5.43)$$

Substituting from Eq. (5.36) into Eq. (5.41), one can show that $\tilde{M}_Q$ may be expressed in the form

$$\tilde{M}_Q[\mathbf{K}_0, \mathbf{K}_1] = (2\pi)^3 \tilde{h}_Q^* \left[-\mathbf{K}_1 - \frac{1}{2}\mathbf{K}_0\right] \tilde{h}_Q \left[-\mathbf{K}_1 + \frac{1}{2}\mathbf{K}_0\right]. \quad (5.44)$$

Substituting from this expression into Eq. (5.43), and changing the variable of integration from $\mathbf{K}$ to $-\mathbf{K}$, we arrive at the result that

$$\begin{aligned}\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \int \tilde{h}_Q^* \left[\mathbf{K} - \frac{1}{2}k(\mathbf{s}_2 - \mathbf{s}_1)\right] \tilde{h}_Q \left[\mathbf{K} + \frac{1}{2}k(\mathbf{s}_2 - \mathbf{s}_1)\right] \\ \times \tilde{\mu}_Q^B \left[\mathbf{K} + k\left(\frac{\mathbf{s}_1 + \mathbf{s}_2}{2}\right)\right] \mathrm{d}^3K. \quad (5.45)\end{aligned}$$

We have as yet made no approximations, so equation (5.45) is an *exact* expression for $\tilde{W}_Q$, equivalent to our defining formula, Eq. (5.33). Because each of the functions $\tilde{h}_Q(\mathbf{K})$ and $\tilde{\mu}_Q^B(\mathbf{K})$ is the Fourier transform of a function of finite support, each is the boundary value of an entire analytic function in three complex variables ([49], p. 353). A consequence of their analyticity is that if $h_Q(\mathbf{r})$ and $\mu_Q^B(\mathbf{r})$ are both non-null functions (which is true if $W_Q(\mathbf{r}_1, \mathbf{r}_2) \not\equiv 0$), then both $\tilde{h}_Q(\mathbf{K})$ and $\tilde{\mu}_Q^B(\mathbf{K})$ are functions of infinite support; neither may vanish over a domain in $\mathbf{K}$-space larger than a two-dimensional manifold. This property will be seen to be of great importance in the theory of nonradiating quasi-homogeneous sources.

Although $\tilde{h}_Q(\mathbf{K})$ is not of finite support, it must be negligible for large values of $|\mathbf{K}|$, because its Fourier transform exists (which implies that it decays sufficiently rapidly for large $\mathbf{K}$). Let us assume that $\tilde{h}_Q(\mathbf{K})$ is sufficiently narrow so that the integrand in Eq. (5.45) is negligible for values of $|\mathbf{K}|$ larger than some parameter α, i.e. that

$$\tilde{h}_Q(\mathbf{K}) \approx 0 \quad \text{for all } |\mathbf{K}| \geq \alpha \quad . \tag{5.46}$$

This requirement suggests that $h_Q(\mathbf{r})$ is slowly varying over spatial distances on the order of $1/\alpha$. It is then not difficult to show that the integrand in Eq. (5.45) will be appreciable only for vectors $\mathbf{K}$ such that $|\mathbf{K}| \leq \alpha$. We have already noted that $\tilde{\mu}_Q^B\left[k\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)+\mathbf{K}\right]$ is the boundary value of an entire analytic function of three complex variables; it follows that it is differentiable to all orders and can therefore be expanded in a Taylor series around the point $\mathbf{K}=0$, i.e. that

$$\tilde{\mu}_Q^B\left[k\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)+\mathbf{K}\right] = \sum_{n=0}^{\infty}\frac{1}{n!}(\mathbf{K}\cdot\nabla_{K'})^n\tilde{\mu}_Q^B(\mathbf{K}')|_{\mathbf{K}'=k(\mathbf{s}_1+\mathbf{s}_2)/2} \quad , \tag{5.47}$$

where $\nabla_{K'}$ is the gradient with respect to $\mathbf{K}'$.

If α is sufficiently small, the first term of this series will dominate the integral in Eq. (5.45). This contribution, which we denote by $\tilde{W}_Q^0$, may be written as

$$\begin{aligned}\tilde{W}_Q^0(-k\mathbf{s}_1,k\mathbf{s}_2) = \int \tilde{h}_Q^*\left[\mathbf{K}-\frac{1}{2}k(\mathbf{s}_2-\mathbf{s}_1)\right]\tilde{h}_Q\left[\mathbf{K}+\frac{1}{2}k(\mathbf{s}_2-\mathbf{s}_1)\right] \\ \times\, \tilde{\mu}_Q^B\left[k\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)\right]\mathrm{d}^3K.\end{aligned} \tag{5.48}$$

The term involving $\tilde{\mu}_Q^B$ is now independent of $\mathbf{K}$, and may be removed from the integrand. The integral may then be evaluated using the definition of the Fourier transform of h_Q, and $\tilde{W}_Q^0$ may be written as

$$\tilde{W}_Q^0(-k\mathbf{s}_1,k\mathbf{s}_2) = \tilde{I}_Q\left[k(\mathbf{s}_2-\mathbf{s}_1)\right]\tilde{\mu}_Q^B\left[k\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)\right], \tag{5.49}$$

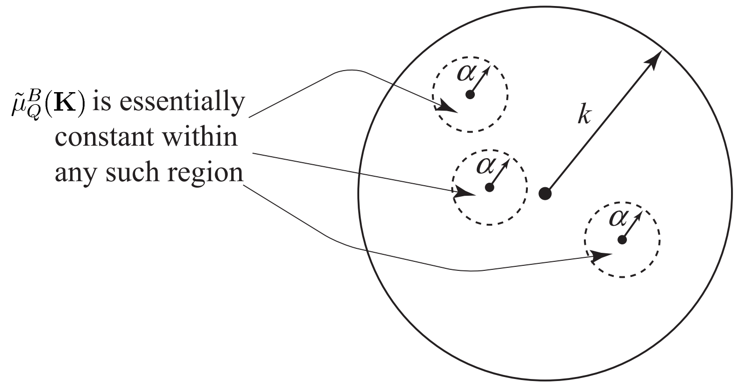


Figure 5.3: Illustrating the requirement for the validity of the quasi-homogeneous approximation. The figure represents the region $|\mathbf{K}| \leq k$ in $\mathbf{K}$-space. If the quasi-homogeneous approximation is valid, then at any point $|\mathbf{K}| \leq k$ the function $\tilde{\mu}_Q^B(\mathbf{K})$ must be well-represented by the zeroth order term of its Taylor expansion within a sphere of radius α. Note that the function $\tilde{\mu}_Q^B(\mathbf{K})$ may vary considerably over distances k, if $\alpha \ll k$.

where

$$\tilde{I}_Q(\mathbf{K}) = \frac{1}{(2\pi)^3} \int I_Q(\mathbf{r}) e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}} \mathrm{d}^3 r \tag{5.50}$$

is the Fourier transform of the source intensity $I_Q(\mathbf{r})$.

Equation (5.49) is equivalent to the Fourier transform of the cross-spectral density of a quasi-homogeneous source, as can be seen by comparison with Eq. (5.7). It is to be noted that this result differs from the usual statement of the quasi-homogeneous approximation by the appearance of the function $\tilde{\mu}_Q^B$, rather then the ill-defined function $\tilde{\mu}_Q$. The quasi-homogeneous approximation therefore consists of using only the first term in the Taylor series expansion of $\tilde{\mu}_Q^B$ in the Fourier domain.

As can be seen by considering Eq. (5.45), this approximation is only valid when $\tilde{\mu}_Q^B(\mathbf{K})$ is constant within a sphere of radius α centered on $\mathbf{K} = k(\mathbf{s}_1+\mathbf{s}_2)/2$, where α is defined by Eq. (5.46). If the approximation is to be valid for all directions $\mathbf{s}_1$ and $\mathbf{s}_2$, then $\tilde{\mu}_Q^B(\mathbf{K})$ must be nearly constant within every sphere of radius α for all $\mathbf{K}$-values such that $|\mathbf{K}| \leq k$. Thus the value of $\tilde{\mu}_Q^B(\mathbf{K})$ cannot change significantly over a distance α in $\mathbf{K}$-space, although it may, for small values of α, change considerably over a distance of order k (see figure 5.3). This assumption will form the basis of our analysis of the inverse quasi-homogeneous source problem, which we will discuss in section 5.4.

Furthermore, suppose that $\tilde{\mu}_Q^B(\mathbf{K})$ is approximately constant for all values of $|\mathbf{K}| \leq k$. Then the spectral degree of coherence is practically indistinguishable from a delta function, whose Fourier transform is a constant for *all* values of $\mathbf{K}$. Our analysis has therefore provided a criterion for when it is appropriate to approximate a source cross-spectral density as a delta-correlated source.

It is to be noted that these statements are in agreement with the usual justification of the quasi-homogeneous approximation, because of the reciprocal nature of a function and its transform. If $\tilde{\mu}_Q^B(\mathbf{K})$ is varies slowly over distances comparable to the width of $\tilde{h}_Q(\mathbf{K})$, then $h_Q(\mathbf{r})$ must be vary slowly over distances comparable to the width of $\mu_Q^B(\mathbf{r})$. This relation is demonstrated in Appendix IV.

By use of higher-order terms of the Taylor series (5.47), we are now in a position to calculate correction terms to the quasi-homogeneous approximation for general source cross-spectral densities. These correction terms may be used to derive conditions under which the approximation is valid. We will discuss this problem in Appendix V.

5.3 Uniqueness of the inverse source problem for quasi-homogeneous sources

We may draw two immediate conclusions from our analysis of the quasi-homogeneous approximation. First, it is to be noted that a given source cross-spectral density will factorize in the form of Eq. (5.49) only if $\tilde{\mu}_Q^B(\mathbf{K})$ is constant for all values of $\mathbf{K}$. Formally, the inverse Fourier transform of a constant is proportional to a delta function, and therefore a cross-spectral density will only factorize if it is delta-correlated. For any other source with a sufficiently narrow correlation function $\mu_Q^B(\mathbf{r})$, this factorization is only approximate.

Second, it is to be noted that since the functions $\tilde{\mu}_Q^B$ and $\tilde{I}_Q$ are each the boundary value of an entire analytic function in three complex variables, neither function may vanish throughout a region of $\mathbf{K}$-space with dimensionality greater than that of a surface, and likewise their product may only vanish on surfaces in $\mathbf{K}$-space. It is therefore not possible for $\tilde{W}_Q^0$ to vanish for all pairs of directions $\mathbf{s}_1$ and $\mathbf{s}_2$, unless W_Q^0 vanishes identically. Therefore *nonradiating quasi-homogeneous sources do not exist.*

This result has important consequences for the inverse source problem. The nonexistence of nonradiating quasi-homogeneous sources suggests that, if a source is quasi-homogeneous, *some* unique information about the source structure may be determined from measurements of the radiated field outside the source. In the next section we will discuss what structural information can be recovered.

5.4 The inverse problem for quasi-homogeneous sources

In section 5.2, we derived the quasi-homogeneous approximation through a careful analysis of radiation from globally incoherent sources. Using this derivation we demonstrated the non-existence of nonradiating quasi-homogeneous sources. This result suggests that the radiation generated by every quasi-homogeneous source possesses a unique "signature" that distinguishes it from every other, and that by measurements of the radiation emitted by such a source we may determine some of its structural features. We now briefly consider what sort of structural information may be obtained.

We have seen that when a source is quasi-homogeneous, the function $\tilde{\mu}_Q^B(\mathbf{K}+\mathbf{K}_0)$ must be effectively constant for all $|\mathbf{K}| \leq \alpha$, for every $|\mathbf{K}_0| \leq k$ [α being defined in Eq. (5.46)]. The cross-spectral density of the far field is then proportional to $\tilde{W}_Q^0(-k\mathbf{s}_1, k\mathbf{s}_2)$, given by Eq. (5.49).

Let us assume that measurements of the cross-spectral density of the field of a quasi-homogeneous source have been made for all directions $\mathbf{s}_1$ and $\mathbf{s}_2$. If we consider only field data for directions of observation such that

$$\left| k\frac{\mathbf{s}_1 + \mathbf{s}_2}{2} \right|^2 \leq \alpha^2, \tag{5.51}$$

the Fourier transform of the spectral degree of coherence will be effectively constant over this range and may be replaced by its value at the origin, $\tilde{\mu}_Q^B(0)$. The function $\tilde{W}_Q^0(-k\mathbf{s}_1, k\mathbf{s}_2)$ may then be expressed in the form

$$\tilde{W}_Q^0(-k\mathbf{s}_1, k\mathbf{s}_2) = \tilde{I}_Q\left[k(\mathbf{s}_2 - \mathbf{s}_1)\right] \tilde{\mu}_Q^B(0). \tag{5.52}$$

The inequality (5.51) is equivalent to considering only directions of observation

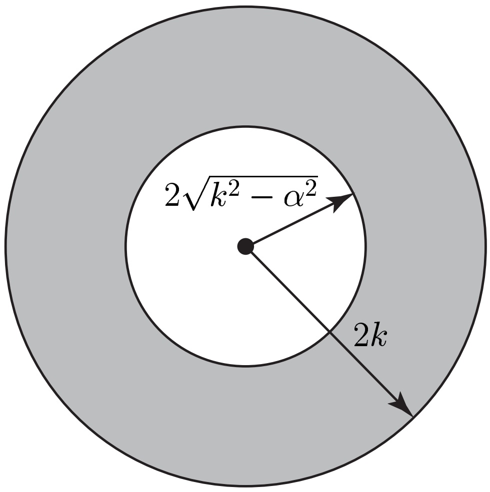


Figure 5.4: Showing those Fourier components of $\tilde{I}_Q(\mathbf{K})$ which may be determined by field correlation measurements for a quasi-homogeneous source. The parameter α is defined by Eq. (5.46).

in the range

$$4k^2 - 4\alpha^2 \leq k^2|\mathbf{s}_2 - \mathbf{s}_1|^2 \leq 4k^2, \tag{5.53}$$

where the upper bound is determined by the maximum value of $k|\mathbf{s}_2 - \mathbf{s}_1|$.

Using the values of $k(\mathbf{s}_2 - \mathbf{s}_1)$ given by inequality (5.53), we may determine, up to an arbitrary multiplicative constant $\tilde{\mu}_Q^B(0)$, the Fourier components of $\tilde{I}_Q[\mathbf{K}]$ whose $\mathbf{K}$-vectors lie within the spherical shell defined by Eq. (5.53) (see figure 5.4). This data may be Fourier inverted to reconstruct a "high pass" filtered version of the intensity function, $I_Q(\mathbf{r})$.

This reconstruction procedure has only two undetermined parameters which cannot be obtained from field measurements: the value of $\tilde{\mu}_Q^B(0)$, as mentioned above, and α, which determines the allowed Fourier components, as in Eq. (5.53). Earlier inversion methods for quasi-homogeneous sources described in the literature require the knowledge of the value of $\tilde{\mu}_Q(\mathbf{K})$ over a continuous domain, either throughout the volume $|\mathbf{K}| \leq k$ [17] or along a radial line within that volume [18].

It is to be noted, however, that in deriving the quasi-homogeneous approximation, we have assumed that $\tilde{h}_Q(\mathbf{K})$ is negligible for all $|\mathbf{K}| > \alpha$; this assumption suggests that $\tilde{I}_Q(\mathbf{K})$ is negligible for all $|\mathbf{K}| > 2\alpha$.[2] In order, then, that our reconstruction contains non-negligible Fourier components of the source intensity $I_Q(\mathbf{r})$, we require that $4k^2 - 4\alpha^2 \leq 4\alpha^2$, i.e. that

$$\alpha^2 \geq \frac{1}{2}k^2. \tag{5.54}$$

This reconstruction represents a "worst-case scenario", for it assumes that the spectral degree of coherence may be considered constant over distances in the space-frequency domain no greater than α. Such sources would barely satisfy the quasi-homogeneous approximation. For many more sources, though, $\tilde{\mu}_Q^B$ may be considered to be constant over a spatial-frequency distance $\gamma > \alpha$, and we may then reconstruct I_Q using data from directions of observation such that

$$4k^2 - 4\gamma^2 \leq k^2|\mathbf{s}_2 - \mathbf{s}_1|^2 \leq 4k^2. \tag{5.55}$$

It is to be noted that if $\gamma \geq k$, then *all* directions of observation may be used to reconstruct I_Q, as for an incoherent source (see section 2.3).

If γ is not greater than k, but is of comparable magnitude, another approximate reconstruction method may be used. Because $\tilde{\mu}_Q^B(\mathbf{K})$ is slowly varying over spatial-frequency distances comparable to k, it is well-approximated for all $|\mathbf{K}| \leq k$ by the

[2]That this is so can be shown by taking the Fourier transform of $I_Q(\mathbf{r}) = |h_Q(\mathbf{r})|^2$), and using the convolution theorem.

first few terms of its Taylor series,

$$\tilde{\mu}_Q^B\,[k\mathbf{s}] \approx \tilde{\mu}_Q^B(0) + k\mathbf{s}\cdot\nabla_{K'}\tilde{\mu}_Q^B[\mathbf{K}']|_{K'=0} + \frac{1}{2}\left(k\mathbf{s}\cdot\nabla_{K'}\right)^2\tilde{\mu}_Q^B[\mathbf{K}']|_{K'=0}. \quad (5.56)$$

In our earlier, simpler, solution to the inverse problem, we used only the first term of this expansion, and we used as prior knowledge the value of $\tilde{\mu}_Q^B(0) \equiv \tilde{\mu}_0$. Let us now suppose we have, as prior knowledge, not only the value of $\tilde{\mu}_0$ but also the width σ_μ of the spectral degree of coherence. We may estimate the functional form of $\tilde{\mu}_Q^B$ by assuming it is a Gaussian of width σ_μ and zero value $\tilde{\mu}_0$,

$$\tilde{\mu}_Q^B(\mathbf{K}) = \tilde{\mu}_0 e^{-K^2\sigma_\mu^2/2}, \quad (5.57)$$

in which case the Taylor expansion of $\tilde{\mu}_Q^B$ becomes

$$\tilde{\mu}_Q^B\,[k\mathbf{s}] \approx \tilde{\mu}_0\left(1 - \frac{(k\sigma_\mu)^2}{2}\,[k\mathbf{s}]^2\right). \quad (5.58)$$

This expression for the Fourier transform of the spectral degree of coherence may be used to make better reconstructions of quasi-homogeneous sources. It is to be noted, however, that this expression may only be used if $k\sigma_\mu$ is appreciably less than unity.

So far we have only considered reconstruction of the intensity of the source; we now briefly examine the possibility of reconstructing its spectral degree of coherence. Let us assume that we know the source intensity I_Q. For a quasi-homogeneous source, the Fourier transform of the source intensity is negligible for all $|\mathbf{K}| > 2\alpha$; therefore the only field data available for reconstructing the degree of coherence are those for which

$$k|\mathbf{s}_2 - \mathbf{s}_1| \leq \min[2\alpha, 2k] \equiv \beta. \quad (5.59)$$

The inequality (5.59) may be rewritten to show that the only non-negligible field data are those for which

$$\sqrt{k^2 - \frac{\beta^2}{4}} \leq k \left| \frac{\mathbf{s}_1 + \mathbf{s}_2}{2} \right| \leq k. \tag{5.60}$$

This formula defines a spherical shell which contains all of the Fourier components which may be used to reconstruct the spectral degree of coherence. The radial width of this shell, however, is always comparable to α. For the quasi-homogeneous approximation to be valid, $\tilde{\mu}_Q^B$ must be constant across any radial distance α. The Fourier information available for reconstruction of the spectral degree of coherence, then, contains little or no information about the radial structure of the function, and will not give an accurate reconstruction. From this argument it seems evident that, for quasi-homogeneous sources, the spectral degree of coherence cannot be reliably reconstructed.

5.5 An example

In this section we give a simple example to illustrate the inversion possiblilities mentioned previously. We will consider sources with a Gaussian intensity profile, i.e.

$$I_Q(\mathbf{r}) = I_0 e^{-r^2/2\sigma_I^2}, \tag{5.61}$$

and a spectral degree of coherence of exponential form, i.e.

$$\mu_Q(\mathbf{r}) = e^{-2r/\sigma_\mu}. \tag{5.62}$$

For simplicity, we will use the knowledge that the source intensity and the spectral degree of coherence are spherically symmetric; the relevant reconstruction then involves only one-dimensional integrations. We will also assume that the values of $\tilde{\mu}_Q(0)$ and σ_μ are known.

The data to be used for inversion was computed numerically by substituting from Eq. (5.62) into Eq. (5.15), using various values of the parameters $k\sigma_I$ and $k\sigma_\mu$. Although we are using an exponential spectral degree of coherence instead of a Gaussian spectral degree of coherence, it is reasonable to assume that Eqs. (5.27) and (5.28) will be good indicators of the validity of the quasi-homogeneous approximation.

We first consider a source for which $k\sigma_I = 4$, $k\sigma_\mu = 0.01$. For such a source the quasi-homogeneous approximation will be satisfied, and furthermore the Fourier transform of the spectral degree of coherence will be approximately constant over all values of $\mathbf{K}$ for which $|\mathbf{K}| \leq 2k$. We may, therefore, treat the source as incoherent and reconstruct it as described in section 2.3. The result of such a reconstruction is shown in figure 5.5. It is seen that the actual intensity function and the reconstructed intensity function are in good agreement with each other.

Next we consider a source for which $k\sigma_I = 4$, $k\sigma_\mu = 0.6$. Such a source may still be considered quasi-homogeneous, but the spectral degree of coherence may no longer be considered to be constant for all values of $|\mathbf{K}| \leq 2k$. We therefore expect that reconstructing this source as an incoherent source will not provide a good reconstruction. However, because $k\sigma_\mu < 1$, we may use Eq. (5.58) to provide a better estimate of the effect of the spectral degree of coherence. The "incoherent reconstruction" and the "quasi-homogeneous reconstruction" are shown in figures 5.6 and 5.7, respectively. It is seen that the quasi-homogeneous reconstruction does, in fact, provide a better reconstruction than the incoherent one. Note that this

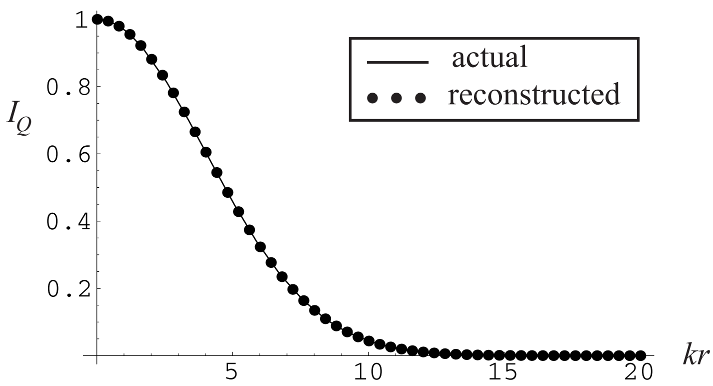


Figure 5.5: The actual intensity function and the reconstructed intensity function of a Gaussian Schell-model source with an exponential spectral degree of coherence, for $k\sigma_I = 4$, $k\sigma_\mu = 0.01$.

reconstruction was made by assuming a Gaussian correlation function, even though the true correlation function is exponential.

Finally we consider a source for which $k\sigma_I = 4$, $k\sigma_\mu = 2$. Such a source is not quasi-homogeneous, as the width of the spectral degree of coherence is comparable to the width of the source intensity. An "incoherent reconstruction" of the source should be poor, and such a reconstruction is shown in figure 5.8. Equation (5.58) cannot be used to improve upon this, because $k\sigma_\mu > 1$.

These examples are not intended to provide an exhaustive set of inversion methods for quasi-homogeneous sources; rather they are intended to demonstrate that good reconstructions may be obtained for sources which are quasi-homogeneous but which may not be considered incoherent. Furthermore, it shows that, under certain circumstances, very simple assumptions about the spectral degree of coherence may be made which allow a considerable improvement of the reconstruction.

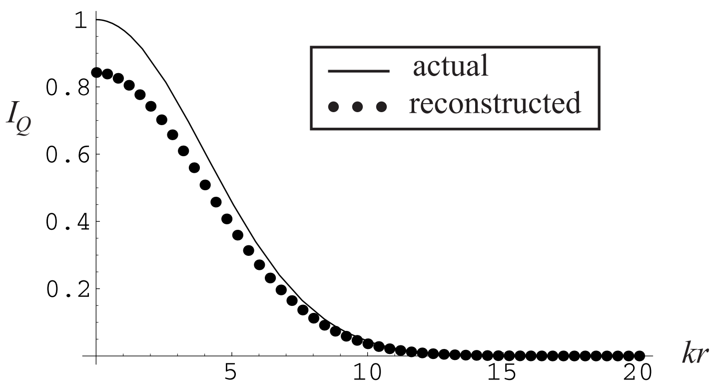


Figure 5.6: The actual intensity function and the reconstructed intensity function of a Gaussian Schell-model source with an exponential spectral degree of coherence, for $k\sigma_I = 4$, $k\sigma_\mu = 0.6$. The source was assumed to be incoherent for the reconstruction.

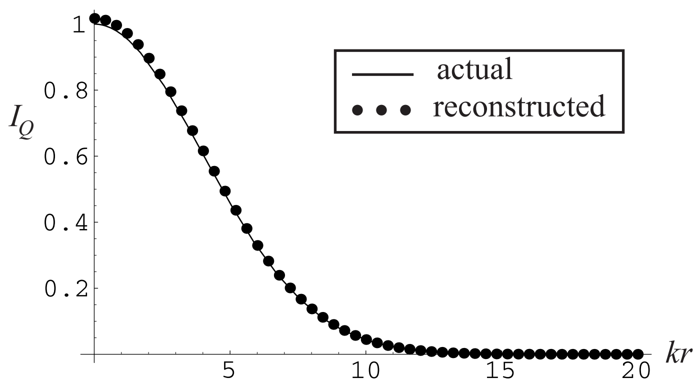


Figure 5.7: The actual intensity function and the reconstructed intensity function of a Gaussian Schell-model source with an exponential spectral degree of coherence, for $k\sigma_I = 4$, $k\sigma_\mu = 0.6$. Equation (5.58) was used as an approximate form for the spectral degree of coherence.

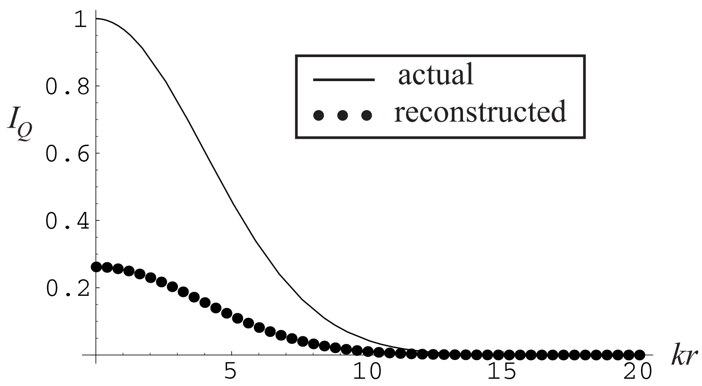


Figure 5.8: The actual intensity function and the reconstructed intensity function of a Gaussian Schell-model source with an exponential spectral degree of coherence, for $k\sigma_I = 4$, $k\sigma_\mu = 2$. The reconstruction, which assumes incorrectly that the source is incoherent, produces an inaccurately reconstructed source.

5.6 Globally incoherent sources

One objection to the preceding analysis that might be raised is that it is highly dependent upon the Schell-model nature of the source distribution. Within the domain of the source, the spectral degree of coherence μ_Q of a Schell-model source is dependent only upon the distance between the points $\mathbf{r}_1$ and $\mathbf{r}_2$ and not dependent upon the relative location within the source, given by $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$. If the results of the previous sections are dependent upon the correlations being *strictly* homogeneous within the domain of the source, then those results will be of limited use. In this section we investigate a broader class of globally incoherent sources and demonstrate that under certain conditions they are well-approximated by quasi-homogeneous sources.

Let us write the spectral degree of coherence in the form[3]

$$\mu_Q(\mathbf{r}_1, \mathbf{r}_2) = \mu_Q\left(\frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \mathbf{r}_2 - \mathbf{r}_1\right). \tag{5.63}$$

This is always possible as we are simply expressing it in terms of a different set of orthogonal coordinates. On substituting from Eq. (5.63) into Eq. (5.6), and introducing the coordinates

$$\mathbf{R} \equiv (\mathbf{r}_1 + \mathbf{r}_2)/2, \quad \mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1, \tag{5.64}$$

the Fourier transform of the cross-spectral density of the source may be written as

$$\begin{aligned}\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \frac{1}{(2\pi)^6} \iint h_Q\left(\mathbf{R} + \frac{\mathbf{r}}{2}\right) h_Q\left(\mathbf{R} - \frac{\mathbf{r}}{2}\right) \\ \times\, \mu_Q(\mathbf{R}, \mathbf{r}) e^{-\mathrm{i}k(\mathbf{s}_2 - \mathbf{s}_1)\cdot\mathbf{R}} e^{-\mathrm{i}k\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}\right)\cdot\mathbf{r}} \mathrm{d}^3 r \mathrm{d}^3 R.\end{aligned} \tag{5.65}$$

It is to be noted that, in a manner similar to that done in section 5.2, Eq. (5.38), we may replace $\mu_Q(\mathbf{R}, \mathbf{r})$ by a function $\mu_Q^B(\mathbf{R}, \mathbf{r}) = B(\mathbf{R}, \mathbf{r})\mu_Q(\mathbf{R}, \mathbf{r})$, where

$$B(\mathbf{R}, \mathbf{r}) = \begin{cases} 1 & \text{when} \quad h_Q(\mathbf{R} + \mathbf{r}/2)h_Q(\mathbf{R} - \mathbf{r}/2) \neq 0, \\ 0 & \text{when} \quad h_Q(\mathbf{R} + \mathbf{r}/2)h_Q(\mathbf{R} - \mathbf{r}/2) = 0. \end{cases} \tag{5.66}$$

This is necessary, as before, because $\mu_Q(\mathbf{R}, \mathbf{r})$ is undefined in regions where $h_Q(\mathbf{r})$ vanishes.

Now, noting that h_Q and μ_Q^B both have Fourier representations, i.e.

$$h_Q(\mathbf{r}) = \int e^{\mathrm{i}\mathbf{K}\cdot\mathbf{r}} \tilde{h}_Q(\mathbf{K}) \mathrm{d}^3 K, \tag{5.67}$$

$$\mu_Q^B(\mathbf{R}, \mathbf{r}) = \iint \tilde{\mu}_Q^B(\mathbf{K}_1, \mathbf{K}_2) e^{\mathrm{i}\mathbf{K}_1\cdot\mathbf{R}} e^{\mathrm{i}\mathbf{K}_2\cdot\mathbf{r}} \mathrm{d}^3 K_1 \mathrm{d}^3 K_2, \tag{5.68}$$

[3]Similar generalizations of quasi-homogeneous sources have been used in investigations of radiometry. See, for instance, [95].

the Fourier transform of the cross-spectral density may be written as

$$\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \iint\iint\iint \tilde{h}_Q^*(\mathbf{K}_1)\tilde{h}_Q(\mathbf{K}_2)\tilde{\mu}_Q^B(\mathbf{K}_3,\mathbf{K}_4)e^{-\mathrm{i}(\mathbf{K}_1-\mathbf{K}_2-\mathbf{K}_3+\mathbf{K}_-)\cdot\mathbf{R}} \times e^{-\mathrm{i}(\mathbf{K}_1/2+\mathbf{K}_2/2-\mathbf{K}_4+\mathbf{K}_+)\cdot\mathbf{r}}\{\mathrm{d}K_i\}\mathrm{d}^3r\mathrm{d}^3R, \qquad (5.69)$$

where $\{\mathrm{d}K_i\}$ represents the integration over all $\mathbf{K}_i$, $i = 1, 2, 3, 4$, and $\mathbf{K}_- = k(\mathbf{s}_2 - \mathbf{s}_1)$, $\mathbf{K}_+ = k(\mathbf{s}_1 + \mathbf{s}_2)/2$. The $\mathbf{r}$, $\mathbf{R}$ integrations may be carried out by use of the Fourier representation of the Delta function,

$$\delta^{(3)}(\mathbf{K}) = \frac{1}{(2\pi)^3}\int e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}}\mathrm{d}^3r, \qquad (5.70)$$

and the expression (5.69) may then be simplified by integration over $\mathbf{K}_1$, $\mathbf{K}_2$. The resulting expression is then

$$\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \iint \tilde{h}_Q^*(\mathbf{K}_4 + \mathbf{K}_3/2 - \mathbf{K}_-/2 - \mathbf{K}_+)\tilde{h}_Q(\mathbf{K}_4 - \mathbf{K}_3/2 + \mathbf{K}_-/2 - \mathbf{K}_+) \times \tilde{\mu}_Q^B(\mathbf{K}_3,\mathbf{K}_4)\mathrm{d}^3K_3\mathrm{d}^3K_4. \qquad (5.71)$$

Let us further modify Eq. (5.71) by changing the origin of the integration variable $\mathbf{K}_4$ so that $\mathbf{K}_4 \to \mathbf{K}_4 + \mathbf{K}_+$; then Eq. (5.71) becomes

$$\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \iint \tilde{h}_Q^*(\mathbf{K}_4 + \mathbf{K}_3/2 - \mathbf{K}_-/2)\tilde{h}_Q(\mathbf{K}_4 - \mathbf{K}_3/2 + \mathbf{K}_-/2) \times \tilde{\mu}_Q^B(\mathbf{K}_3,\mathbf{K}_4 + \mathbf{K}_+)\mathrm{d}^3K_3\mathrm{d}^3K_4. \qquad (5.72)$$

This expression should be compared to Eq. (5.45). It should be clear that equation (5.72) is comparable to the corresponding Schell-model equation, save for the integration over an additional variable $\mathbf{K}_3$. Note that if $\mu_Q^B(\mathbf{R},\mathbf{r})$ is independent of $\mathbf{R}$, then the Fourier transform with respect to $\mathbf{R}$ results in a delta function $\delta^{(3)}(\mathbf{K}_3)$, and Eq. (5.72) reduces exactly to the Schell-model result.

Now let us assume as before that the source intensity $h_Q(\mathbf{r})$ is slowly varying, so that

$$\tilde{h}_Q(\mathbf{K}) \approx 0 \quad \text{for all } |\mathbf{K}| \geq \alpha \quad . \tag{5.73}$$

If $\tilde{\mu}_Q^B(\mathbf{K}_i, \mathbf{K}_j)$ is a slowly varying function in $\mathbf{K}_j$ over distances α for *all* values of $\mathbf{K}_i$, then it may be replaced by the first term of its Taylor expansion in the second variable. The $\mathbf{K}_4$ integration may then be carried out explicitly, and we have

$$\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \int \tilde{I}_Q(\mathbf{K}+\mathbf{K}_-)\tilde{\mu}_Q^B(\mathbf{K}, \mathbf{K}_+)\mathrm{d}^3K. \tag{5.74}$$

This differs from the quasi-homogeneous approximation in that there remains an integration over $\mathbf{K}$. However, we note that if

$$\tilde{\mu}_Q^B(\mathbf{K}_1, \mathbf{K}_2) \approx 0 \quad \text{for all } |\mathbf{K}_1| \geq \beta \ll \alpha \quad , \tag{5.75}$$

then we may treat $\tilde{\mu}_Q^B$ as a delta function with respect to its first variable and write that

$$\tilde{W}_Q(-k\mathbf{s}_1, k\mathbf{s}_2) = \tilde{I}_Q(\mathbf{K}_-)\tilde{\mu}_Q^B(0, \mathbf{K}_+), \tag{5.76}$$

which has the form of the quasi-homogeneous approximation. Equation (5.75) suggests that the spectral degree of coherence $\tilde{\mu}_Q^B$ must be (on average) slowly varying compared to $\tilde{h}_Q$. If this requirement is satisfied, along with the usual requirements for the quasi-homogeneous approximation derived previously, then the quasi-homogeneous approximation will be a good model for a globally incoherent source.

Chapter 6

Conclusions

This thesis has been concerned with the properties of so-called nonradiating sources and their one and two-dimensional counterparts. We have attempted to answer several outstanding questions concerning such sources and their properties.

The investigation of nonpropagating excitations was undertaken with the goal of describing possible experiments to demonstrate that sources of this kind can be realized. It was shown that such excitations produce observable effects even when significant damping forces exist or the applied force density is not strictly monochromatic. A simple experiment to be performed on a flexible string was proposed which would indirectly demonstrate the existence of nonpropagating excitations. Some differences between the one-dimensional and three-dimensional properties of such localized excitations were noted. Furthermore, the problem of two-dimensional nonpropagating excitations was investigated, and it was shown that such excitations exist and are comparable to their one and three-dimensional counterparts.

Relatively little work has been done concerning partially coherent nonradiating sources. This deficit was in part remedied in chapter 4, where unusual radiation effects were demonstrated which are closely connected with the existence of nonradiating sources. It was shown that a fully coherent field may be produced by sources of nearly arbitrary states of coherence, even by sources which are fairly incoherent. It was pointed out that nonradiating sources are a particular type of correlation-induced spectral change; the consistency of correlation-induced spectral changes with energy conservation was demonstrated for statistically stationary electromagnetic sources and fields. An example of a partially polarized source which may produce an almost completely polarized field was given.

Examples of this kind demonstrate further the nonuniqueness of the inverse source problem, proving that it is not possible, in general, to determine the state of coherence of the source, the spectrum of the source, or the degree of polarization of the source from measurements of the field outside the source domain. However, in the process of examining such properties, it was observed that certain globally incoherent sources must radiate. An investigation of a certain class of such sources, so-called quasi-homogeneous sources, was undertaken, and as a consequence of this investigation it was demonstrated that nonradiating quasi-homogeneous sources do not exist. This result suggests that the inverse source problem is unique for such sources, and some simple inversion schemes were described and demonstrated. The schemes described require much less prior knowledge than those described previously to reconstruct the source intensity. Contrary to earlier results, it was shown that it is not possible in practice to determine the spectral degree of coherence from field correlation measurements, even if the intensity function of the source is known. Globally incoherent sources, which are a generalization of quasi-homogeneous sources, were also investigated, and some circumstances under which such sources may be considered quasi-homogeneous were described.

These results were intended to show, at least in part, that nonradiating sources are not simply a theoretical curiosity and an endless source of experimentally unverifiable papers on their mathematical properties. In quasi-homogeneous sources we have found a partial explanation of why nonradiating sources are not normally observed (most natural sources may be considered quasi-homogeneous) and in nonpropagating excitations we have shown, for the first time, a relatively simple method of producing such a localized excitation. Hopefully these results will lead to a more enlightened, and physical, examination of the nonradiating phenomenon and those objects that may be considered "invisible".

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Appendices

I Calculations relating to nonpropagating excitations

I.1 Continuity of $y(x)$ and $\mathrm{d}y/\mathrm{d}x$

Let $q(x)$ be a piecewise continuous function on a finite interval $a \leq x \leq b$, with discontinuities only at the points $x_0 = a, x_1, x_2, \ldots, x_n = b$. It is clear from the integral representation for $y(x)$, given by Eq. (3.7), that $y(x)$ is continuous at every point on the line, and one can readily deduce from that equation that its first derivative is continuous at every point with the possible exception of the points x_j, $j = 0, 1, \ldots, n$, i.e. those points for which $q(x)$ is discontinuous. Now we consider the relation

$$\int_{x_j-\epsilon_1}^{x_j+\epsilon_2} y''(x)\mathrm{d}x + k^2 \int_{x_j-\epsilon_1}^{x_j+\epsilon_2} y(x)\mathrm{d}x = \int_{x_j-\epsilon_1}^{x_j+\epsilon_2} q(x)\mathrm{d}x \quad (j = 0, 1, \ldots, n), \qquad \text{(I.1)}$$

with $\epsilon_1 \geq 0$, $\epsilon_2 \geq 0$, which follows from the differential equation (3.4). From this equation it is clear that

$$y'(x)|_{x_j+\epsilon_2} - y'(x)|_{x_j-\epsilon_1} = -k^2 \int_{x_j-\epsilon_1}^{x_j+\epsilon_2} y(x)\mathrm{d}x + \int_{x_j-\epsilon_1}^{x_j+\epsilon_2} q(x)\mathrm{d}x, \quad (j = 0, 1, \ldots, n). \qquad \text{(I.2)}$$

Because $y(x)$ is continuous and $q(x)$ piecewise continuous, it is evident on proceeding to the limits $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ that $y'(x)$ is also continuous at the points of discontinuity of $q(x)$.

I.2 Derivation of the displacement cross-spectral density, W_y

In order to derive the formula (3.63) we will use the space-frequency domain formulation of coherence theory, discussed in [72],[73]. According to this formulation, the cross-spectral density of a fluctuating wavefield such as $y(x,t)$ may be represented as an average taken over a suitably chosen ensemble of strictly monochromatic wavefields $\{\hat{y}(x,\omega)e^{-\mathrm{i}\omega t}\}$, all of the same angular frequency ω, in the form

$$W_y(x_1, x_2, \omega) = \langle \hat{y}^*(x_1, \omega)\hat{y}(x_2, \omega)\rangle . \tag{I.3}$$

A similar representation exists for the cross-spectral density of the force density, i.e.

$$W_f(x_1, x_2, \omega) = \left\langle \hat{f}^*(x_1, \omega)\hat{f}(x_2, \omega)\right\rangle . \tag{I.4}$$

It follows from Eqs. (3.7) and (3.8) that to the right of the source ($x > b$),

$$\hat{y}(x, \omega) = -\frac{e^{\mathrm{i}kx}}{2\mathrm{i}kT}\int_a^b \hat{f}(x', \omega)e^{-\mathrm{i}kx'}\mathrm{d}x'. \tag{I.5}$$

Upon substituting this result into the left-hand side of Eq. (I.3), and using Eq. (I.4), we find that

$$\begin{aligned} W_y(x_1, x_2, \omega) &= \frac{e^{\mathrm{i}k(x_2-x_1)}}{(2kT)^2}\int_a^b\int_a^b W_f(x_1', x_2', \omega)e^{\mathrm{i}k(x_1'-x_2')}\mathrm{d}x_1'\mathrm{d}x_2' \\ &= \frac{(2\pi)^2}{(2kT)^2}\tilde{W}_f(-k, k, \omega)e^{\mathrm{i}k(x_2-x_1)}, \end{aligned} \tag{I.6}$$

where $\tilde{W}_f$ is defined by Eq. (3.64).

II Derivation of an orthonormal set of nonradiating sources*

In this appendix we will construct a set of functions ϕ^{NR}_{lmN} which satisfy the nonradiating condition (4.10) and which are orthonormal over the spherical domain $r \leq a$. The orthonormality condition may be written as

$$\int_D \phi^{NR*}_{lmN}(\mathbf{r}')\phi^{NR}_{l'm'N'}(\mathbf{r}')\mathrm{d}^3r' = \delta_{ll'}\delta_{mm'}\delta_{NN'}, \tag{II.1}$$

where $\delta_{nn'}$ is the Kronecker delta function.

The most useful form of the nonradiating condition for our purposes was proven as Theorem 2.2: a function ϕ^{NR}_{lmN} is nonradiating if and only if

$$\int_D \phi^{NR}_{lmN}(\mathbf{r}')j_0(k|\mathbf{r}-\mathbf{r}'|)\mathrm{d}^3r' = 0 \tag{II.2}$$

for all $\mathbf{r}$.

To construct an orthonormal set of functions, it will be useful to begin with a set of functions which form an orthonormal basis for the sphere. It is well-known that a complete set of functions $\Lambda_{lmn}(\mathbf{r})$ within the sphere are given by the solutions of the partial differential equation†

$$\left[\nabla^2 + k^2_{lmn}\right]\Lambda_{lmn}(\mathbf{r}) = 0, \tag{II.3}$$

subject to the boundary condition

$$\Lambda_{lmn}(a) = 0. \tag{II.4}$$

*This work is based upon an unpublished derivation by A.I. Nachman of a complete set of nonradiating fields for the sphere.

†See [71], particularly chapters 5 and 8.

The normalized form of these solutions can be shown to be

$$\Lambda_{lmn}(\mathbf{r}) = N_{ln} j_l(k_{ln} r) Y_l^m(\theta, \phi), \tag{II.5}$$

where

$$N_{ln} = \left[\frac{2}{a^3}\right]^{1/2} \frac{1}{|j_{l+1}(k_{ln} a)|}, \tag{II.6}$$

Y_l^m are the spherical harmonics, and k_{ln} are the solutions to the equation

$$j_l(k_{ln} a) = 0. \tag{II.7}$$

We would like to write our nonradiating condition (II.2) in terms of these basis functions. It can be shown‡ that the kernel of Eq. (II.2) can be expanded in the form

$$j_0(k|\mathbf{r} - \mathbf{r}'|) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} j_l(kr) j_l(kr') Y_l^m(\theta, \phi) Y_l^{m*}(\theta', \phi'). \tag{II.8}$$

Furthermore, since the functions Λ_{lmn} form a complete basis, we may expand each of the nonradiating functions in terms of this basis, i.e.

$$\phi_{lmN}^{NR}(\mathbf{r}') = \sum_{n'=0}^{\infty} \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} C_{l'm'n'}^{lmN} \Lambda_{l'm'n'}(\mathbf{r}'), \tag{II.9}$$

where the coefficients $C_{l'm'n'}^{lmN}$ may be determined, given ϕ_{lmN}^{NR}, by the relation

$$C_{l'm'n'}^{lmN} = \int_D \phi_{lmN}^{NR}(\mathbf{r}') \Lambda_{l'm'n'}^*(\mathbf{r}') \mathrm{d}^3 r'. \tag{II.10}$$

It is to be noted, on examination of the kernel of Eq. (II.2), that a source with an angular dependence determined by a spherical harmonic Y_l^m will generate a field with an angular dependence determined by the *same* spherical harmonic Y_l^m. Therefore, for a spherical source, only sources with the same angular dependence may generate fields which destructively interfere. There are, in principle, many

‡This is easily shown by taking the real part of Eq. (2.32).

ways to select the coefficients $C^{lmN}_{l'm'n'}$ to satisfy Eqs. (II.1) and (II.2); let us restrict ourselves to a set of nonradiating modes such that

$$\phi^{NR}_{lmN}(\mathbf{r}') = \sum_{n=0}^{N} D^{lmN}_{n} \Lambda_{lmn}(\mathbf{r}'). \tag{II.11}$$

The nonradiating condition (II.2) may therefore be written, by use of Eqs. (II.8) and (II.11), in the form

$$\sum_{n=0}^{N}\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} D^{lmN}_{n} N_{ln} Y^{m'}_{l'}(\theta,\phi) j_{l'}(kr) \times \int_D j_{l'}(kr') Y^{m'*}_{l'}(\theta',\phi') j_l(k_{ln}r') Y^{m}_{l}(\theta',\phi') \mathrm{d}^3 r' = 0. \tag{II.12}$$

We may use the orthonormality conditions for the spherical harmonics Y^m_l,

$$\int_\Omega Y^{m'}_{l'}(\theta',\phi') Y^{m*}_{l}(\theta',\phi') \mathrm{d}\Omega = \delta_{ll'}\delta_{mm'}, \tag{II.13}$$

where the integration is over all θ', ϕ', to simplify the nonradiating condition (II.12). That condition may then be written in the form

$$\sum_{n=0}^{N} D^{lmN}_{n} N_{ln} \left[\int_0^a j_l(kr') j_l(k_{ln}r') r'^2 \mathrm{d}r'\right] Y^{m}_{l}(\theta,\phi) j_l(kr) = 0. \tag{II.14}$$

Because the Y^m_l are linearly independent, and because condition (II.14) must be satisfied for all $\mathbf{r}$, the series on the left-hand side of Eq. (II.14) must vanish identically, i.e.

$$\sum_{n=0}^{N} D^{lmN}_{n} N_{ln} \left[\int_0^a j_l(kr') j_l(k_{ln}r') r'^2 \mathrm{d}r'\right] = 0. \tag{II.15}$$

The integral in Eq. (II.15) may be evaluated exactly, the result being

$$\int_0^a j_\nu(kr) j_\nu(k_{\nu n}r) r^2 \mathrm{d}r = \frac{a^2}{k^2 - k^2_{\nu n}} \left\{k_{\nu n} j_\nu(ka) j'_\nu(k_{\nu n}a)\right\}, \tag{II.16}$$

where

$$j'_\nu(u) \equiv \frac{\mathrm{d}}{\mathrm{d}u} j_\nu(u). \tag{II.17}$$

Using Eq. (II.16), the nonradiating condition takes on the simple form

$$\left[\frac{2}{a^3}\right]^{1/2} a^2 j_l(ka) \sum_{n=0}^{N} D_n^{lmN} \alpha_{ln} = 0, \tag{II.18}$$

where

$$\alpha_{ln} = \frac{1}{|j_{l+1}(k_{ln}a)|} \frac{k_{ln}}{k^2 - k_{ln}^2} j_l'(k_{ln}a). \tag{II.19}$$

This expression may be further simplified to take the form

$$\sum_{n=0}^{N} D_n^{lmN} \alpha_{ln} = 0, \tag{II.20}$$

provided $j_l(ka) \neq 0$. We will discuss at the end of this appendix what must be done if $j_l(ka) = 0$. It is clear from Eq. (II.20) that this condition can be satisfied only for $N \geq 1$.

Using Eq. (II.11), the orthonormality condition for our set of nonradiating modes may be written as two equations, given by

$$\sum_{n=0}^{N} D_n^{lmN} D_n^{lmP} = 0, \quad N > P \geq 1, \tag{II.21a}$$

$$\sum_{n=0}^{N} D_n^{lmN} D_n^{lmN} = 1. \tag{II.21b}$$

The first of these equations is the requirement that a nonradiating mode of order N must be orthogonal to every mode of order $P < N$; this guarantees that the entire set of modes will be mutually orthogonal. The second of these equations is the normalization condition. If we satisfy Eqs. (II.21a) and (II.21b), together with Eq. (II.20), our modes $\phi_{lmN}^{NR}(\mathbf{r})$ will be nonradiating and mutually orthogonal.

If we choose the coefficients D_n^{lmN} such that

$$D_n^{lmN} = \begin{cases} \beta_N \alpha_{ln}, & n < N, \\ -\frac{\beta_N}{\alpha_{lN}} \sum_{n'=0}^{N-1} [\alpha_{ln'}]^2, & n = N, \end{cases} \tag{II.22}$$

it is not difficult to show that Eqs. (II.20) and (II.21a) will be satisfied. On substituting the expression (II.22) into the normalization condition, Eq. (II.21b), one finds that the normalization coefficient β_N is given by the expression

$$\beta_N = \frac{|\alpha_{lN}|}{\sqrt{\left[\sum_{n=0}^{N-1}[\alpha_{ln}]^2\right]\left[\sum_{n=0}^{N}[\alpha_{ln}]^2\right]}}. \tag{II.23}$$

An orthonormal set of nonradiating modes are therefore given by Eq. (II.11), where the coefficients D_n^{lmN} are given by Eqs. (II.22) and (II.23), and the functions $\Lambda_{lmn}(\mathbf{r})$ are given by Eqs. (II.5) and (II.6).

This construction must be modified somewhat if $j_l(ka) = 0$ for some value of l, say l_0. Since k is now a zero of the l_0-th spherical Bessel function, the functions $\Lambda_{l_0mn}(\mathbf{r})$ are already each orthogonal to $j_0(k|\mathbf{r}-\mathbf{r}'|)$, save for the n_0-th functions for which $k_{l_0n_0} = k$. Therefore, for $l = l_0$, the functions $\Lambda_{l_0mn}(\mathbf{r})$, for $n \neq n_0$, are already an orthonormal set of nonradiating functions, and the construction beginning with Eq. (II.9) is unnecessary.

III Non-negative definiteness and Hermiticity of the polarization cross-spectral tensor

For a tensor $W_{ij}^{(P)}$ to be a valid cross-spectral density tensor of a source polarization, it must be Hermitian,

$$\left[W_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega)\right]^* = W_{ji}^{(P)}(\mathbf{r}_2,\mathbf{r}_1,\omega), \tag{III.1}$$

and it must be non-negative definite, i.e.

$$\int_D\int_D f_i^*(\mathbf{r}_1)f_j(\mathbf{r}_2)W_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega)\mathrm{d}^3r_1\mathrm{d}^3r_2 \geq 0, \tag{III.2}$$

for all well-behaved vector functions $f_i(\mathbf{r})$ (see [28], section 6.6.1).

Let us consider the source distribution described by equation (4.59), namely,

$$W_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega) \approx S^{(P)}\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2},\omega\right)\left\{\delta_{ij}A(\mathbf{r}_2-\mathbf{r}_1,\omega)+a_ia_jB(\mathbf{r}_2-\mathbf{r}_1,\omega)\right\}. \tag{III.3}$$

This tensor will be Hermitian if the functions A, B are chosen to be real and to be dependent only upon the magnitude of the difference vector $\mathbf{r}_2-\mathbf{r}_1$, i.e. one must choose A, B such that

$$A(\mathbf{r}_2-\mathbf{r}_1,\omega) = A(|\mathbf{r}_2-\mathbf{r}_1|,\omega), \quad B(\mathbf{r}_2-\mathbf{r}_1,\omega) = B(|\mathbf{r}_2-\mathbf{r}_1|,\omega). \tag{III.4}$$

As regards non-negative definiteness, it is to be noted that the Kronecker delta δ_{ij} may be written as the direct product of three orthogonal unit vectors,

$$\delta_{ij} = a_ia_j + a_i^{(2)}a_j^{(2)} + a_i^{(3)}a_j^{(3)}, \tag{III.5}$$

where a_i is the same unit vector as in equation (III.3). We may use this expression to rewrite equation (III.3) in the form

$$W_{ij}^{(P)}(\mathbf{r}_1, \mathbf{r}_2, \omega) = S^{((P)}\left(\frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \omega\right) \left\{[A(\mathbf{r}, \omega) + B(\mathbf{r}, \omega)]\, a_i a_j \right.$$
$$\left. + \; A(\mathbf{r}, \omega)\left[a_i^{(2)} a_j^{(2)} + a_i^{(3)} a_j^{(3)}\right]\right\}, \quad \text{(III.6)}$$

where $\mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1$. Substituting this expression into the non-negative definiteness condition (III.2), we find that

$$\int_D \int_D S^{(P)}\left(\frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \omega\right) \left\{[A(\mathbf{r}, \omega) + B(\mathbf{r}, \omega)] f^*_{a^{(1)}}(\mathbf{r}_1) f_{a^{(1)}}(\mathbf{r}_2) \right.$$
$$\left. + A(\mathbf{r}, \omega)\left[f^*_{a^{(2)}}(\mathbf{r}_1) f_{a^{(2)}}(\mathbf{r}_2) + f^*_{a^{(3)}}(\mathbf{r}_1) f_{a^{(3)}}(\mathbf{r}_2)\right]\right\} \geq 0, \quad \text{(III.7)}$$

where $f_{a^{(i)}}(\mathbf{r})$ is the component of the function f along the unit vector $a^{(i)}$.

Because the components of f are arbitrary, the total tensor given by Eq. (III.3) will be non-negative definite if each of the two functions $A(\mathbf{r}, \omega)$ and $[A(\mathbf{r}, \omega) + B(\mathbf{r}, \omega)]$ are chosen to be non-negative definite scalar functions.

IV A reciprocity relation involving a pair of functions and their Fourier transforms

In section 5.2, the following assertion is made: if $\tilde{f}(\mathbf{K})$ is a slowly varying function over distances comparable to the width of the function $\tilde{g}(\mathbf{K})$, then $g(\mathbf{r})$ is a slowly varying function over distances comparable to the width of the function $f(\mathbf{r})$. In this Appendix we prove this assertion for continuous functions $f(\mathbf{r})$ and $g(\mathbf{r})$.

Let us consider a continuous function $f(\mathbf{r})$ confined to a finite domain D such that

$$\langle \mathbf{r} \rangle_f \equiv \int_D \mathbf{r}|f(\mathbf{r})|^2 \mathrm{d}^3 r = 0; \tag{IV.1}$$

that is, the function $f(\mathbf{r})$ is centered on the origin. The "width" of $f(\mathbf{r})$ may then be described by its second moment $\langle r^2 \rangle$, i.e.

$$\langle r^2 \rangle_f = \frac{\int_D r^2 |f(\mathbf{r})|^2 \mathrm{d}^3 r}{\int_D |f(\mathbf{r})|^2 \mathrm{d}^3 r}. \tag{IV.2}$$

Because $f(\mathbf{r})$ is a continuous function confined to a finite domain, it has a Fourier representation,

$$f(\mathbf{r}) = \int \tilde{f}(\mathbf{K}) e^{\mathrm{i}\mathbf{K}\cdot\mathbf{r}} \mathrm{d}^3 K, \tag{IV.3}$$

where $\tilde{f}$ is the Fourier transform of $f(\mathbf{r})$. On substituting from Eq. (IV.3) into Eq. (IV.2), we find that

$$\langle r^2 \rangle_f \int_D |f(\mathbf{r})|^2 \mathrm{d}^3 r = \iint \int_D r^2 \tilde{f}^*(\mathbf{K}) \tilde{f}(\mathbf{K}') e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}} e^{\mathrm{i}\mathbf{K}'\cdot\mathbf{r}} \mathrm{d}^3 K \mathrm{d}^3 K' \mathrm{d}^3 r. \tag{IV.4}$$

It is to be noted that

$$\mathbf{r} e^{\mathrm{i}\mathbf{K}\cdot\mathbf{r}} = -\mathrm{i}\nabla_K e^{\mathrm{i}\mathbf{K}\cdot\mathbf{r}}, \tag{IV.5}$$

where ∇_K is the gradient with respect to the vector $\mathbf{K}$. We may then rewrite Eq. (IV.4) in the form

$$\langle r^2\rangle_f \int_D |f(\mathbf{r})|^2 \mathrm{d}^3r = \iint \int_D \tilde{f}^*(\mathbf{K})\tilde{f}(\mathbf{K}')\nabla_K e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}} \cdot \nabla_{K'} e^{\mathrm{i}\mathbf{K}\cdot\mathbf{r}} \mathrm{d}^3K \mathrm{d}^3K' \mathrm{d}^3r. \quad \text{(IV.6)}$$

The integrand may be rewritten in the form

$$\begin{aligned}\tilde{f}^*(\mathbf{K})\tilde{f}(\mathbf{K}')\nabla_K e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}} \cdot \nabla_{K'} e^{\mathrm{i}\mathbf{K}\cdot\mathbf{r}} &= \tilde{f}(\mathbf{K}')\nabla_{K'} e^{\mathrm{i}\mathbf{K}'\cdot\mathbf{r}} \cdot \nabla_K \left\{\tilde{f}^*(\mathbf{K})e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}}\right\} \\ &- \tilde{f}(\mathbf{K}')\nabla_{K'} e^{\mathrm{i}\mathbf{K}'\cdot\mathbf{r}} e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}} \cdot \nabla \tilde{f}^*(\mathbf{K}). \quad \text{(IV.7)}\end{aligned}$$

The first term of this integrand is a complete gradient with respect to the $\mathbf{K}$-variable. The integration over $\mathbf{K}$ for this term may be converted to a surface integral which vanishes because $\tilde{f}(\mathbf{K})$ is a well-behaved Fourier transform. The second term may also be rewritten as the sum of a complete gradient with respect to the $\mathbf{K}'$-variable and another term, and the complete gradient may be eliminated as for $\mathbf{K}$. Finally, Eq. (IV.6) may be expressed as

$$\langle r^2\rangle_f \int_D |f(\mathbf{r})|^2 \mathrm{d}^3r = \iint \int_D \nabla_K \tilde{f}^*(\mathbf{K}) \cdot \nabla_{K'} \tilde{f}(\mathbf{K}') e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}} e^{\mathrm{i}\mathbf{K}'\cdot\mathbf{r}} \mathrm{d}^3K \mathrm{d}^3K' \mathrm{d}^3r. \quad \text{(IV.8)}$$

By use of the Fourier representation of a delta function, viz.

$$\delta(\mathbf{K}-\mathbf{K}') = \frac{1}{(2\pi)^3}\int e^{\mathrm{i}(\mathbf{K}-\mathbf{K}')\cdot\mathbf{r}} \mathrm{d}^3r, \quad \text{(IV.9)}$$

equation (IV.8) may be readily simplified to the form

$$\langle r^2\rangle_f \int_D |f(\mathbf{r})|^2 \mathrm{d}^3r = (2\pi)^3 \int |\nabla_K \tilde{f}(\mathbf{K})|^2 \mathrm{d}^3K. \quad \text{(IV.10)}$$

We may then make use of the Parseval formula,

$$\int_D |f(\mathbf{r})|^2 \mathrm{d}^3r = (2\pi)^3 \int |\tilde{f}(\mathbf{K})|^2 \mathrm{d}^3K, \quad \text{(IV.11)}$$

and express Eq. (IV.10) in the simple form

$$\langle r^2\rangle_f = \frac{\int |\nabla_K \tilde{f}(\mathbf{K})|^2 \mathrm{d}^3K}{\int |\tilde{f}(\mathbf{K})|^2 \mathrm{d}^3K}. \tag{IV.12}$$

The right side of this equation may be seen to represent the ratio of the average value of the squared modulous of the derivative of the function $\tilde{f}(\mathbf{K})$ to the average value of the squared modulous of the function f. Let us define the distance $\overline{K}_f$ by the formula

$$\frac{1}{\overline{K}_f^2} = \frac{\int |\nabla_K \tilde{f}(\mathbf{K})|^2 \mathrm{d}^3K}{\int |\tilde{f}(\mathbf{K})|^2 \mathrm{d}^3K}. \tag{IV.13}$$

The distance $\overline{K}_f$ characterizes the rate of variation of $\tilde{f}(\mathbf{K})$; over distances significantly less than $\overline{K}_f$, the function $\tilde{f}(\mathbf{K})$ is effectively constant. We have thus demonstrated that

$$\langle r^2\rangle_f = \frac{1}{\overline{K}_f^2}. \tag{IV.14}$$

This argument can clearly be reversed, by beginning with the width of a function $g(\mathbf{r})$ in Fourier space,

$$\langle K^2\rangle_g = \frac{\int K^2 |\tilde{g}(\mathbf{K})|^2 \mathrm{d}^3K}{\int |\tilde{g}(\mathbf{K})|^2 \mathrm{d}^3K}. \tag{IV.15}$$

It follows then that

$$\langle K^2\rangle_g = \frac{\int |\nabla g(\mathbf{r})|^2 \mathrm{d}^3r}{\int |g(\mathbf{r})|^2 \mathrm{d}^3r} = \frac{1}{\overline{r}_g^2}, \tag{IV.16}$$

where $\overline{r}_g$ characterizes the rate of variation of the function $g(\mathbf{r})$.

Let us now suppose that $\tilde{f}(\mathbf{K})$ is a slowly varying function over distances comparable to the width of the function $\tilde{g}(\mathbf{K})$, i.e. that

$$\overline{K}_f^2 \gg \langle K^2\rangle_g \tag{IV.17}$$

By use of Eqs. (IV.12) and (IV.16), we may rewrite Eq. (IV.17) in the form

$$\frac{1}{\langle r^2 \rangle_f} \gg \frac{1}{\bar{r}_g^2}, \tag{IV.18}$$

or

$$\bar{r}_g^2 \gg \langle r^2 \rangle_f. \tag{IV.19}$$

Equation (IV.19) implies that $g(\mathbf{r})$ is a slowly varying function over distances comparable to the width of $f(\mathbf{r})$. The proof of our assertion is therefore complete.

V Conditions for quasi-homogeneity

We have seen that the requirement for the validity of the quasi-homogeneous approximation is that the function $\tilde{\mu}_Q^B[\mathbf{K}+\mathbf{K}_0]$ be effectively constant for every value of $\mathbf{K}$ such that $|\mathbf{K}| \leq \alpha$, for any $|\mathbf{K}_0| \leq k$. In this Appendix we will express this condition in a form which may be used in a straightforward manner to determine if a given correlation function may be considered quasi-homogeneous.

Instead of using the complete Taylor expansion for the spectral degree of coherence given by Eq. (5.47), let us consider the finite Taylor expansion given by

$$\tilde{\mu}_Q^B[\mathbf{K}_0+\mathbf{K}] = \tilde{\mu}_Q^B[\mathbf{K}_0] + \int_0^1 \frac{\partial}{\partial l}\tilde{\mu}_Q^B[l\mathbf{K}+\mathbf{K}_0]\,dl. \tag{V.1}$$

This expansion of $\tilde{\mu}_Q^B$ can be verified directly by carrying out the integration on the right-hand side of Eq. (V.1). The first term of this expansion results in the quasi-homogeneous approximation, and the second term is the correction to this approximation. It is therefore clear that a requirement for the validity of the quasi-homogeneous approximation is that

$$\frac{\left|\int_0^1 \frac{\partial}{\partial l}\tilde{\mu}_Q^B[l\mathbf{K}+\mathbf{K}_0]\,dl\right|}{\left|\tilde{\mu}_Q^B(\mathbf{K}_0)\right|} \ll 1 \quad \text{for all} \quad |\mathbf{K}| \leq \alpha, |\mathbf{K}_0| \leq k. \tag{V.2}$$

This requirement guarantees that the quasi-homogeneous term will dominate the integral in Eq. (5.45) for all directions of observation $\mathbf{s}_1$ and $\mathbf{s}_2$. How small, in fact, the correction term must be to obtain a good solution to the inverse problem will evidently depend upon the desired accuracy of the reconstruction. In this sense the correction term may be considered to represent "noise" in the field data.

We may use the triangle inequality, viz.

$$\left|\int_0^1 \frac{\partial}{\partial l}\tilde{\mu}_Q^B\left[l\mathbf{K}+\mathbf{K}_0\right]dl\right| \le \int_0^1 \left|\frac{\partial}{\partial l}\tilde{\mu}_Q^B\left[l\mathbf{K}+\mathbf{K}_0\right]\right|dl. \tag{V.3}$$

to simplify Eq. (V.2). It follows then that a weaker condition for the validity of the quasi-homogeneous approximation is that

$$\frac{\int_0^1 \left|\frac{\partial}{\partial l}\tilde{\mu}_Q^B\left[l\mathbf{K}+\mathbf{K}_0\right]\right|dl}{\left|\tilde{\mu}_Q^B(\mathbf{K}_0)\right|} \ll 1 \quad \text{for all} \quad |\mathbf{K}| \le \alpha, |\mathbf{K}_0| \le k. \tag{V.4}$$

Because $\tilde{\mu}_Q^B(\mathbf{K})$ is the boundary value of an entire analytic function, it is everywhere continuous. It follows from this result that $\left|(\partial/\partial l)\tilde{\mu}_Q^B(l\mathbf{K}+\mathbf{K}_0)\right|$ is a continuous real function of the integration variable l. Therefore by the fundamental theorem of calculus, we may write

$$\int_0^1 \left|\frac{\partial}{\partial l}\tilde{\mu}_Q^B\left[l\mathbf{K}+\mathbf{K}_0\right]\right|dl = \left|\frac{\partial}{\partial l}\tilde{\mu}_Q^B(l\mathbf{K}+\mathbf{K}_0)\right|_{l=l_1(\mathbf{K})}, \quad \text{where} \quad 0 \le l_1(\mathbf{K}) \le 1. \tag{V.5}$$

This theorem states that the value of this dimensionless integral is equal to the value of the integrand evaluated at some point within the range of the integration. Using the chain rule for differentiation, we may rewrite this derivative in the form

$$\frac{\partial}{\partial l}\tilde{\mu}_Q^B(l\mathbf{K}+\mathbf{K}_0) = \mathbf{K}\cdot\nabla_{\mathbf{K}'}\tilde{\mu}_Q^B(\mathbf{K}'+\mathbf{K}_0)|_{\mathbf{K}'=l\mathbf{K}}; \tag{V.6}$$

our condition for the quasi-homogeneous approximation then becomes

$$\frac{\left|\mathbf{K}\cdot\nabla_{\mathbf{K}'}\tilde{\mu}_Q^B(\mathbf{K}'+\mathbf{K}_0)|_{\mathbf{K}'=l_1(\mathbf{K})\mathbf{K}}\right|}{\left|\tilde{\mu}_Q^B(\mathbf{K}_0)\right|} \ll 1 \quad \text{for all} \quad |\mathbf{K}| \le \alpha, |\mathbf{K}_0| \le k. \tag{V.7}$$

This inequality may be simplified further. We note that the maximum value of $|\mathbf{K}|$ is α; hence if

$$\alpha\frac{\left|\hat{\mathbf{K}}\cdot\nabla_{\mathbf{K}'}\tilde{\mu}_Q^B(\mathbf{K}'+\mathbf{K}_0)|_{\mathbf{K}'=l_1(\mathbf{K})\mathbf{K}}\right|}{\left|\tilde{\mu}_Q^B(\mathbf{K}_0)\right|} \ll 1 \quad \text{for all} \quad |\mathbf{K}| \le \alpha, |\mathbf{K}_0| \le k \tag{V.8}$$

is satisfied, then Eq. (V.7) is satisfied. Here $\hat{\mathbf{K}}$ is a unit vector in the direction of $\mathbf{K}$. Also, if inequality (V.8) is satisfied for every l in the range $(0,1)$, and not just at l_1, then the inequality (V.8) will be satisfied. A final condition is therefore

$$\frac{\left|\hat{\mathbf{K}} \cdot \nabla_{\mathbf{K}'} \tilde{\mu}_Q^B(\mathbf{K}')|_{\mathbf{K}'=\mathbf{K}+\mathbf{K}_0}\right|}{\left|\tilde{\mu}_Q^B(\mathbf{K}_0)\right|} \ll \frac{1}{\alpha} \quad \text{for all} \quad |\mathbf{K}| \leq \alpha, |\mathbf{K}_0| \leq k. \tag{V.9}$$

This condition suggests that the absolute value of the gradient of $\tilde{\mu}_Q$ must be sufficiently small for all measurable values of $\mathbf{K}$. This is essentially the same statement, expressed mathematically, that $\tilde{\mu}_Q^B$ must be smoothly varying over all measurable values of $\mathbf{K}$. For this reason, we may consider Eq. (V.9) to be a general requirement for the validity of the quasi-homogeneous approximation. We may use this equation to determine whether a given correlation function is well-described by this approximation.