Complete representation of a correlation singularity in a partially coherent beam

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An understanding of phase singularities of correlation functions is important in optical coherence theory and imaging science, but to date such singularities have only been theoretically studied in a single transverse plane, at most. In this Letter we evaluate the complete structure of a correlation singularity of a partially coherent Laguerre–Gauss beam, describing it in both the transverse and the propagation directions. These results agree with previously found solutions, and introduce new aspects of correlation singularities. © 2014 Optical Society of America

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In recent years, the study of singular optics has been of increasing interest to scientists both as an unexplored area of classical electromagnetic theory as well as a possible source for new technologies. Singular optics traditionally focuses on the study of coherent wavefields which have regions of zero amplitude and therefore undefined phase [1,2]; such regions are known as *phase singularities* and typically manifest as lines in three-dimensional space, around which the phase exhibits a circulating or helical structure. Because of this, these structures are referred to as *optical vortices*.

Besides being of purely topological interest, singular optics presents a new way to look at wavefields and has led to a number of applications. These include using optical vortices as optical tweezers and spanners [3], in spatial and temporal coherence filtering [4], in optical vortex coronagraphy [5], in phase contrast microscopy [6], and as information carriers in free-space optical communications [7].

There are advantages to using partially coherent light in a number of these applications; for example, it has been demonstrated that partially coherent beams are more resistant to atmospheric turbulence degradation than their fully coherent counterparts [8], making them ideal for free-space communications. However, there are conceptual challenges to extending the study of singular optics to partially coherent fields; among them is the lack of a well-defined phase in fields that are not spatially coherent [9]. For some time now, researchers have been investigating phase singularities in correlation functions of wavefields both theoretically [10–12] and experimentally [13–15], and have demonstrated the existence of singularities of the spatial correlation function, known as *correlation singularities*.

Notably, it has been shown that there is a connection between the optical vortices created by a linear optical system and the correlation singularities produced by the same system [16]. A good physical understanding of this connection is still lacking, however, due at least in part to the complicated nature of correlation singularities. Because the correlation function is a measure of the statistical correlations between two arbitrary points in a field, a complete description of correlations in three-dimensional space requires the specification of the correlation function in six variables. Most previous work on correlation singularities has focused on simple two variable projections of the function, with the exception of one recent article that studied the complete structure of a correlation singularity in four variables in the waist plane of a partially coherent beam [17]. The evolution of this singular structure on propagation has not been addressed.

In this Letter, we undertake such a description, using an analytic model of a correlation singularity. This model has been previously shown [16] to be representative of the generic form of screw dislocations in partially coherent waves. It is shown that the geometric structure of the singularity undergoes nontrivial changes on propagation, depending strongly on the size of the beam and the state of coherence. We describe the geometric form of correlation singularities in the full six-variable correlation space, and look at selected projections of these correlation singularities. Finally, we offer some thoughts on the significance of these results.

In the time domain, the mutual coherence function $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ of a statistically stationary, fluctuating wavefield $U(\mathbf{r}, t)$ can be written in the form [18, Section 4.3]

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = \langle U^*(\mathbf{r}_1, t) U(\mathbf{r}_2, t+\tau) \rangle, \tag{1}$$

where the angle brackets represent a time average or, equivalently, an ensemble average.

For present purposes, it is simpler to work with the temporal Fourier transform of the mutual coherence function, known as the cross-spectral density [18, Section 4.3.2], defined as

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) \exp[i\omega\tau] \mathrm{d}\tau. \tag{2}$$

The cross-spectral density characterizes both the intensity and the spatial coherence properties of the field as a function of frequency ω, and contains all the information of the mutual coherence function. For a quasi-monochromatic field of frequency ω_0, the complete coherence properties of the field are well approximated by the value of the cross-spectral density at ω_0.

It has been shown that the cross-spectral density of an arbitrary partially coherent field at frequency ω may always be expressed as the average of an ensemble of monochromatic realizations of the field [19], i.e.,

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) = \langle U^*(\mathbf{r}_1, \omega) U(\mathbf{r}_2, \omega)\rangle_\omega, \tag{3}$$

where the subscript ω denotes averaging with respect to this particular ensemble. The advantage of this method is that it allows for the construction of partially coherent field models in the frequency domain without first having to find the more complicated mutual coherence function. We use this representation to construct a model of a quasi-monochromatic, partially coherent field.

In the absence of sources, the cross-spectral density satisfies a pair of Helmholtz wave equations:

$$[\nabla_j^2 + k^2] W(\mathbf{r}_1, \mathbf{r}_2, \omega) = 0, \qquad j = 1, 2, \tag{4}$$

where ∇_j represents the gradient with respect to the variable $\mathbf{r}_j$. From this it follows that, with one observation point fixed, the cross-spectral density behaves exactly like a monochromatic wave field and can possess vortices with the same topological and phase properties as such fields. These vortices are known as *correlation vortices*, also referred to in some earlier work as *coherence vortices*. Such singularities are distinct from optical vortices in both their behavior and their interpretation. While optical vortices are phase singularities of a one-point monochromatic wave field, coherence vortices are singularities in a two-point correlation function of the field. This means, among other things, that the location of any correlation vortices is dependent on the choice of reference point.

In principle, the formalism described above is exact; correlation singularities can exist independently at each frequency of a partially coherent wave field. For the remainder of this Letter, we will restrict ourselves to a quasi-monochromatic wave field of center frequency ω, and will thus suppress the expression of this frequency for notational brevity. Furthermore, we will use the vector $\mathbf{r}$ to denote the three-dimensional position vector, where $\mathbf{r} = (x, y, z) = (\boldsymbol{\rho}, z)$, thus allowing the exploration of vortices not just in the source plane, but at any point in the propagation of the beam.

We consider the simple model for a coherence vortex first described in Ref. [20]. We take a Laguerre–Gauss beam of order $n = 0$ and topological charge $m = \pm 1$, whose central vortex core is a random function of position. A field with $m = +1$ is a right-handed vortex, while $m = -1$ is a left-handed vortex. This model may be expressed as

$$W(\mathbf{r}_1, \mathbf{r}_2) = \int f(\boldsymbol{\rho}_0) U_\pm^*(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_0, z_1) U_\pm(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_0, z_2) \mathrm{d}^2\boldsymbol{\rho}_0, \tag{5}$$

where

$$U_\pm(x, y, z) = \sqrt{\frac{4}{\pi}} \frac{x \pm iy}{w^2(z)} \exp\left[\frac{-(x^2 + y^2)}{w^2(z)}\right] \times \exp\left[\frac{-ik(x^2 + y^2)}{2R(z)}\right] \exp[-2i\xi(z)]. \tag{6}$$

Here, $\xi(z)$ represents the Gouy phase, and $w(z)$ and $R(z)$ are the propagation-dependent width and wavefront curvature of the beam, respectively,

$$w(z) = w_0 \sqrt{1 + (z/z_R)^2}, \tag{7}$$

$$R(z) = z + \frac{z_R^2}{z}, \tag{8}$$

with $z_R = \pi w_0^2/\lambda$ indicating the Rayleigh range and w_0 the width of the beam in the waist plane. The function $f(\boldsymbol{\rho}_0)$ is the probability density for the position $\boldsymbol{\rho}_0$ of the vortex core, taken to be a Gaussian function,

$$f(\boldsymbol{\rho}_0) = \frac{1}{\pi\delta^2} \exp\left[-\rho_0^2/\delta^2\right]. \tag{9}$$

The parameter δ represents average wander of the vortex core. In the limit $\delta \to 0$, the beam does not wander at all and is therefore fully coherent. Increasing δ represents decreasing spatial coherence.

The integral above in Eq. (5) can be evaluated analytically and results in the expression

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z_1, z_2) = Q\left\{\left[\frac{1}{\alpha_2^2}\left(x_1 \mp iy_1\right) - \frac{1}{\sigma_2^2}(x_2 \mp iy_2)\right] \times \left[\frac{1}{\alpha_1^{2*}}(x_2 \pm iy_2) - \frac{1}{\sigma_1^{2*}}(x_1 \pm iy_1)\right] + \frac{1}{\beta_2^2}\right\}, \tag{10}$$

where

$$\frac{1}{\sigma_n^2} = \frac{1}{w_n^2} - \frac{ik}{2R_n}, \qquad n = 1, 2, \tag{11}$$

$$\frac{1}{\alpha_n^2} = \frac{1}{\delta^2} + \frac{1}{\sigma_n^2}, \qquad n = 1, 2, \tag{12}$$

$$\frac{1}{\beta_2^2} = \frac{1}{\delta^2} + \frac{1}{\sigma_1^{2*}} + \frac{1}{\sigma_2^2}, \tag{13}$$

and Q is a complex position-dependent factor that plays no role in the location of zeros, to be suppressed from now on. With these equations, where $w_n \equiv w(z_n)$, $R_n \equiv R(z_n)$ and so forth, it can be shown that the cross-spectral density only equals zero when $z_1 = z_2 \equiv z$. To study the singularities of the correlation function we therefore need only consider this case for which

$$\frac{1}{\alpha_1^2} = \frac{1}{\alpha_2^2} = \frac{1}{\alpha^2}, \tag{14}$$

$$\frac{1}{\sigma_1^2} = \frac{1}{\sigma_2^2} = \frac{1}{\sigma^2}. \tag{15}$$

The cross-spectral density is in general a complex quantity; singularities of the cross-spectral density therefore only exist in locations where the real and imaginary parts of $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z, z)$ are simultaneously zero. It can be readily demonstrated that the topology of the zeros is exceedingly complex when expressed in terms of $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$. However, it reduces to a simple form by changing the coordinate system to the sum and difference variables,

$$\boldsymbol{\rho}_+ \equiv \frac{\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2}{2}, \qquad \boldsymbol{\rho}_- \equiv \boldsymbol{\rho}_2 - \boldsymbol{\rho}_1, \tag{16}$$

and then introducing the coefficients

$$\eta = \frac{1}{\alpha^2 \sigma^{*2}}, \tag{17}$$

$$2\chi = \left(\frac{1}{|\alpha|^4} + \frac{1}{|\sigma|^4}\right), \tag{18}$$

$$2Z = \left(\frac{1}{|\alpha|^4} - \frac{1}{|\sigma|^4}\right). \tag{19}$$

With these choices, the real and imaginary parts of the cross spectral density may be written as

$$\mathrm{Re}(W) = -2\,\mathrm{Re}(\eta)\left[x_+^2 + y_+^2 + \frac{x_-^2 + y_-^2}{4}\right] + 2\chi\left[x_+^2 + y_+^2 - \frac{x_-^2 + y_-^2}{4}\right] + \frac{1}{\beta_2^2}, \tag{20}$$

$$\mathrm{Im}(W) = 2i\,\mathrm{Im}(\eta)[x_+ x_- + y_+ y_-] \pm i2Z[x_+ y_- + y_+ x_-]. \tag{21}$$

By making a further transformation into polar coordinates, $\boldsymbol{\rho}_+ = (\rho_+, \phi_+)$, and an appropriate use of trigonometric identities, it can be shown that the imaginary portion of the cross-spectral density only vanishes when

$$\tan\Phi = \mp \frac{\mathrm{Im}(\eta)}{Z}, \tag{22}$$

where $\Phi \equiv \phi_+ - \phi_-$. It should be noted that these transformations give us an imaginary part of W that depends solely on the angle between $\boldsymbol{\rho}_+$ and $\boldsymbol{\rho}_-$. Similarly, the real part becomes

$$(-2\,\mathrm{Re}(\eta) + 2\chi)\rho_+^2 + \frac{1}{2}(-\mathrm{Re}(\eta) - \chi)\rho_-^2 + \frac{1}{\beta_2^2} = 0, \tag{23}$$

which depends solely on ρ_+, ρ_-, without any angular dependence.

Equations (22) and (23) form the main results of this Letter. They indicate that the singularity has a well-defined structure in terms of the transverse sum and difference vectors $\boldsymbol{\rho}_+$ and $\boldsymbol{\rho}_-$. Equation (23) can be shown to represent the equation of a hyperbola for all propagation distances, though the shape of the hyperbola evolves on propagation. An example of this is illustrated in Fig. 1(a). The hyperbolas are plotted at distances $z = 10$ m, $z = 100$ m, and $z = 1$ km, for the case of $w_0 = 1$ mm, $\delta = 2$ mm. For this value of δ, the beam has little wander and is relatively coherent.

As the beam propagates, the hyperbolas angle away from the ρ_+ axis and asymptotically approach the lines $\rho_- = \pm 2\rho_+$. This is noteworthy, as a fully coherent beam would have zeros only when either $\rho_1 = 0$ or $\rho_2 = 0$, which can be seen from Eq. (16) to be the same condition. This is in agreement with the classic van Cittert–Zernike theorem [18, Section 4.4.4] that indicates that a field will become more coherent on propagation. Remarkably, this suggests that a vortex, even in a randomized beam, will in a sense reconstitute itself after a sufficient propagation distance.

A second example, with $\delta = 8$ mm, is shown in Fig. 1(b). With this value of δ, the beam wanders greatly and can be considered significantly incoherent. At short propagation distances, the hyperbola very nearly meet at the $\rho_- = 0$ line; this indicates that, for incoherent vortex beams, correlation singularities can be found at any radial distance from the center of the beam, provided $\rho_1 \approx \rho_2$. As the field propagates, again the hyperbolas angle away and asymptotically approach the coherent limit. However, it can be readily shown from Eq. (23) that the upper and lower branches of the hyperbola will remain separated by a distance $\Delta\rho_- = 2\delta$ for large values of z. This implies that, for a large value of observation point $\boldsymbol{\rho}_1$, the field will appear to have a coherent vortex at the origin in $\boldsymbol{\rho}_2$. When $\boldsymbol{\rho}_1$ is close to the origin, however, the zero hyperbolas deviate significantly from the straight line coherent case. It could be said that, after propagation, a correlation vortex will look coherent only when one observation point is away from the origin.

The angular positions of these singularities undergo a striking behavior on propagation, as illustrated in Fig. 1(c). As the field propagates, the angle between the vectors $\boldsymbol{\rho}_+$ and $\boldsymbol{\rho}_-$ quickly diverges from $\Phi = 0$ and, for highly incoherent initial beams, approaches $\mp\pi/2$, corresponding to a vortex charge of $m = \pm 1$. This "kink" in the location of the correlation singularities

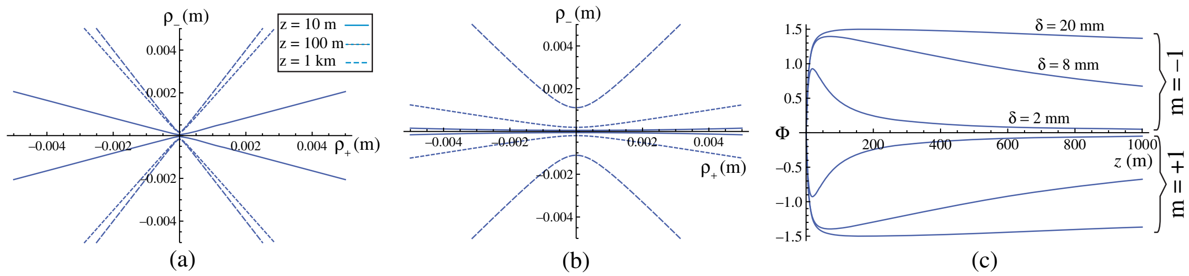

Fig. 1. Zero manifolds of a partially coherent vortex beam on propagation. (a) Radial positions on propagation for $\delta = 2$ mm, (b) radial positions on propagation for $\delta = 8$ mm, and (c) angular positions on propagation. In all figures, $w_0 = 1$ mm, $\lambda = 500$ nm.

eventually returns to $\Phi = 0$ over long propagation distances, though for highly incoherent fields it may never do so for any practical distance (for $\delta = 20$ mm, we still have $\Phi = 0.1$ even at $z = 50$ km).

The origin of this kink in the angle is a result of what may loosely be called an interaction between the phase due to the wavefront curvature $R(z)$ and the phase of the original vortex beam. At large distances, the wavefront of the beam becomes locally flat, i.e., the surface of a large sphere, and the phase is approximately planar near the z axis. At intermediate propagation distances, however, the two phases mix to produce a charge-dependent change in the structure of the singularity. It is noted that this angular shift may provide a noninterferometric method for measuring the topological charge of a partially coherent vortex beam, as the charge can be determined from the location of the zeros, rather than the phase structure of the beam.

These results represent the first theoretical treatment of the full propagation characteristics of a partially coherent vortex beam. We have found that the beam evolves some features of a coherent vortex beam on sufficient propagation, and that there exists a nontrivial phase kink on propagation that is directly dependent on the beam's topological charge.

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