Null-field radiationless sources

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It is shown that it is in principle possible to produce combined sources of polarization and magnetization that are not only radiationless but that have any (and sometimes several) of the four microscopic or macroscopic electromagnetic fields exactly zero. The conditions that such a "null-field radiationless source" must satisfy are derived, and examples are given for several cases. The implications for transformation optics and invisibility physics in general are discussed.

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Long before the introduction of cloaking devices in 2006 [1,2] and the flurry of investigations they sparked [3], researchers had invested much effort into the understanding of simpler types of invisible objects. The earliest of these is the class of so-called nonradiating sources (see, for instance, [4,5], and the review of [6]), sources whose fields are confined within the domain of excitation and are identically zero outside this domain. The theory of such sources, though relatively simple in comparison with the sophisticated mathematics of transformation optics [7], allows some effects that are evidently not possible using such coordinate transformation design techniques.

A good example of this was illustrated in a 1910 paper by Ehrenfest [8], in what may be the earliest paper dedicated to radiationless sources. Ehrenfest provided a general prescription for constructing nonradiating sources using the (not obvious) assumption that the magnetic field could be set identically to zero. Such null-field sources were later considered by van Bladel [9], who suggested that they could be constructed with electric and magnetic currents, and most recently, Nikolova and Rickard [10] briefly discussed the possibility of making sources with vanishing $\mathbf{E}$ or $\mathbf{H}$.

In this Letter we build on previous observations and introduce a full theory of null-field radiationless polarization and magnetization sources that can have any of the four electromagnetic fields $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$ or $\mathbf{B}$ identically zero within the source domain—and often multiple fields zero simultaneously. The results are derived through the use of Maxwell's equations and verified through the use of a Green's dyadic formalism. Illustrative examples are given, and the implications of the results for invisibility physics are discussed using the known relationship between the electromagnetic radiation and scattering problems.

We begin with the monochromatic macroscopic form of Maxwell's equations in Gaussian units, and assume that there exist no free currents and charges, i.e.,

$$\nabla \cdot \mathbf{D} = 0, \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2}$$

$$\nabla \times \mathbf{E} = ik\mathbf{B}, \tag{3}$$

$$\nabla \times \mathbf{H} = -ik\mathbf{D}, \tag{4}$$

where k is the wavenumber. We use the definitions of the auxiliary fields $\mathbf{D}$ and $\mathbf{H}$,

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \tag{5}$$

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}, \tag{6}$$

and substitute these expressions into Maxwell's equations to write the latter entirely in terms of the $\mathbf{E}$ and $\mathbf{H}$ fields, as well as the polarization $\mathbf{P}$ and magnetization $\mathbf{M}$,

$$\nabla \cdot \mathbf{E} = -4\pi\nabla \cdot \mathbf{P}, \tag{7}$$

$$\nabla \cdot \mathbf{H} = -4\pi\nabla \cdot \mathbf{M}, \tag{8}$$

$$\nabla \times \mathbf{E} = ik\mathbf{H} + 4\pi ik\mathbf{M}, \tag{9}$$

$$\nabla \times \mathbf{H} = -ik\mathbf{E} - 4\pi ik\mathbf{P}. \tag{10}$$

By use of the curl of Eqs. (10) and (8), we arrive at a monochromatic electromagnetic wave equation with both polarization and magnetization sources,

$$\nabla \times (\nabla \times \mathbf{H}) - k^2\mathbf{H} = 4\pi k^2\mathbf{M} - 4\pi ik\nabla \times \mathbf{P}. \tag{11}$$

The source term on the right of this equation suggests an intriguing possibility: if there are no additional free-propagating fields in the system, there will be no source of magnetic waves, and therefore no magnetic fields at all, if the following condition is satisfied,

$$ik\mathbf{M} + \nabla \times \mathbf{P} = 0. \tag{12}$$

Equation (12) may be considered our condition for a null-$\mathbf{H}$ source. Because a propagating electromagnetic wave requires both $\mathbf{E}$ and $\mathbf{H}$ fields, this also implies that the source must produce no electromagnetic waves outside its domain.

We may also take the curl of Eq. (9) and simplify using Eq. (7); we then arrive at a similar wave equation for the electric field $\mathbf{E}$,

$$\nabla \times (\nabla \times \mathbf{E}) - k^2\mathbf{E} = 4\pi k^2\mathbf{P} + 4\pi ik\nabla \times \mathbf{M}. \tag{13}$$

Two observations result from this equation. First, we can see that we will have a null-**E** source if the following condition is satisfied,

$$ik\mathbf{P} - \nabla \times \mathbf{M} = 0. \tag{14}$$

Second, we can see that Eqs. (12) and (14) are distinct equations, implying that, in general, a null-**E** source will not be a null-**H** source, and vice-versa.

These results are similar to those reported by van Bladel and Nikolova and Rickard previously. However, we can go further and express Maxwell's equations entirely in terms of **D** and **B** instead of **E** and **H**, which then leads to a different pair of wave equations,

$$\nabla \times (\nabla \times \mathbf{B}) - k^2\mathbf{B} = -4\pi\nabla \times (ik\mathbf{P} - \nabla \times \mathbf{M}), \tag{15}$$

$$\nabla \times (\nabla \times \mathbf{D}) - k^2\mathbf{D} = 4\pi\nabla \times (ik\mathbf{M} + \nabla \times \mathbf{P}). \tag{16}$$

In analogy with the previous results, we see that we can create null-**B** and null-**D** sources, respectively, if the polarization and magnetization satisfy

$$\nabla \times (ik\mathbf{P} - \nabla \times \mathbf{M}) = 0, \tag{17}$$

$$\nabla \times (ik\mathbf{M} + \nabla \times \mathbf{P}) = 0. \tag{18}$$

On comparison of these new conditions with Eqs. (12) and (14), it is apparent that a null-**E** source is automatically a null-**B** source and a null-**H** source is automatically a null-**D** source, but the converse is not true.

We can confirm the nonradiating nature of sources that satisfy these expressions by directly calculating the fields from the sources using Hertz vectors ([11], Section 2.2.2), defined as

$$\pi_e = \int_V \mathbf{P}(\mathbf{r}')G(|\mathbf{r} - \mathbf{r}'|)\mathrm{d}^3r', \tag{19}$$

$$\pi_m = \int_V \mathbf{M}(\mathbf{r}')G(|\mathbf{r} - \mathbf{r}'|)\mathrm{d}^3r', \tag{20}$$

where V is the domain of the source and $G(R) = \exp[ikR]/R$ is the Green's function of the scalar Helmholtz equation. The electric field **E** and the magnetic field **H** then satisfy

$$\mathbf{E}(\mathbf{r}) = \nabla \times (\nabla \times \pi_e) + ik\nabla \times \pi_m - 4\pi\mathbf{P}, \tag{21}$$

$$\mathbf{H}(\mathbf{r}) = \nabla \times (\nabla \times \pi_m) - ik\nabla \times \pi_e - 4\pi\mathbf{M}. \tag{22}$$

The behavior of the fields in the far-zone ($kr \gg 1$) can be readily determined by noting that $\pi_e(r\hat{\mathbf{s}}) = \tilde{\mathbf{P}}(k\hat{\mathbf{s}})e^{ikr}/r$ and $\pi_m(r\hat{\mathbf{s}}) = \tilde{\mathbf{M}}(k\hat{\mathbf{s}})e^{ikr}/r$, where r is the distance from the origin, and $\hat{\mathbf{s}}$ is the unit vector pointing from the origin, and by using the simple far-zone rule that $\nabla \to ik\hat{\mathbf{s}}$. With these substitutions, one can demonstrate that the **E** and **H** fields, as well as **D** and **B**, are identically zero in the far-zone.

The fields within null-field sources can also be calculated explicitly using Hertz vectors, though care must be taken in the interchange of derivatives and integrals in the calculation, as has been long known [12]. A simpler observation comes from Eqs. (17) and (18): since a null-**H** source, labeled by $H0$, automatically has $\mathbf{D} = 0$ and a null-**E** source, labeled by $E0$, automatically has $\mathbf{B} = 0$, it follows from Eqs. (5) and (6) that

$$\mathbf{E}_{H0} = -4\pi\mathbf{P}, \tag{23}$$

$$\mathbf{H}_{E0} = -4\pi\mathbf{M}. \tag{24}$$

To design a null-field source, the polarization and magnetization must be chosen to satisfy one of the four conditions given above. A null-**H** source, for instance, may be designed by choosing functions for **P** and **M** that satisfy Eq. (12). A spherical example of such a source is given by the choice

$$\mathbf{P}(\mathbf{r}) = \hat{z}f(r), \tag{25}$$

$$f(r) = \begin{cases} \cos^2(\pi r^2/2), & |\mathbf{r}| \leq 1, \\ 0, & \text{otherwise}, \end{cases} \tag{26}$$

$$\mathbf{M}(\mathbf{r}) = -\frac{1}{ik}\nabla \times \mathbf{P}(\mathbf{r}). \tag{27}$$

It is to be noted that the function $f(r)$ has been taken to be continuous; this is not a necessary requirement, but prevents a magnetization singularity on the outer surface of the sphere. Such singularities can be incorporated, however, and are discussed by van Bladel [13]. An illustration of this source is given in Fig. 1; the electric field is directly proportional to the polarization.

The previous source, as noted, is also a null-**D** source. With a slight modification, it can be converted into a source with a zero **D** field but a nonzero **H** field. To do so, we introduce a new magnetization $\mathbf{M}'(\mathbf{r}) = \mathbf{M}(\mathbf{r}) + \nabla\phi(r)$, with $\phi(r)$ a function that is continuous and that possesses a continuous first derivative (again to avoid requiring singular sources).

A simple example of this is given by choosing

$$\phi(r) = \frac{1}{ik}\begin{cases} \cos^2(\pi r^2/2), & |\mathbf{r}| \leq 1, \\ 0, & \text{otherwise}, \end{cases} \tag{28}$$

and the modified magnetization functions are shown in Fig. 2.

This example suggests other possibilities for null-field sources. If the polarization density is taken to be of the form $\mathbf{P}(\mathbf{r}) = \nabla\phi(r)$, then it will automatically satisfy Eq. (12), the null-**H** condition, without any magnetization at all. Similarly, a source with magnetization $\mathbf{M}(\mathbf{r}) = \nabla\phi(r)$ will satisfy Eq. (14), the null-**E** condition, without any polarization. Such sources produce only longitudinal fields, and cannot therefore produce any transverse

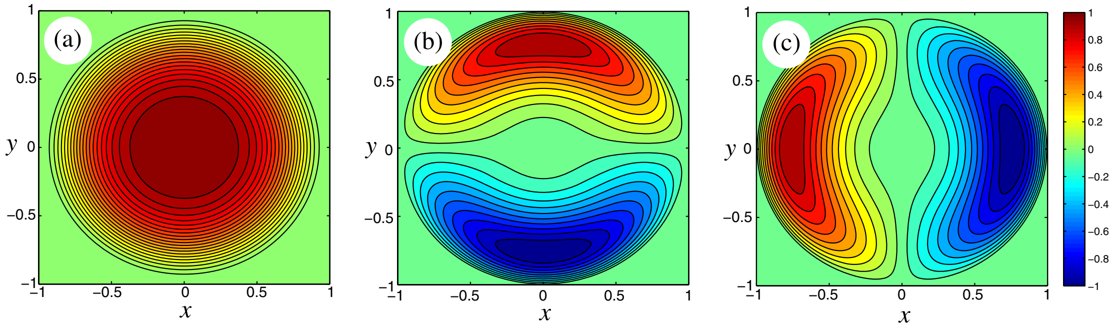

Fig. 1. (a) Polarization P_z and magnetization (b) iM_x and (c) iM_y of the null-**H** source given by Eq. (27), with $k = 1$.

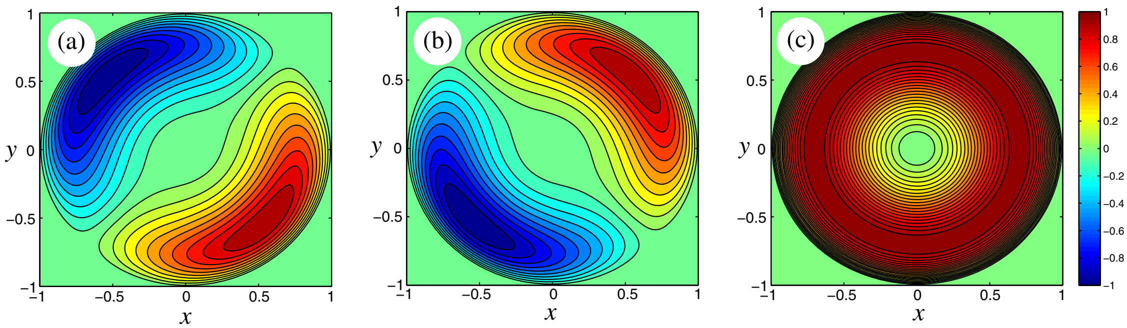

Fig. 2. (a) Magnetization iM_x, (b) iM_y, and (c) iM_z of the null-**D** source with nonzero **H**, with $k = 1$. The quantity P_z is the same as in Fig. 1.

radiation. They may be considered a generalization of a source considered by Ehrenfest in [8], in which he noted that a source with a purely radial current density, like a radially pulsating sphere, cannot produce radiation.

Finally, we may look again at the conditions for null-**E** and null-**H** simultaneously. These equations, written as

$$ik\mathbf{P} - \nabla \times \mathbf{M} = 0, \tag{29}$$

$$ik\mathbf{M} + \nabla \times \mathbf{P} = 0, \tag{30}$$

respectively, in fact mirror Maxwell's equations, (3) and (4), for an electromagnetic wave propagating in a source-free region. If we choose **P** and **M** to have the same form as the electric and magnetic fields of a free-propagating field, the actual electromagnetic fields will be identically zero in the source.

This cannot be done over a complete source region without introducing singular boundary sources. We can, however, design a source with a null-field region inside and a gradual outer transition region, as the following example shows. For the polarization and magnetization, we use a source of outer radius a and inner radius b, with

$$P_x(\mathbf{r}) = \exp[ikz]\begin{cases} \cos^2[g(r)], & b \le |r| \le a, \\ 1, & |r| < b, \\ 0, & |r| > a, \end{cases} \tag{31}$$

$$M_y(\mathbf{r}) = -\exp[ikz]\begin{cases} \cos^2[g(r)], & b \le |r| \le a, \\ 1, & |r| < b, \\ 0, & |r| > a, \end{cases} \tag{32}$$

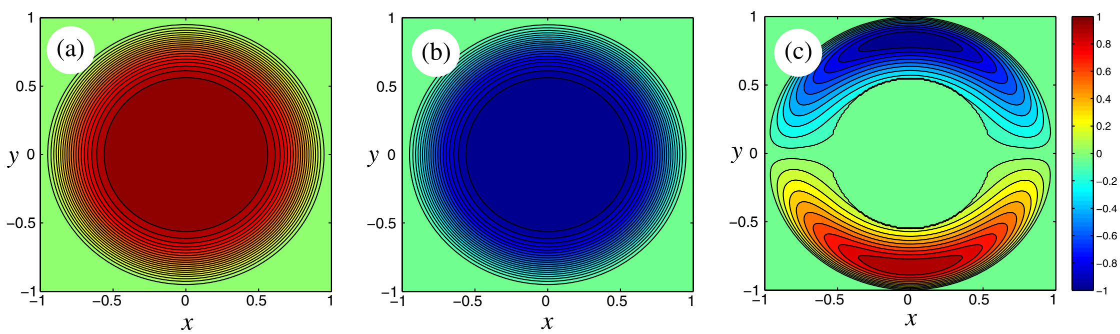

Fig. 3. (a) Polarization P_x, magnetization (b) M_y, and (c) iM_z of the complete null source, with $k = 1$, $a = 1$, $b = 0.3$, and $z = 0$.

$$M_z(\mathbf{r}) = -\exp[ikz]\begin{cases} \frac{\pi(r-b)}{(a-b)^2}\frac{y}{ikr}h(r), & b \le |r| \le a, \\ 0, & \text{otherwise}, \end{cases} \tag{33}$$

with

$$g(r) = \frac{\pi}{2}\left(\frac{r-b}{a-b}\right)^2, \tag{34}$$

$$h(r) = \sin[2g(r)]. \tag{35}$$

This source satisfies Eqs. (30) and (29) simultaneously within the sphere $|r| < b$, implying that it has simultaneously zero **E** and **H** in that region, but only satisfies Eq. (30) in the intermediate region, making it a null-**H** source in that region and radiationless overall. The resulting polarization and magnetization are illustrated in Fig. 3.

This array of results is of some significance to the theory of invisibility through metamaterials. A primary source of polarization and magnetization is mathematically equivalent to a scattering object with spatially varying permittivity and permeability excited by an electromagnetic wave; the fields produced by the primary source are equivalent to the scattered fields produced by the interaction. A null-field radiationless source is then equivalent to a scattering object that produces no scattered field—it is invisible—and whose interior fields take on an exceedingly simple form. A null-**E** source, for instance, is equivalent to a scattering object whose interior electric field is exactly equal to the electric field of the illuminating wave. These results are potentially of use in recent studies of lossless open resonators, which use similar interference effects for light confinement [14,15].

It does not seem likely that such null-field scattering objects can be produced by the techniques of transformation optics. Null-field sources therefore suggest that the class of invisible and cloaked objects is broader than previously realized.

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