Designing directional cloaks from localized fields

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Designs of cloaking devices and more general invisible objects have primarily applied the techniques of transformation optics and scattering cancellation to derive material structures that achieve the desired effects. In this Letter, we note that it is also possible to construct a broader class of invisible objects directly from the defining wave equation. The technique is demonstrated in a scalar formalism with illustrative examples.

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Since the groundbreaking theoretical research on optical cloaking devices published in 2006 [1,2], there has been significant activity devoted to studying such devices and investigating their implications beyond concealment [3]. Because of the complexity of the original designs and the extreme optical properties in their construction, much work has focused on finding alternatives to ideal three-dimensional cloaks, usually sacrificing performance for simplicity. For instance, so-called "carpet cloaks" function with relatively small variations in refractive index by shielding an object on a surface from observation from above [4]. More recently, it has been shown [5,6] that directional cloaks, which hide an object from only a single direction of illumination, can be constructed with low anisotropy and no magnetic materials.

The design of cloaking devices has primarily been done with two broad techniques. The most prominent of these is transformation optics [7], in which a mathematical warping of space is reinterpreted as an optical material. Also popular is scattering cancellation [8], in which two or more layers or structures of a device are tuned to produce opposing scattered fields. Though effective, both of these techniques are complicated and do not represent the complete set of possible cloaked structures.

In this Letter, we show that a wide variety of directionally invisible objects, including directional cloaks, can be constructed directly and without approximation from the governing wave equation itself, subject to a number of boundary conditions. We apply this technique to the construction of invisible objects for scalar monochromatic waves, and demonstrate their validity using computational wave calculations. The implications of this method for cloak design, and its extension to vector electromagnetic fields and other types of devices, are discussed.

We begin by considering a system in which a localized inhomogeneous object of refractive index $n(\mathbf{r})$ is illuminated by a scalar monochromatic plane wave $U_i(\mathbf{r}) = U_0 \exp[ik\hat{\mathbf{s}}_0 \cdot \mathbf{r}]$ propagating in the direction $\hat{\mathbf{s}}_0$, as illustrated in Fig. 1. The total field $U(\mathbf{r})$ satisfies the Helmholtz equation with an inhomogeneous wavenumber,

$$[\nabla^2 + n^2(\mathbf{r})k^2]U(\mathbf{r}) = 0, \tag{1}$$

where $k = \omega/c$, ω being the frequency and c the vacuum speed of light. As is typically done in scattering theory ([9], Section 13.1), we may introduce a scattering potential, defined as

$$F(\mathbf{r}) = \frac{k^2}{4\pi}[n^2(\mathbf{r}) - 1], \tag{2}$$

and with this rewrite the Helmholtz equation in the inhomogeneous form,

$$[\nabla^2 + k^2]U(\mathbf{r}) = -4\pi F(\mathbf{r})U(\mathbf{r}). \tag{3}$$

To simplify this problem, we express the total field as a sum of the incident field $U_i(\mathbf{r})$ and the scattered field $U_s(\mathbf{r})$. The incident field satisfies the homogeneous Helmholtz equation,

$$[\nabla^2 + k^2]U_i(\mathbf{r}) = 0, \tag{4}$$

which immediately leads us to the standard wave equation for the scattered field,

$$[\nabla^2 + k^2]U_s(\mathbf{r}) = -4\pi F(\mathbf{r})U(\mathbf{r}). \tag{5}$$

This differential equation is in general difficult to solve because of the presence of the scattered field on both sides. It is, however, quite convenient to apply in the design of invisible objects.

We look in particular for objects that are invisible with respect to the particular direction of illumination $\hat{\mathbf{s}}_0$. We first note that Eq. (5) is structurally similar to the wave equation for the field $V(\mathbf{r})$ produced by a scalar primary radiation source $q(\mathbf{r})$,

$$[\nabla^2 + k^2]V(\mathbf{r}) = -4\pi q(\mathbf{r}). \tag{6}$$

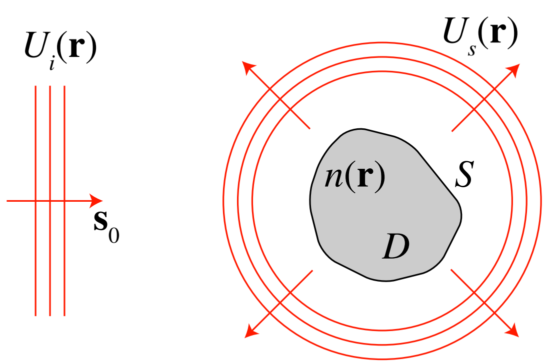

Fig. 1. Illustrating the notation related to scalar scattering.

We can therefore take advantage of techniques used to construct so-called *nonradiating sources*, time-harmonic primary sources that produce no field outside their domain of excitation [10], to design an object that produces no external scattered field. It has been shown [11,12] that a necessary and sufficient condition for a scalar field $V(\mathbf{r})$, localized to a domain D bounded by a surface S, to represent a nonradiating distribution is for $V(\mathbf{r})$ and $\nabla V(\mathbf{r})$ to be continuous throughout the domain and for

$$V(\mathbf{r})|_S = 0, \qquad \left.\frac{\partial V(\mathbf{r})}{\partial n}\right|_S = 0, \tag{7}$$

where $\partial/\partial n$ represents the derivative normal to the surface. In other words, the field and its normal derivative must go smoothly to zero at the boundary of the domain D. If any distribution satisfying these conditions is found, the source $q(\mathbf{r})$ producing the radiationless field can be derived from Eq. (6).

We can apply the same reasoning to derive the exact scattering potential of a nonscattering scatterer. Let us express the scattered field as

$$U_s(\mathbf{r}) = U_i(\mathbf{r})U_{\mathrm{loc}}(\mathbf{r}), \tag{8}$$

where $U_{\mathrm{loc}}(\mathbf{r})$ is a field constructed to satisfy both of Eqs. (7). By analogy with Eq. (6), this implies that the scattered field will be localized and the scattering object will be invisible. With our choice of U_s, the total field in the scattering case may be written as

$$U(\mathbf{r}) = [1 + U_{\mathrm{loc}}(\mathbf{r})]U_i(\mathbf{r}), \tag{9}$$

and on substitution into Eq. (5) we may directly solve for the scattering potential,

$$F(\mathbf{r}) = -\frac{1}{4\pi}\frac{\nabla^2 U_{\mathrm{loc}}(\mathbf{r}) + 2ik\hat{\mathbf{s}}_0\cdot\nabla U_{\mathrm{loc}}(\mathbf{r})}{1 + U_{\mathrm{loc}}(\mathbf{r})}. \tag{10}$$

This expression, together with Eq. (7) applied to U_s, is the main result of this Letter. It demonstrates that we can design a scalar scattering potential that produces a localized scattered field for a given direction of incidence $\hat{\mathbf{s}}_0$; the object is invisible to illumination from that direction. The scattering potential will be finite provided that $U_{\mathrm{loc}}(\mathbf{r}) \neq -1$ anywhere within the domain D, a condition easily avoided.

A few observations can be made about this method before providing examples. First, it is to be noted that, because Eqs. (7) are necessary and sufficient conditions for a localized field, all possible invisible objects for the scalar wave equation are encompassed by the method, including any that are nonscattering for multiple directions of incidence. However, how easy it is to find the corresponding field U_{loc} remains an open question.

It is also important to note that the field and potential relationship defined by Eq. (10) is *exact*, and does not use any approximation such as the Born approximation. It should also be noted that the separation given in Eq. (8) is not essential, but is convenient to avoid constructing potentials with rapid variations on the scale of a wavelength.

Finally, we note that equation (10) will generally result in a complex potential, and the potential will therefore typically have regions with both positive and negative real and imaginary parts. This suggests that the most general directional cloaks for the scalar wave equation will incorporate both gain and loss, and will also have regions where the refractive index, related to the potential by Eq. (10), is less than unity. The presence of gain and loss in invisible structures is strikingly similar to the directional invisibility observed recently in PT-symmetric structures [13,14] and, in fact, gain-loss structures have been studied in recent years both with transformation optics [15] and with scattering cancellation [16].

Gain and loss add significant complexity to the structure of an invisible object, and it is desirable to reduce or eliminate them entirely. If we consider a complex field $U_{\mathrm{loc}}(\mathbf{r})$ in Eq. (10), we can readily derive a differential condition for the imaginary part of $F(\mathbf{r})$ and, hence $n(\mathbf{r})$, to vanish. Assuming $\mathbf{s}_0 = \hat{\mathbf{x}}$, we have

$$\begin{aligned}&[(1 + U_R)\nabla^2 U_I - U_I\nabla^2 U_R + 2k[U_I\partial_x U_I\\ &\quad + (1 + U_R)\partial_x U_R] = 0,\end{aligned} \tag{11}$$

where we have written $U_{\mathrm{loc}} = U_R + iU_I$ and $\partial_x \equiv \partial/\partial x$. This equation is nontrivial, and it is not immediately obvious that it even has a solution. However, it provides a clear and general condition for the existence of real refractive index invisibility objects.

We now consider a pair of examples, one a simple directional invisible object and the second a directional cloak. For simplicity, we take the wavelength $\lambda = 1$ and work with two-dimensional fields and scatterers, with $\mathbf{r} = (x, y)$. We design the scatterers to be nonscattering for an incident field in the x-direction, so that $U_s(\mathbf{r}) = U_{\mathrm{loc}}(\mathbf{r})\exp[ikx]$, and choose rotationally symmetric U_{loc} for simplicity. It is to be noted that a rotationally symmetric U_{loc} cannot satisfy Eq. (11) because of the presence of the x-derivative.

For the first example, we construct a scatterer localized to a circle of radius $r = 1$, and choose

$$U_{\mathrm{loc}}(\mathbf{r}) = \left[-\frac{1}{2} + r^2 - \frac{1}{2}r^4\right] + i\left[-\frac{1}{2} + r^2 - \frac{1}{2}r^4\right]. \tag{12}$$

This particular field has been tailored to reduce the imaginary part of $F(\mathbf{r})$, with $-0.6 \le \mathrm{Im}\{F\} \le 0.3$, and the field can readily be shown to satisfy Eqs. (7).

The scattered field is calculated numerically using a Green's function technique, analogous to ones used in electromagnetic theory [17]. Figure 2(a) shows the field illuminated by a plane wave propagating in the x-direction, while Fig. 2(b) shows the field illuminated by a plane wave propagating 45° to the x-axis. It can be seen that the object is nonscattering in the former case, and significantly scattering in the latter.

We can go further and design a directional cloaking device by implementing an additional boundary condition. If we require the scattered field on the boundary of a centralized cloaking region to satisfy the constraint $U_s = -U_i$, the scattered field within will completely

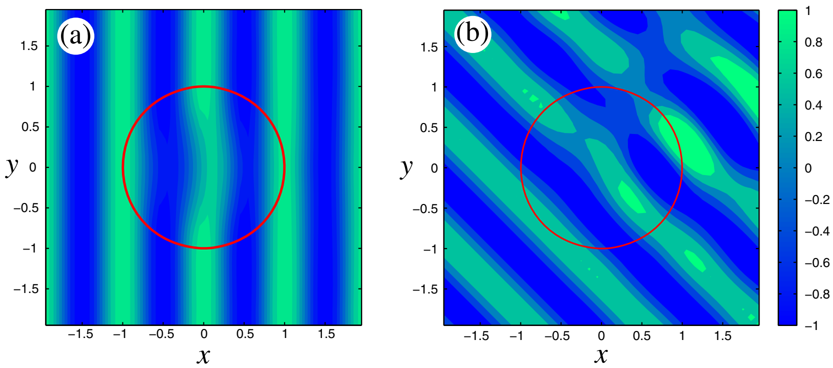

Fig. 2. Total field produced on scattering from the object defined by the field of Eq. (12), for an incident wave propagating (a) in the x-direction, and (b) along the diagonal between x and y.

cancel the incident field. It is to be noted that this condition results in the scattering potential becoming singular on this boundary, as discussed earlier, analogous to the singular behavior that appears in the original three-dimensional electromagnetic cloaks [2]. We consider an object with inner cloaked region $r < 1/2$ and an outer boundary $r = 1$, and use the following as our localized field:

$$U_{\mathrm{loc}}(\mathbf{r}) = -\frac{16}{27} - \frac{32}{9}r^2 + \frac{80}{9}r^4 - \frac{128}{27}r^6. \qquad (13)$$

This U_{loc} satisfies Eqs. (7) at $r = 1$, as well as the aforementioned cloaking constraint at $r = 1/2$.

The fields for a pair of plane waves are shown in Fig. 3, again calculated with a Green's function technique. It can be seen that the device is, in fact, acting as a directional cloak, completely excluding fields (within numerical accuracy) from the cloaked region for the appropriate direction of illumination.

It is to be noted that this technique can be easily modified to study "nearly perfect" cloaks which possess no singular behavior. For example, a "95% perfect" cloak could be constructed by changing the inner boundary condition to $U_s = -0.95U_i$.

These simple examples illustrate that directionally invisible objects can be readily constructed for scalar wavefields directly from the governing wave equation. An analogous method can be used to construct directional electromagnetic cloaks, starting instead with the electromagnetic wave equation,

$$\nabla \times (\nabla \times \mathbf{E}_s) - k^2\mathbf{E}_s = 4\pi\mathbf{F}\cdot\mathbf{E}, \qquad (14)$$

where $\mathbf{F}$ is the scattering dyadic, defined as

$$\mathbf{F} = \frac{k^2}{4\pi}[\epsilon/\epsilon_0 - \mathbf{I}], \qquad (15)$$

ϵ is the (generally anisotropic) permittivity, and $\mathbf{I}$ is the identity dyadic. The equation is derived from the monochromatic forms of Faraday's law and the Maxwell-Ampére law,

$$\nabla \times \mathbf{E} = i\omega\mathbf{B}, \qquad (16)$$

$$\nabla \times \mathbf{B} = -i\omega\mu_0\epsilon\cdot\mathbf{E}, \qquad (17)$$

assuming that $\mu = \mu_0$ for the scatterer. Following arguments similar to the scalar case and requiring all components of the electric field to vanish smoothly at the boundary of the scatterer, we can construct an expression for the scattering potential; this potential must in general be anisotropic because the double curl of Eq. (14) mixes the electric field components.

For example, if the illuminating field is a plane wave propagating in the z-direction with polarization in the x-direction and the scattered field is assumed to have only an x-component as well, three elements of the scattering dyadic are determined from the wave equation,

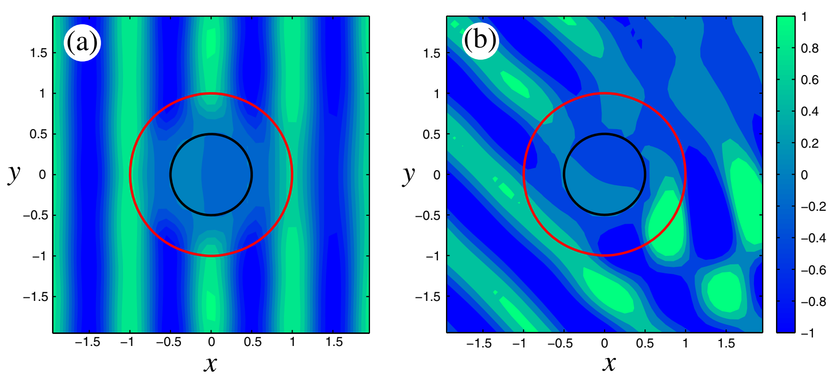

Fig. 3. Total field produced on scattering from the object defined by the field of Eq. (13), for an incident wave propagating (a) in the x-direction, and (b) along the diagonal between x and y.

$$F_{xx} = \frac{[\partial_x^2 - \nabla^2 - k^2]E_x^s}{4\pi[E_x^i + E_x^s]}, \tag{18}$$

$$F_{yx} = \frac{\partial_x \partial_y E_x^s}{4\pi[E_x^i + E_x^s]}, \tag{19}$$

$$F_{zx} = \frac{\partial_x \partial_z E_x^s}{4\pi[E_x^i + E_x^s]}, \tag{20}$$

where E_x^i and E_x^s are the incident and scattered electric fields, respectively. The other elements of **F** are only constrained by the physical properties of the scattering dyadic.

The potential power in deriving invisible objects directly from wave equations comes from the freedom of choice in designing the localized field. In additional to designing single directional fields, it should be possible to make scatterers that are nonscattering for N multiple directions of incidence by tailoring the resulting scatterer to be N-fold symmetric. Such objects would be more general versions of the "nonscattering scatterers," introduced some time ago in the context of weak scattering [18]. In the electromagnetic case, an appropriate tailoring of all three localized field components could potentially be used to construct an isotropic directional scatterer, or to reproduce the original anisotropic all-directions cloak. Though finding the appropriate symmetries of the localized fields might be challenging, determining the potential from these fields involves a straightforward derivative relation.

As all invisible objects must satisfy the governing wave equations, the discovery of new invisible and cloaking structures is limited only by the ingenuity of the designer.

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