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Hilbert's Hotel in polarization singularities

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We demonstrate theoretically how the creation of polarization singularities by the evolution of a fractional nonuniform polarization optical element involves the peculiar mathematics of countably infinite sets in the form of "Hilbert's Hotel." Two distinct topological processes can be observed, depending on the structure of the fractional optical element.

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Over the past several years, a number of papers have come out that have demonstrated curious connections between optical vortices and the mathematical concept of Hilbert's Hotel, which illustrates the strange behavior of countably infinite sets. Hilbert's Hotel, originally attributed to David Hilbert in a 1924 lecture, but popularized by George Gamow some years later [1], imagines a hotel with an infinite number of rooms, all occupied, and labeled with the natural numbers. Though the hotel is completely booked, it is always possible to make a vacancy by asking each guest to move to the next room higher. Any finite number of vacancies M may be created by asking each guest to move M rooms up, i.e., the Nth guest performs the move $N \rightarrow N + M$, and a countably infinite number of vacancies may be created, for example, by asking the guest in the Nth room to move to the $2N$th room, i.e., $N \rightarrow 2N$. Therefore, Hilbert's Hotel is simultaneously completely occupied and infinitely vacant.

One would intuitively expect that such strange behaviors would have no place in experimental reality, but it has now been demonstrated that systems with optical vortices can manifest realizations of Hilbert's Hotel in a number of different ways, as described in detail further below. An optical vortex is a line in three-dimensional space on which the intensity of a wavefield is zero and the phase is consequently singular; it is well-known that the phase takes on a circulating or helical form around such a line. The phase change in a counterclockwise path around a vortex is necessarily an integer multiple of 2π, this integer being known as the *topological charge*, and the topological charge is generally a conserved quantity under the effect of any perturbations to the wavefield. Since the properties of vortices were first elucidated by Nye and Berry [2], the study of singularities in wavefields of all types has become its own subfield of optics known as *singular optics* [3].

The discrete nature of optical vortices evidently makes them natural structures to manifest Hilbert's Hotel. Potoček *et al.* [4] have demonstrated that a system that is used to spatially separate vortex modes in a beam also acts physically like the $N \rightarrow 2N$ version of Hilbert's Hotel. Soon after, building on original calculations by Berry [5], Gbur [6] noted that the creation of integer vortices on transmission through a fractional vortex phase plate reproduces almost perfectly the example of Hilbert's Hotel with a finite number of rooms.

Optical vortices typically manifest only in scalar wavefields or in electromagnetic wavefields, where the state of polarization is uniform throughout the entire transverse cross section of the field. When the state of polarization is nonuniform, the typical singularities are not of phase, but of the vector state of the electric field. It was shown by Nye [7] that the most common singularities in a transverse plane of a paraxial field are L-lines, which separate regions of left-handed and right-handed elliptical polarization from each other, and C-points, points of left- and right-handed circular polarization. The latter points are singularities of the direction of the major axis of the polarization ellipse, and the direction of the ellipse must rotate by a half-integer multiple of 2π in a clockwise path around the singularity to maintain the continuity of the field; this integer is known in this case as the topological index n. A detailed review of polarization singularities can be found in Ref. [8].

It is natural to ask whether polarization singularities can manifest Hilbert Hotel-like behaviors in analogy with those that appear for optical vortices, and whether there are any new topological effects that arise. In this Letter, we consider theoretically the manifestation of Hilbert's Hotel when fields pass through a pair of distinct fractional nonuniform polarization optical elements. The behavior of the system depends significantly on the type of element used, and a new Hotel behavior is shown. The results further illustrate how singular optics can be used as a laboratory to study the behavior of countably infinite sets.

The simplest fractional nonuniform polarization element that can be imagined is one that, based on a fractional parameter α, goes from uniform horizontal polarization for $\alpha = 0$ to radial polarization for $\alpha = 1$. As the topological index of a radially polarized beam is $n = 1$, one can envision the parameter α as the fractional topological index. It is assumed that an incident field of uniform horizontal polarization is incident upon the element, and is perfectly transmitted through the system, but undergoes a rotation of polarization $\alpha\phi$, where ϕ is the azimuthal angle in the transverse plane. Therefore, we may express the output field immediately after the element as

$$\mathbf{E}_\alpha(\rho, \phi) = \hat{\mathbf{x}} \cos(\alpha\phi) + \hat{\mathbf{y}} \sin(\alpha\phi). \tag{1}$$

We may readily write the sine and cosine in this expression in their exponential forms, i.e.,

$$\mathbf{E}_\alpha(\rho, \phi) = \left[\frac{\exp[i\alpha\phi] + \exp[-i\alpha\phi]}{2}\right]\hat{\mathbf{x}} + \left[\frac{\exp[i\alpha\phi] - \exp[-i\alpha\phi]}{2i}\right]\hat{\mathbf{y}}. \tag{2}$$

The advantage in doing this is that we may then apply formulas originally derived by Berry [5] to determine the behavior of such a field on propagation. The strategy is to first expand the complex exponentials, which are multi-valued for fractional α in terms of their single-valued Fourier series components, i.e.,

$$\exp[\pm i\alpha\phi] = \frac{\exp[\pm i\pi\alpha]\sin(\pm\pi\alpha)}{\pi}\sum_{m=-\infty}^{\infty}\frac{\exp[im\phi]}{\pm\alpha - m}. \tag{3}$$

This is a decomposition of a "fractional" vortex field into an infinite series of integer vortex fields. Then it can be shown that each integer vortex field will have the following form on propagation:

$$U_{\pm m}(\rho, \phi, z) = \exp[ikz \pm im\phi + ik\rho^2/4z]\sqrt{\frac{\pi}{8}}(-i)^{m/2}\sqrt{\frac{k\rho^2}{z}} \times [J_{(m-1)/2}(k\rho^2/4z) - iJ_{(m+1)/2}(k\rho^2/4z)]. \tag{4}$$

Therefore, a fractional vortex field which arises from a source field of $\exp[\pm i\alpha\phi]$ may be written as

$$U_{\pm\alpha}(\rho, \phi, z) = \pm\frac{\exp[\pm i\pi\alpha]\sin\pi\alpha}{\pi}\sum_{m=-\infty}^{\infty}\frac{U_m(\rho, \phi, z)}{\pm\alpha - m}. \tag{5}$$

The evolution of the electric field given by Eq. (2) may be determined by assuming that each exponential in that expression evolves according to Eq. (5). On substitution, and the use of some algebraic manipulations, one may write that

$$\mathbf{E}_\alpha(\rho, \phi, z) = \frac{\sin(\pi\alpha)}{\pi}\sum_{m=-\infty}^{\infty}\frac{V_m(\rho, \phi, z)}{\alpha - m} \times [\cos(\pi\alpha + m\phi)\hat{\mathbf{x}} + \sin(\pi\alpha + m\phi)\hat{\mathbf{y}}], \tag{6}$$

where we have defined $V_m \equiv \exp[-im\phi]U_m$ for convenience. With this expression, we can evaluate how the $n = 1$ singularity at the center of a radially polarized field is created when the fractional parameter α is increased continuously from $\alpha = 0$ to $\alpha = 1$.

Typically, plots showing a nonuniformly polarized field will show either a field of polarization ellipses or the major axes of those ellipses. However, this depiction makes it difficult to see individual singularities when many are present. Instead, we plot the angle of the major axis with color, with a range $-\pi/2 \leq \theta \leq \pi/2$. Figure 1 illustrates the typical polarization

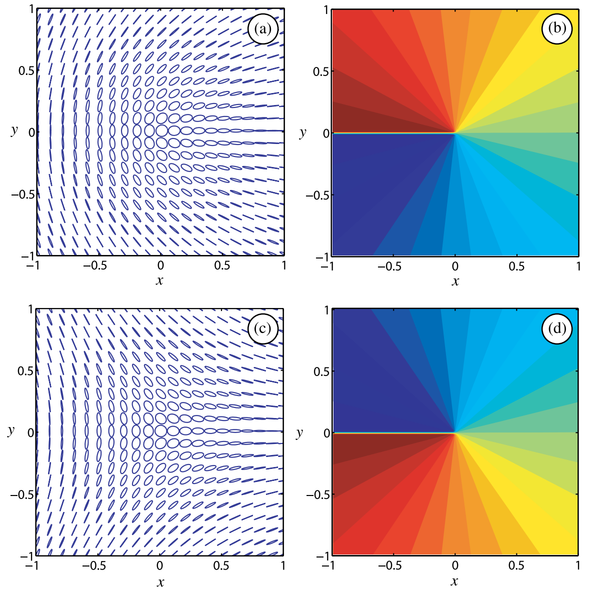

Fig. 1. Illustration of the state of polarization around (a), (b) a lemon and (c), (d) a star. Blue represents $\theta = -\pi/2$, while red represents $\theta = +\pi/2$.

singularities, a lemon (topological index $n = +1/2$) and a star (topological index $n = -1/2$), both with polarization ellipses and colors. The point at which all color contours converge is a point of circular polarization and a singularity of polarization ellipse major axis direction.

We now evaluate the behavior of our fractional nonuniform polarization element as the fractional angle α is increased. By analogy with the vortex realization of Hilbert's Hotel, one might intuitively expect that two Hotel creation processes will occur, one near $\alpha = 1/2$, resulting in the creation of a lemon (the generic polarization singularity of index $n = 1/2$) and one near $\alpha = 1$, resulting in the creation of a second lemon that joins with the first to form radial polarization.

The actual process, however, is somewhat different, as illustrated in Fig. 2. As α approaches $\alpha = 1/2$, we find that we first get two pairs of lemons and stars, and then additional pairs of lemons and stars appear on either side of the horizontal axis. At $\alpha = 1/2$; we evidently have an infinite set of lemon-star pairs along that axis. Then, as α becomes greater than 1/2, each star annihilates with its neighbor on its right. The end result is two lemons, for a total topological index of $n = +1$; we never pass through a state where $n = +1/2$.

Therefore, the system acts as an $N \to N + 2$ version of Hilbert's Hotel, creating two "vacancies" (or, in our system, polarization singularities) through the use of infinite mathematics. The conservation law for a topological index, which indicates that new polarization singularities must be created in the pairs of opposite index, is skirted by passing through a state where an infinite number of singularities exist.

The absence of a pure $n = +1/2$ state for our system can be traced to the fact that, at $\alpha = +1/2$, the phase of the electric field in Eq. (1) is discontinuous by π at the $\phi = 0, 2\pi$ boundary. This discontinuity evolves on propagation as a rapid additional rotation of the major axis by π, making the total topological index $n = +1$.

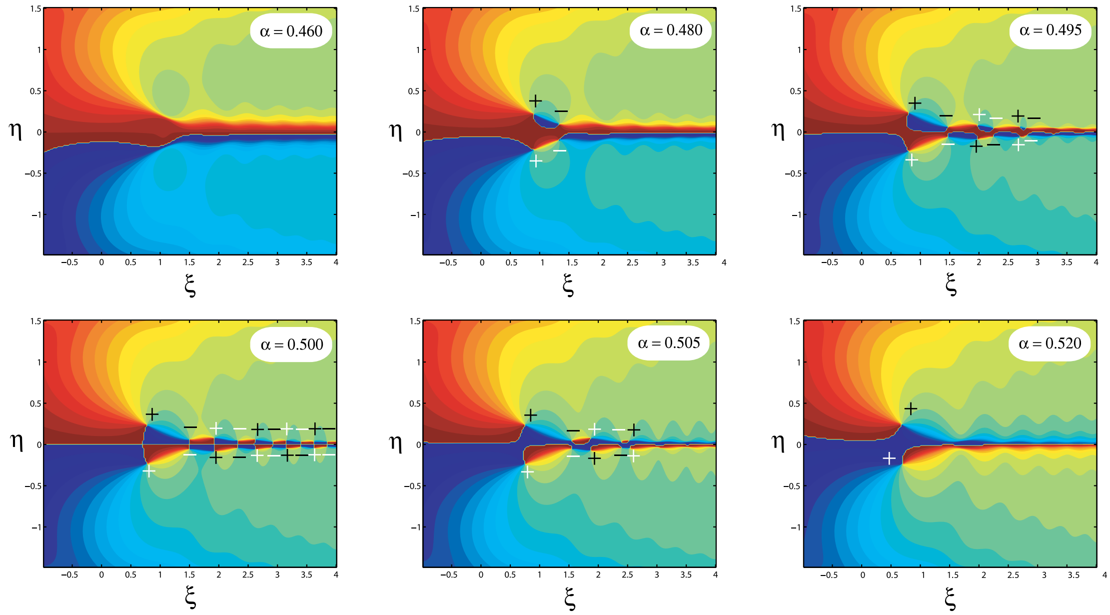

Fig. 2. Illustration of the evolution of the polarization singularities as a function of α. Here $\xi = x\sqrt{k/4z}$ and $\eta = y\sqrt{k/4z}$. The pairs of singularities created together are designated by white or black.

The existence of an $N \to N + M$ Hilbert Hotel in vortex fields was demonstrated in Ref. [6], but there it was created by making the fractional phase plate M-fold symmetric. Here the $N \to N + 2$ Hotel arises from the choice of topology of the phase plate. We can create a simpler $N \to N + 1$ Hotel by a different fractional nonuniform polarization element which properly emulates a lemon singularity. One local form of the electric field for a simple lemon singularity may be written as (Section 7.5 [3])

$$\begin{aligned}\mathbf{E}(x, y) &= (\hat{\mathbf{x}} + i\hat{\mathbf{y}})(x - iy) + (\hat{\mathbf{x}} - i\hat{\mathbf{y}}) \\ &= \hat{\mathbf{x}}[\rho \exp[-i\phi] + 1] + i\hat{\mathbf{y}}[\rho \exp[-i\phi] - 1], \end{aligned} \quad (7)$$

where we have written the expression in polar coordinates in the latter case. To envision this as the effect of a fractional nonuniform polarization element on an incident plane wave, we simply remove the radial amplitude dependence ρ and add α to the phase exponentials, so that

$$\mathbf{E}_\alpha(\rho, \phi) = \hat{\mathbf{x}}[\exp[-i\alpha\phi] + 1] + i\hat{\mathbf{y}}[\exp[-i\alpha\phi] - 1]. \quad (8)$$

With the field in exponential form, we may again apply Eqs. (3)–(5) to determine the behavior of the field on propagation. The evolution as a function of α is shown in Fig. 3 around the value $\alpha = 1/2$. It is to be noted that $\alpha = 1$ in this case implies topological index 1/2, where, in the previous case, $\alpha = 1$ implies topological index 1.

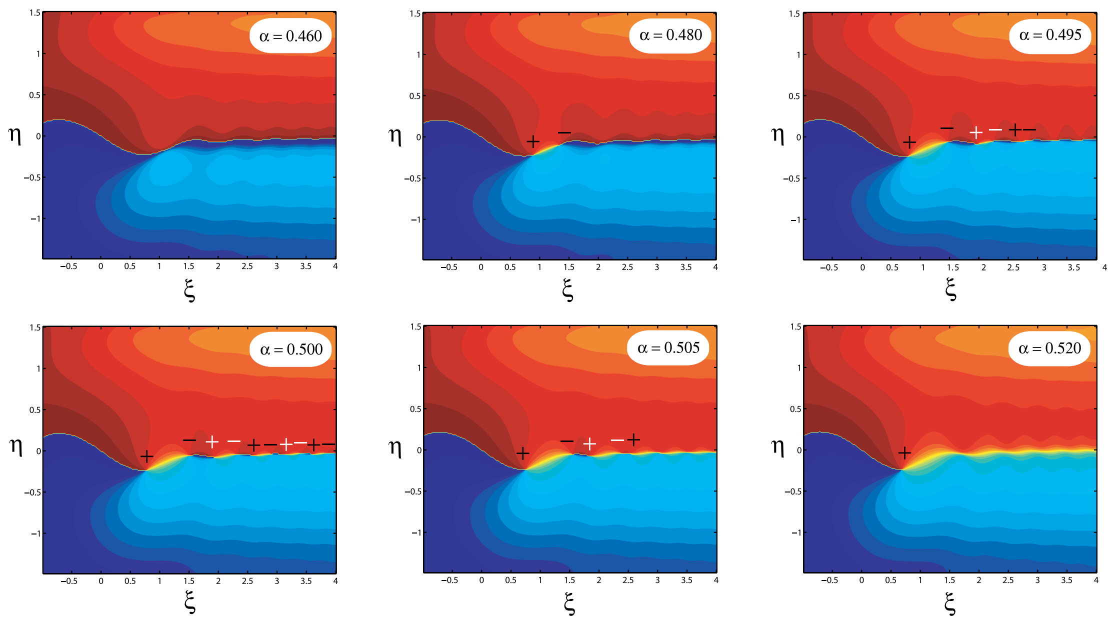

Fig. 3. Illustration of the evolution of the polarization singularities as a function of α. Here $\xi = x\sqrt{k/4z}$ and $\eta = y\sqrt{k/4z}$.

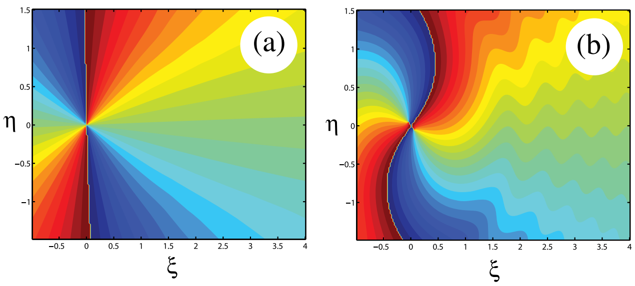

Fig. 4. Polarization of the field for (a) the radial polarization element at $\alpha = 0.99$ and (b) the lemon polarization element at $\alpha = 1.99$.

We now see an "ordinary" Hilbert Hotel $N \to N + 1$ evolution, with the result being a single lemon created through the annihilation that proceeds after $\alpha > 0.5$.

What is striking in the different topological processes represented by the optical elements of Eqs. (2) and (8) is that both result in a radially polarized field from a uniformly polarized field through the change of a continuous parameter α. In the latter case, however, one jumps first to a pure lemon state of index $n = +1/2$ while, in the former case, one jumps directly to index $n = +1$. The end result is topologically the same, as illustrated in Fig. 4.

It is of interest to note the behavior of L-lines in both of these cases. In the system of Eq. (2), a single L-line lies along the horizontal axis for $\alpha = 0.5$, making all the C-points above the axis left-handed and all those below right-handed. (It is important to recognize that the handedness of the polarization at a C-point is independent of the handedness of the polarization singularity, represented by the topological index.) In the system of Eq. (8), an L-line envelopes all the C-points produced, making them all left-handed.

Two further points are worth emphasizing here. First, it is to be noted that this ideal Hilbert Hotel behavior will only exist in a wavefield which is infinite in transverse extent; however, as noted in Ref. [6], in the case of vortices, the "signature" of such events even appears in finite beams, though the infinite line of singularities is lost in the darkness of the beam's edge. In fact, the original Hotel-like behavior was observed experimentally some years ago [9–11], though it was not recognized as such at the time. Secondly, though the fractional nonuniform polarization elements described here are only theoretical as yet it should be possible to fabricate them using Pancharatnam–Berry phase optical elements, which exhibit both phase and polarization sensitivity [12,13].

These examples show that the Hilbert Hotel behavior that was previously demonstrated for optical vortices in the creation of singularities also applies to polarization singularities and suggests that systems with other discrete topological defects will show similar Hotel behavior. The two distinct ways to arrive at radial polarization indicates that there is significant variation in the behavior of polarization sensitive optical elements, and may be useful in designing novel devices, such as ones that can switch rapidly from one polarization singularity state to another.

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