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Greg Gbur, "Partially coherent vortex beams," Proc. SPIE 10549, Complex Light and Optical Forces XII, 1054903 (22 February 2018); doi: 10.1117/12.2287170

SPIE.

Event: SPIE OPTO, 2018, San Francisco, California, United States

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ABSTRACT

Beams with structured phase and beams with partial spatial coherence have both been considered for a variety of applications for many years, and it would seem natural to combine their advantageous properties. However, there is a conceptual difficulty in doing so because partially coherent beams lack a definite phase, which would seem to rule out a direct synthesis. Nevertheless, research over the past decade has clarified the relationship between phase singularities of coherent fields and correlation singularities of two-point correlation functions, showing that coherent vortices survive the disruption of spatial coherence. In this presentation we discuss a number of theoretical models of what we refer to as "partially coherent vortex beams," considering their relative merits.

Keywords: Coherence, singular optics, optical vortex

1. INTRODUCTION

Over the past decade, the rapid growth of the field of singular optics[1] has led to an increased interest in applying the unique properties of beams possessing optical vortices in a variety of technologies. Such beams include the familiar Laguerre-Gauss (LG) modes; the field of such beams may be written in the waist plane $z = 0$ as

$$U_{nm}(\mathbf{r}) = \sqrt{\frac{2n!}{\pi w_0^2 (n+|m|)!}} \left(\frac{\sqrt{2}}{w_0}\right)^{|m|} L_n^{|m|}\left(\frac{2r^2}{w_0^2}\right) (x \pm iy)^{|m|} e^{-r^2/w_0^2}, \tag{1}$$

where w_0 is the RMS width in the waist plane, $\mathbf{r} = (x, y)$, $r = \sqrt{x^2 + y^2}$, L_n^m are the associated Laguerre polynomials, and the $\pm$ depends on the sign of m. The $(x \pm iy)^{|m|}$ term represents an intensity null of the field on the z-axis, on which the phase is undefined, or singular, and around which it has a helical structure. This can be seen clearly by writing

$$(x \pm iy)^{|m|} = r^{|m|} e^{im\phi}, \tag{2}$$

where ϕ is the azimuthal angle.

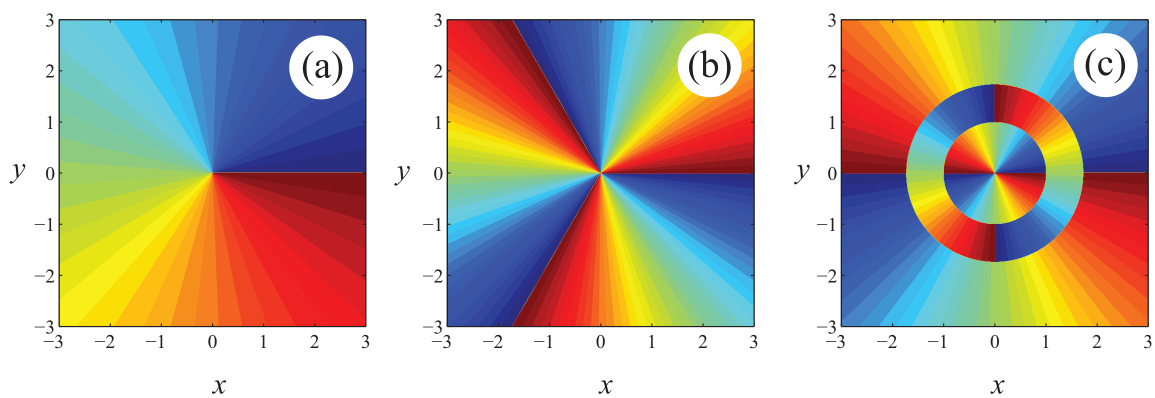

Figure 1. The phase of three Laguerre-Gauss modes, of order (a) $n = 0$, $m = 1$, (b) $n = 0$, $m = -3$, (c) $n = 2$, $m = 2$.

The index m is often referred to as the order, or the *topological charge*, of the vortex, and it is always an integer value. This is due to the requirement that the field be continuous and single-valued at every point in

Complex Light and Optical Forces XII, edited by Enrique J. Galvez, David L. Andrews,
Jesper Glückstad, Proc. of SPIE Vol. 10549, 1054903 · © 2018 SPIE
CCC code: 0277-786X/18/$18 · doi: 10.1117/12.2287170

space, which requires the phase to increase or decrease by integer multiples of 2π in closed paths around the vortex core. Several examples of the phase of Laguerre-Gauss modes are shown in Fig. 1. Furthermore, it can be shown that this topological charge is a conserved quantity, and resistant to amplitude and phase perturbations of the field. Furthermore, vortices can only be created or annihilated in pairs of equal and opposite charge, except under certain very unusual circumstances.[2]

With these two properties – discreteness and robustness – many researchers have studied the use of optical vortices as information carriers in free-space optical communications systems.[3–5] Since the different vortex modes are orthogonal to each other, it is in principle possible to transmit information simultaneously using different vortex orders through the same propagation channel, allowing a very significant increase in the data transmission rate. The conservation of topological charge furthermore suggests that it will be robust to atmospheric turbulence. However, such turbulence can nevertheless degrade the performance of such systems, by causing the vortex to wander out of the detector area or by creating pairs of additional vortices through self-interference.[6]

Turbulence resistance has, however, been achieved to some degree by using fields with a reduced spatial coherence.[7] The decrease in spatial coherence reduces the self-interference effects of the beam, in turn reducing the strength of intensity fluctuations (scintillations) that the beam experiences.

It is natural, then, to ask if it is possible to improve the propagation characteristics of vortex beams by reducing their spatial coherence. It has recently been demonstrated that an entire class of partially coherent vortex beams (PCVBs) can be modeled as a simple randomization of their coherent counterparts.[8] But this is not the only class of partially coherent beams with a circulating structure that exist. Based on the orbital angular momentum properties of the beams, we can identify three "pure" classes of PCVBs. It is the properties of these classes that this paper is concerned with.

We begin in Section 2 by talking about how vortices manifest in partially coherent fields, and the definition of orbital angular momentum in such cases. Then we introduce the three classes of pure PCVBs in Section 3; in Section 4 we give some observations and thoughts about future research.

2. SINGULARITIES IN PARTIALLY COHERENT WAVEFIELDS

In attempting to study the singular optics of partially coherent fields, one immediately runs into a conceptual problem. An optical vortex has a well-defined phase structure, but partially coherent fields, due to their nature as randomly fluctuating in space and time, do not. To describe a partially coherent field at a frequency ω, one cannot use a monochromatic field $U(\mathbf{r})$, but must use the cross-spectral density $W(\mathbf{r}_1, \mathbf{r}_2)$, which describes correlations between two points of a field at frequency ω. This cross-spectral density may be defined[9] as an average $\langle \cdots \rangle_\omega$ over an ensemble of monochromatic realizations of the field, as

$$W(\mathbf{r}_1, \mathbf{r}_2) = \langle U^*(\mathbf{r}_1) U(\mathbf{r}_2) \rangle_\omega. \tag{3}$$

The cross-spectral density, unlike the field itself, has a well-defined phase with respect to the two measurement points. Early investigations of the singular optics of partially coherent fields therefore focused on investigating phase singularities of the cross-spectral density, i.e. those pairs of points $\mathbf{r}_1$ and $\mathbf{r}_2$ for which $W(\mathbf{r}_1, \mathbf{r}_2) = 0$. Such *correlation singularities* were first predicted for Young's two-pinhole experiment;[10] the singularities found, however, were not typical, or generic, singularities of partially coherent fields. Almost immediately after this first paper, others were published that considered the generic case.[11–14] A consistent picture emerged from these studies: as the spatial coherence of a field is decreased, any vortices that are present evolve from being vortices of the field to being vortices of the correlation function. These vortices can be seen by fixing one observation point, say $\mathbf{r}_1$, and looking at the phase of the cross-spectral density with respect to $\mathbf{r}_2$. Because the location of the singularities depend on the position of the observation point, one may loosely say that they are nonlocalized singularities.

Because of this nonlocal nature, the topological charge of any PCVB can evidently only be determined by an interference experiment. It is likely more convenient, then, to instead consider and study the orbital angular momentum of the beam, which can be measured directly by means of optical elements that perform a geometric transformation of the beam, converting an azimuthal phase twist of the beam into a linear displacement of the beam.[15–17] For a partially coherent paraxial beam, the orbital angular momentum flux density along the z-axis may be written as[18]

$$M_{orbit}(x,y) = -\frac{\epsilon_0}{2k}\mathrm{Im}\left\{y\partial'_x W_{yy}(\mathbf{r},\mathbf{r}',\omega) - x\partial'_y W_{xx}(\mathbf{r},\mathbf{r}',\omega) - x\partial'_y W_{yy}(\mathbf{r},\mathbf{r}',\omega) + y\partial'_x W_{xx}(\mathbf{r},\mathbf{r}',\omega)\right\}_{\mathbf{r}=\mathbf{r}'}, \quad (4)$$

where ∂'_x represents the partial derivative with respect to x', and so forth. If the field is taken to be unpolarized, so that $W_{xx} = W_{yy} = W$, this expression reduces to the form

$$M_{orb}(x,y) = -\frac{\epsilon_0}{k}\mathrm{Im}\left\{y\partial'_x W(\mathbf{r},\mathbf{r}',\omega) - x\partial'_y W(\mathbf{r},\mathbf{r}',\omega)\right\}. \quad (5)$$

In the next section, we will see that three classes of PCVBs may be distinguished, based on the spatial distribution of their orbital angular momentum flux density.

3. PARTIALLY COHERENT VORTEX BEAMS

We now consider each of the three classes of PCVBs in turn, starting with the most recently derived class which we will refer to as *Rankine vortex beams*. This terminology will be made clear momentarily.

3.1 RANKINE VORTEX MODEL

The simplest way to imagine generating a partially coherent vortex beam is to randomize a coherent vortex beam. The first method for analytically doing so was introduced[19] in 2004, and is referred to as the *beam wander model*. We begin with a fully coherent Laguerre-Gauss beam $U(\mathbf{r})$ and imagine that its central axis position $\mathbf{r}_0$ is allowed to wander, while remaining parallel to the z-axis. The probability that the beam axis is found at position $\mathbf{r}_0$ is defined as $f(\mathbf{r}_0)$; then the cross-spectral density, which is the ensemble average all positions, may be written as

$$W(\mathbf{r}_1,\mathbf{r}_2) = \int \tilde{U}(\mathbf{r}_1-\mathbf{r}_0)U(\mathbf{r}_2-\mathbf{r}_0)f(\mathbf{r}_0)d^2r_0, \quad (6)$$

where a tilde is used to represent complex conjugation and the integral is over the entire (x_0, y_0) plane. This model is roughly analogous to the manner in which an optical beam wanders on propagation through atmospheric turbulence.

For simplicity, we restrict our attention to Laguerre-Gauss modes of order $n = 0$, which we write as

$$U(x,y) = C(x \pm iy)^m \exp\left[-\frac{1}{w_0^2}(x^2+y^2)\right], \quad (7)$$

with

$$C \equiv \sqrt{\frac{2}{\pi w_0^2 |m|!}}\left(\frac{\sqrt{2}}{w_0}\right)^{|m|}. \quad (8)$$

We now assume that the function $f(\mathbf{r}_0)$ is of Gaussian form,

$$f(\mathbf{r}_0) = \frac{1}{\pi\delta^2}\exp\left[\frac{-(x_0^2+y_0^2)}{\delta^2}\right], \quad (9)$$

and as a probability density it has been normalized to unity when integrated over the entire transverse plane. The quantity δ, which is the RMS width of the wandering, may be considered an inverse measure of the spatial coherence: when $\delta = 0$, the beam does not wander and the field is fully coherent, and as δ increases, the coherence decreases.

With some significant manipulation,[8] this integral can be reduced to the form,

$$\begin{aligned} W(\mathbf{r}_1,\mathbf{r}_2) &= \pi D(\mathbf{r}_1,\mathbf{r}_2)\left\{\sum_{l=0}^{m-1}\binom{m}{l}^2\frac{\Gamma(l+1)}{A^{2m-l+1}}\right. \\ &\times \left[\frac{1}{\alpha^2}(x_2\pm iy_2)-\frac{1}{w_0^2}(x_1\pm iy_1)\right]^{m-l} \\ &\times \left[\frac{1}{\alpha^2}(x_1\mp iy_1)-\frac{1}{w_0^2}(x_2\mp iy_2)\right]^{m-l} \\ &+ \left.\frac{\Gamma(m+1)}{A^{m+1}}\right\}, \end{aligned} \tag{10}$$

where we have introduced the following notation,

$$A \equiv \frac{2}{w_0^2}+\frac{1}{\delta^2}, \tag{11}$$

$$\frac{1}{\alpha^2} \equiv \frac{1}{w_0^2}+\frac{1}{\delta^2}, \tag{12}$$

and

$$D(\mathbf{r}_1,\mathbf{r}_2)=\frac{|C|^2}{\pi\delta^2}\exp\left[-\frac{r_2^2}{Aw_0^2\delta^2}\right]\exp\left[-\frac{r_1^2}{Aw_0^2\delta^2}\right]\times\exp\left[-\frac{|\mathbf{r}_1-\mathbf{r}_2|^2}{Aw_0^4}\right]. \tag{13}$$

The function $D(\mathbf{r}_1,\mathbf{r}_2)$ represents the correlation function of a Gaussian Schell-model beam. The remaining spatial dependence of Eq. (10) therefore represents the vortex properties of the correlation function.

An example of the behavior of one of these PCVBs is shown in Fig. 2. As the coherence decreases, the single fourth-order vortex breaks into four first-order vortices that lie roughly along the line connecting the origin and the observation point $\mathbf{r}_1$. As the coherence decreases further, four first-order vortices of opposite sign approach the origin from infinity.

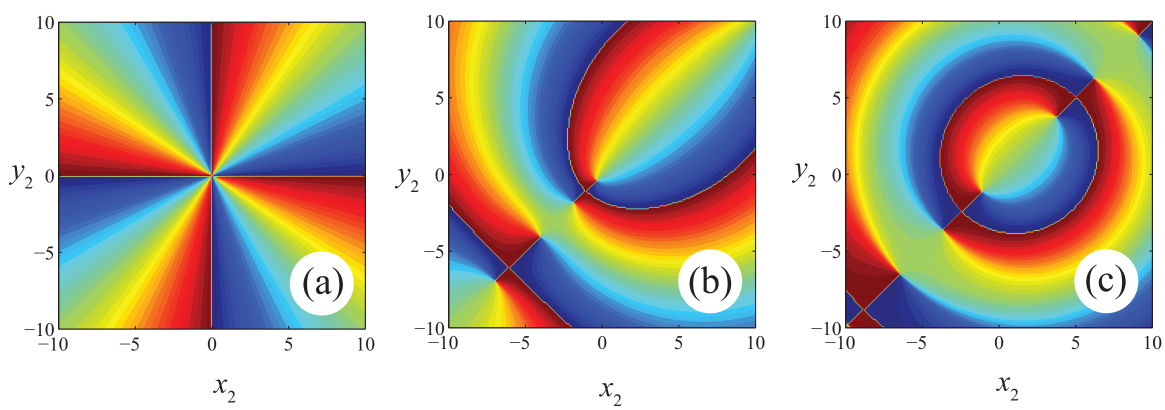

Figure 2. The phase of the cross-spectral density for a beam with $w_0 = 5$ mm, with (a) $\delta = 0.01$ mm, (b) $\delta = 0.5$ mm, (c) $\delta = 1.0$ mm.

There are two noteworthy observations here. First, the vortex properties of the correlation function are non-local, and depend upon the observation point. Second, the net topological charge of the beam, however it is measured, must decrease as the coherence decreases, due to the appearance of the conjugate vortices from infinity. This puts a practical limit on the use of vortices as information carriers in partially coherent beams.

A calculation of the OAM of the beam results in the expression,

$$M_{orb}(x,y)=\beta\frac{\pi\epsilon_0}{k}\exp[-2r^2/w_0^2\beta]\sum_{n}^{m-1}C_n^m(m-n)r^{2(m-n)}, \tag{14}$$

where we have defined

$$C_n^m \equiv \binom{m}{n}^2 \frac{|C|^2}{\pi\delta^2}\frac{\Gamma(n+1)}{A^{n+1}}\beta^{-2(m-n)}, \tag{15}$$

with

$$\beta \equiv \left(1 + \frac{2\delta^2}{w_0^2}\right). \tag{16}$$

By itself, this quantity is difficult to interpret, because it is not only spatially varying but depends on the intensity of the beam. We therefore consider normalized versions of this quantity, where the normalization is the z-component of the Poynting vector, which is of the form

$$S(\mathbf{r}) = \frac{k}{\mu_0\omega}W(\mathbf{r},\mathbf{r}) = \frac{\pi k}{\mu_0\omega}\exp[-2r^2/w_0^2\beta]\sum_{n=0}^{m} C_n^m r^{2(m-n)}. \tag{17}$$

We first calculate the total OAM per photon, which is given by the ratio of the integrated M_{orbit} and $S(\mathbf{r})$,

$$l_{orb} = \frac{\hbar\omega\int M_{orb}(\mathbf{r})d^2r}{\int S(\mathbf{r})d^2r}. \tag{18}$$

It is readily found that the total OAM per photon is simply proportional to the topological charge of the underlying vortex beam, i.e. $l_{orb} = m\hbar$, regardless of coherence. This is not particularly surprising, as every member of the ensemble has the same vortex structure, and it is known that the OAM for such Laguerre-Gauss vortex beams is an intrinsic quantity, independent of the origin of the coordinate system.[20]

More interesting to derive is the normalized orbital angular momentum flux density, which indicates the transverse spatial distribution of OAM; it is defined as

$$l_{orb}(\mathbf{r}) = \frac{\hbar\omega M_{orb}(\mathbf{r})}{S(\mathbf{r})}, \tag{19}$$

and with some work it takes on the form

$$l_{orb}(\mathbf{r}) = \hbar\beta\frac{\sum_{n=0}^{m-1} C_n^m(m-n)r^{2(m-n)}}{\sum_{n=0}^{m-1} C_n^m r^{2(m-n)} + C_m^m}. \tag{20}$$

This flux density has a striking radial dependence. For small values of r, it is approximately quadratic,

$$l_{orb}(r,\omega) \approx \hbar\left(1 + \frac{2\delta^2}{w_0^2}\right)\frac{C_{m-1}^m r^2}{C_m^m}, \tag{21}$$

while for large values of r, it approaches a constant value,

$$l_{orb}(r) \approx m\hbar\left(1 + \frac{2\delta^2}{w_0^2}\right). \tag{22}$$

This is the behavior of what is known as a *Rankine vortex*, with a rigid body rotational behavior near the core and a fluid rotational behavior outside of this. It was previously demonstrated[18,21] that first-order PCVBs generated by the beam wander model have a Rankine-like behavior; it is now confirmed that all orders do. For this reason, we refer to such fields as *Rankine PCVBs*.

An illustration of the Rankine behavior of $l_{orb}(\mathbf{r})$ for various values of δ is shown in Fig. 3.

As a Rankine vortex is a hybrid between a rigid body rotator and a fluid rotator, it seems reasonable to expect that there exist PCVBs that are pure states of the two extremes. We now demonstrate that this is the case.

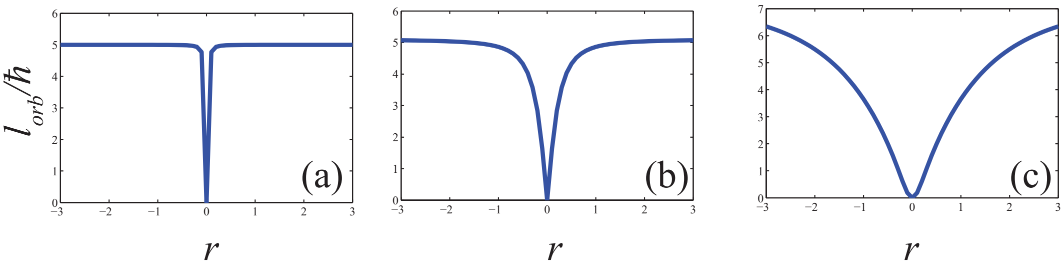

Figure 3. The OAM flux density for a PCVB with $w_0 = 5$ mm, and (a) $\delta = 0.01$ mm, (b) $\delta = 0.1$ mm, (c) $\delta = 0.5$ mm.

3.2 TWISTED GAUSSIAN SCHELL-MODEL

In the 1990s, the class of Gaussian Schell-model beams was generalized to include beams which also have a "handedness" to them, known as *twisted Gaussian Schell-model beams*.[22–24] In the waist plane, such tGSMs have the form

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) = S_0 \exp[-r_1^2/2\sigma_S^2] \exp[-r_2^2/2\sigma_S^2] \exp[-(\mathbf{r}_2 - \mathbf{r}_1)^2/2\sigma_\mu^2] \exp[iku(x_1 y_2 - y_1 x_2)], \tag{23}$$

where σ_S represents the width of the beam and σ_μ represents the transverse correlation length. The parameter u is known as the twist parameter, which characterizes the strength of the beam's "twist," and it must satisfy the constraint,

$$k|u| \leq \frac{1}{\sigma_\mu^2}. \tag{24}$$

The properties of tGSMs have been studied exhaustively; here we focus on their orbital angular momentum properties. If we consider the OAM per photon using Eq. (18), we readily find that

$$l_{orb} = \frac{\sigma_S^2 u}{2c}. \tag{25}$$

Unlike the Rankine case, the OAM spectrum is continuously dependent on the parameter u for tGSMs.

A calculation of the normalized orbital angular momentum flux density results in the expression,

$$l_{orb}(r) = \hbar k u \left[x^2 + y^2\right], \tag{26}$$

which shows that the beam can be viewed a a pure rigid body rotator.

The physically distinct properties of the tGSMs can be elucidated by viewing them as an incoherent superposition of tilted coherent Gaussian beams, as was done some time ago.[25] This model is illustrated in Fig. 4. Much like the beam wander model of Eq. (6), the field is taken to be a superposition of modes of the form,

$$U(\mathbf{r}, \mathbf{r}_0) = \exp\left[-\frac{(x - x_0)^2 + (y - y_0)^2}{X^2} + 2\pi i \alpha (x_0 y - y_0 x)\right], \tag{27}$$

where now not only the origin of the beam depends on (x_0, y_0) but also the degree of tilt; the parameter α can be shown to be proportional to the twist parameter u. Unlike the beam wander model, however, the constituent modes are not vortex beams, but simple Gaussian beams. The circulation within the beam is therefore due to the tilt, not due to any inherent wave vortex.

3.3 SEPARABLE PHASE MODEL

It is also possible to create a vortex beam that acts like a pure fluid rotator, i.e. one that acts like a pure coherent vortex mode with respect to its OAM. This underappreciated class was first introduced in one of the earliest papers on PCVBs,[13] and we call it a *separable phase model*. It may be constructed through the use of the coherent mode representation[9] of the cross-spectral density, i.e.

$$W(\mathbf{r}_1, \mathbf{r}_2) = \sum_n \lambda_n \phi_n^*(\mathbf{r}_1) \phi_n(\mathbf{r}_2), \tag{28}$$

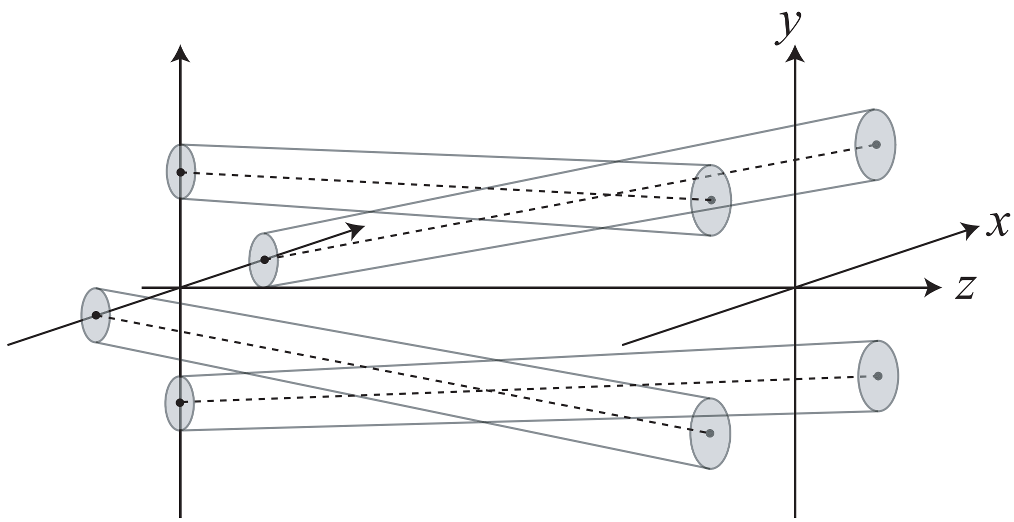

Figure 4. Illustration of the arrangement of tilted Gaussian modes in a tGSM, for positive α. After.[1]

where $\lambda_n > 0$ are eigenvalues and $\phi_n(\mathbf{r})$ are orthonormal eigenmodes of the cross-spectral density with respect to integration in the waist plane. In this general expression, the summation index n may represent one or more actual sums over a finite or infinite set of modes. Using the Laguerre-Gauss modes as the eigenmodes, a separable phase cross-spectral density may be constructed in the form

$$W(\mathbf{r}_1, \mathbf{r}_2) = \sum_{n=0}^{N} \lambda_n U_{nm}^*(\mathbf{r}_1) U_{nm}(\mathbf{r}_2), \tag{29}$$

where m is a fixed constant integer, and the upper limit of sum N as well as the eigenvalues λ_n may be freely chosen. The azimuthal dependence of all the modes is therefore of the form $\exp[im\phi]$, and the entire cross-spectral density may be written as

$$W(\mathbf{r}_1, \mathbf{r}_2) = e^{im(\phi_2 - \phi_1)} F(r_1, r_2), \tag{30}$$

where $F(r_1, r_2)$ is the radial dependence of the resulting correlation function.

The OAM per photon and the OAM flux density can be readily determined simply by converting Eq. (5) into polar coordinates, such that

$$M_{orb}(\mathbf{r}) = \frac{\epsilon_0}{k} \mathrm{Im}\left[\partial_{\phi_2} W(\mathbf{r}_1, \mathbf{r}_2)\right]_{\mathbf{r}_1 = \mathbf{r}_2}. \tag{31}$$

one readily finds that the OAM per photon is $m\hbar$, and the OAM flux density is simply

$$l_{orb}(\mathbf{r}) = m\hbar, \tag{32}$$

a constant. Thus this class of beams act like fluid rotators.

It is important to note that, by nature of their construction, this class of beams will maintain their pure vortex on propagation through free space, as each mode maintains its vortex structure and propagates independently. It is unclear if this pure vortex will remain, however, when the beam is distorted, for example by being passed through a random phase screen or an aberrated lens.

4. OBSERVATIONS

We have therefore seen that there are three distinct classes of partially coherent vortex beams which may be distinguished by their orbital angular momentum characteristics. A comparison of the propagation characteristics of these three classes, both in free space and in turbulence, remains to be done. Furthermore, though these three classes seem in a sense to be "fundamental", it is of interest to find whether there are more general classes of

PCVBs which have significantly different vallues of the OAM flux density $l_{orb}(\mathbf{r})$, in particular having a higher density of OAM in the core of the beam than at the outskirts. Finally, it should be noted that the usefulness of such beams will depend on how the OAM is measured, as any realistic detector will have a finite aperture which will not capture the entire beam.

ACKNOWLEDGMENTS

This work was supported by the Air Force Office of Scientific Research (AFOSR) under grant (FA9550-16-1-0240).

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