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Jones and Stokes–Mueller analogous calculi for OAM-transforming optics

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We introduce a matrix-based approach for characterization of local interactions of optical beams with devices that result in changes of their orbital angular momentum (OAM) content. For deterministic interactions, a method similar to the Jones calculus is developed, while for interactions involving random beams and/or devices, its generalization based on the coherence-OAM matrix is suggested. Applications of the new, to the best of our knowledge, calculus to a spiral plate, a trigonometric grating, and a diffuser are considered. An alternative formulation similar to the Stokes–Mueller calculus is also outlined.

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In dealing with interactions of electromagnetic beams with polarization-sensitive optical devices, the classic or the generalized Stokes–Mueller calculus is typically employed, depending on whether single-point or double-point transformations are of interest [1]. One-point interactions between deterministic light and polarization-changing devices can be characterized by the Jones calculus [2], while cases when either the incident light or the device, or both, are random, can be dealt with by the Stokes–Mueller calculus [3,4]. The classic Stokes–Mueller calculus has a generalization to two-point interactions [5,6], which uses the two-point Stokes vectors and Mueller matrices [7]. These formalisms are key to accurately and efficiently analyzing the behavior of polarization-sensitive optical systems.

The purpose of this Letter is to introduce the counterpart of the Stokes–Mueller calculus for transformations of one-point and two-point orbital angular momentum (OAM) states of scalar stationary light [8] that can be described by a coherence-OAM (COAM) matrix [9]. As the spin angular momentum (SAM) state of light is characterized by the polarization matrix, we expect there to be some analogies between the OAM and polarization calculi, and note the similarities when appropriate. The OAM case, however, is much richer in its possibilities.

We consider first the interaction of a deterministic device with transmission function $o(\mathbf{r})$ and a monochromatic field $U^i(\mathbf{r})$ at frequency ω (omitted), producing a transmitted field:

$$U^t(\mathbf{r}) = o(\mathbf{r}) U^i(\mathbf{r}). \tag{1}$$

We then decompose the optical fields and the transmission function into their spiral Fourier spectra,

$$U^i(\mathbf{r}) = \sum_{l=-\infty}^{\infty} U_l^i(\rho) e^{il\phi}, \quad U^t(\mathbf{r}) = \sum_{k=-\infty}^{\infty} U_k^t(\rho) e^{ik\phi}, \tag{2}$$

$$o(\mathbf{r}) = \sum_{m=-\infty}^{\infty} o_m(\rho) e^{im\phi}, \tag{3}$$

where

$$U_l^i(\rho) = \frac{1}{2\pi} \int_0^{2\pi} U^i(\mathbf{r}) e^{-il\phi} \mathrm{d}\phi, \quad U_k^t(\rho) = \frac{1}{2\pi} \int_0^{2\pi} U^t(\mathbf{r}) e^{-ik\phi} \mathrm{d}\phi, \tag{4}$$

$$o_m(\rho) = \frac{1}{2\pi} \int_0^{2\pi} o(\mathbf{r}) e^{-im\phi} \mathrm{d}\phi. \tag{5}$$

Then substitution from Eqs. (2) and (3) into Eq. (1) yields

$$\sum_{k=-\infty}^{\infty} U_k^t(\rho) e^{ik\phi} = \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} o_m(\rho) U_l^i(\rho) e^{i(l+m)\phi}. \tag{6}$$

Matching the Fourier series terms on the two sides of Eq. (6) with $m = k - l$ implies that

$$U_k^t(\rho) = \sum_{l=-\infty}^{\infty} o_{k-l}(\rho) U_l^i(\rho). \tag{7}$$

Vectors with components U_l^i and U_k^t can be considered as analogs of the Jones vectors of polarization optics [2]. Equation (7) is a linear transformation relating the components of the incident and transmitted fields. In matrix form, Eq. (7) becomes

$$\overrightarrow{U}^t(\rho) = \overleftrightarrow{o}(\rho) \overrightarrow{U}^i(\rho), \tag{8}$$

where $\overleftrightarrow{o}(\rho)$ is the analog of the Jones matrix, with elements $o_{lk}(\rho)$ mapping component $U_l^i(\rho)$ of the incident field to component $U_k^t(\rho)$ of the transmitted field (see Fig. 1 for notations). We note, however, that for devices introducing local (point-by-point) transformations, the o-matrix is *homogeneous*, i.e., its elements depend only on the difference of indices k and l. This therefore mathematically excludes the possibility of designing a local transformation OAM sensitive device that can affect an

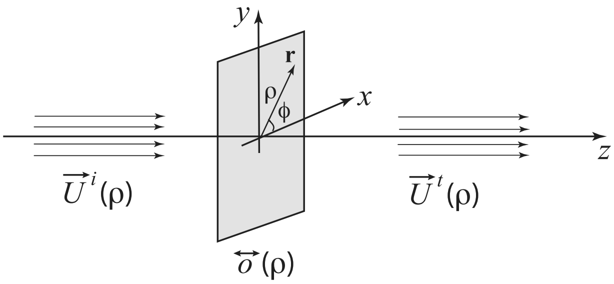

Fig. 1. Transformation of U-vector by o-matrix.

individual mode or any finite combination, e.g., mode sorters or switchers. This includes devices such as fork gratings, which use propagation to diffract and separate modes. We defer the more complicated general case for arbitrary $o_{lk}(\rho)$ for another paper.

The identity transformation for Eq. (7) takes the form:

$$o(\mathbf{r}) = 1, \quad o_{lk}(\rho) = \begin{cases} 1, & k = l \\ 0, & k \neq l, \end{cases} \quad \text{or} \quad \overleftrightarrow{o}(\rho) = \overleftrightarrow{I}^{(\infty)}, \tag{9}$$

i.e., being the infinitely dimensional identity matrix.

If the optical system consists of Q tightly placed devices, then the field transmitted by the last device becomes

$$\overrightarrow{U}^t(\rho) = \prod_{q=1}^{Q} \overleftrightarrow{o}_q(\rho)\overrightarrow{U}^i(\rho). \tag{10}$$

A spiral plate with topological charge n (whole number) is the most familiar device producing a local OAM transformation. Its transmission matrix takes the form $o(\mathbf{r}) = e^{in\phi}$, $o_{lk}(\rho) = \delta_{n,k-l}$, and, hence, Eq. (7) reduces to $U_k^t(\rho) = U_{k-n}^i(\rho)$, implying that all OAM components of the incident beam are shifted by n units. Matrix $\overleftrightarrow{o}(\rho)$ has ones on nth upper or lower diagonal for $n > 0$ or $n < 0$, respectively, and zeroes elsewhere. It reduces to the identity transformation if $n = 0$. Linear combinations of the spiral plates can also be treated, for example, the cosine and sine trigonometric gratings, $o(\mathbf{r}) = \cos(n\phi)$, $o(\mathbf{r}) = \sin(n\phi)$. Euler's formula implies the cosine grating splits each incident OAM component, say l, into two equal parts and assigns them to the components of transmitted field having charges $l \pm n$. This effect is similar to that of a polarizing beam splitter. The matrix $\overleftrightarrow{o}$ then has 1/2 on the upper and lower nth diagonals and zeros otherwise. The action of a sine grating is similar.

All the devices above are functionally independent of ρ and, hence, have constant coefficients $o_m(\rho) = o_m$. Radially varying devices can also be constructed, either with continuous or piecewise continuous ρ-dependence. For instance, one can combine a grating for $\rho < \rho_0$ and a spiral plate for $\rho > \rho_0$.

We consider next the transformation of the COAM matrix by a deterministic OAM transparency. For stationary fields, Eq. (1) can be generalized as:

$$W^t(\mathbf{r}_1, \mathbf{r}_2) = O(\mathbf{r}_1, \mathbf{r}_2)\, W^i(\mathbf{r}_1, \mathbf{r}_2), \tag{11}$$

where W^t, W^i are cross-spectral density matrices and $O(\mathbf{r}_1, \mathbf{r}_2) = o^*(\mathbf{r}_1)o(\mathbf{r}_2)$, where star denotes complex conjugate and we decompose W^i and W^t as [9]

$$W^i(\mathbf{r}_1, \mathbf{r}_2) = \sum_{l=-\infty}^{\infty}\sum_{l'=-\infty}^{\infty} W_{ll'}^i(\rho_1, \rho_2)e^{-il\phi_1}e^{il'\phi_2}, \tag{12}$$

$$W^t(\mathbf{r}_1, \mathbf{r}_2) = \sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty} W_{kk'}^t(\rho_1, \rho_2)e^{-ik\phi_1}e^{ik'\phi_2}, \tag{13}$$

as well as the transparency transmission function O as

$$O(\mathbf{r}_1, \mathbf{r}_2) = \sum_{m=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty} O_{mm'}(\rho_1, \rho_2)e^{-im\phi_1}e^{im'\phi_2}. \tag{14}$$

Here the COAM matrix elements are given by

$$W_{ll'}^i(\rho_1, \rho_2) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} W^i(\mathbf{r}_1, \mathbf{r}_2)e^{-il\phi_1}e^{il'\phi_2}\mathrm{d}\phi_1\mathrm{d}\phi_2, \tag{15}$$

$$W_{kk'}^t(\rho_1, \rho_2) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} W^t(\mathbf{r}_1, \mathbf{r}_2)e^{-ik\phi_1}e^{ik'\phi_2}\mathrm{d}\phi_1\mathrm{d}\phi_2, \tag{16}$$

$$O_{mm'}(\rho_1, \rho_2) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} O(\mathbf{r}_1, \mathbf{r}_2)e^{-im\phi_1}e^{im'\phi_2}\mathrm{d}\phi_1\mathrm{d}\phi_2. \tag{17}$$

On substituting $W_{ll'}^i$, $W_{kk'}^t$, and $O_{mm'}$ into Eq. (14), we get

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty} & W_{kk'}^t(\rho_1, \rho_2)e^{-ik\phi}e^{ik'\phi} \\ &= \sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty}\sum_{l'=-\infty}^{\infty} O_{mm'}(\rho_1, \rho_2) \\ &\times W_{ll'}^i(\rho_1, \rho_2)e^{-i(l+m)\phi}e^{i(l'+m')\phi}.\end{aligned} \tag{18}$$

Matching terms by letting $m = k - l$, $m' = k' - l'$, we get

$$W_{kk'}^t(\rho_1, \rho_2) = \sum_{l=-\infty}^{\infty}\sum_{l'=-\infty}^{\infty} O_{k-l,k'-l'}(\rho_1, \rho_2)\, W_{l,l'}^i(\rho_1, \rho_2). \tag{19}$$

The COAM matrix elements shift k units to left and k' units up and are weighted by $O_{k-l,k'-l'}$, and all contributions are added. The homogeneity of elements $O_{k-l,k'-l'}$ is preserved.

In matrix notation, Eq. (19) can be expressed as

$$\overleftrightarrow{W}^t(\rho_1, \rho_2) = \overleftrightarrow{o}^*(\rho_1)\,\overleftrightarrow{W}^i(\rho_1, \rho_2)\overleftrightarrow{o}^T(\rho_2), \tag{20}$$

where T denotes matrix transpose.

The identity transformation for Eq. (19) takes form of an infinite-dimensional identity matrix:

$$O_{k-l,k'-l'}(\rho_1, \rho_2) = \begin{cases} 1, & k = l,\ k' = l', \\ 0, & k \neq l,\ k' \neq l'. \end{cases} \tag{21}$$

For the spiral plate, the whole COAM matrix shifts n units along the main diagonal:

$$W_{kk'}^t(\rho_1, \rho_2) = W_{k-n,k'-n}^i(\rho_1, \rho_2). \tag{22}$$

In the case when the COAM matrix interacts with a random, stationary transparency, we use the same transformation law as in Eq. (11) but now have the transmission function,

$$O(\mathbf{r}_1, \mathbf{r}_2) = \langle o^*(\mathbf{r}_1) o(\mathbf{r}_2) \rangle_d. \tag{23}$$

Here the angular brackets with subscript d denote an average over the ensemble of transparencies, obtained under the assumption that the statistics of the field and the random transparency are independent. The same steps can be made as in deriving Eq. (19), except the products of the transparency coefficients become scalar correlation functions:

$$O_{k-l,k'-l'}(\rho_1, \rho_2) = \langle o^*_{k-l}(\rho_1) o_{k'-l'}(\rho_2) \rangle_d. \tag{24}$$

A ground-glass diffuser specified by a stationary process with zero mean and having a Gaussian two-point correlation function is a simple yet illustrative example:

$$O(\mathbf{r}_1, \mathbf{r}_2) = \exp\left[-\frac{|\mathbf{r}_1 - \mathbf{r}_2|^2}{2\delta^2}\right], \tag{25}$$

where δ is the correlation width. It may be written as

$$O(\mathbf{r}_1, \mathbf{r}_2) = \sum_{m=-\infty}^{\infty} O_{mm}(\rho_1, \rho_2) e^{-im(\phi_1-\phi_2)}, \tag{26}$$

with

$$O_{mn}(\rho_1, \rho_2) = \exp\left[-\frac{\rho_1^2+\rho_2^2}{2\delta^2}\right] I_m\left(\frac{\rho_1\rho_2}{\delta^2}\right) \delta_{mn}, \tag{27}$$

where I_m is the the first kind, modified Bessel function and δ_{mn} is the Kronecker delta. Inserting Eq. (27) into Eq. (19) yields

$$W^t_{kk'}(\rho_1, \rho_2) = \exp\left[-\frac{\rho_1^2+\rho_2^2}{2\delta^2}\right] \sum_{n=-\infty}^{\infty} I_n\left(\frac{\rho_1\rho_2}{\delta^2}\right) \times W^i_{k-n,k'-n}(\rho_1, \rho_2). \tag{28}$$

If the incident field is in a pure OAM state (single-element COAM matrix), then a COAM matrix with infinitely many diagonal elements is produced, though the number of modes of significant amplitude will depend on δ. If the incident field is not in a pure state and contains off-diagonal elements, then both diagonal and off-diagonal elements will be spread across infinitely many modes. It is worth noting that only a finite number of OAM states can be measured experimentally, making the COAM matrix finite size in practice.

For instance, if the incident field is a Laguerre–Gaussian beam with radial order 0 and azimuthal order l, then

$$U^i_m(\rho) = C_m \rho^{|m|} \exp\left[-\frac{\rho^2}{w^2}\right] \delta_{lm}, \quad C_m = \frac{2^{(1+|m|)/2}}{w^{|m|+1/2}\sqrt{\pi |m|!}}. \tag{29}$$

Then the incident COAM matrix has a single diagonal element,

$$W^i_{mm'}(\rho_1, \rho_2) = C_m C_{m'} \rho_1^{|m|} \rho_2^{|m'|} \exp\left[-\frac{\rho_1^2+\rho_2^2}{w^2}\right] \delta_{lm}\delta_{lm'}. \tag{30}$$

The incident spectral density then has elements $S^i_{mm'}(\rho) = W^i_{mm'}(\rho, \rho)$. Substituting Eq. (30) into Eq. (28) yields

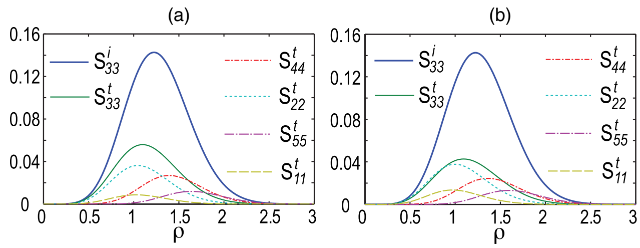

Fig. 2. COAM matrix elements, incident (with $w = 1$) and transmitted through the diffuser with (a) $\delta = 1.2$; (b) $\delta = 0.8$.

$$W^t_{kk'}(\rho_1, \rho_2) = \exp\left[-\frac{\rho_1^2+\rho_2^2}{w_d^2}\right] \sum_{n=-\infty}^{\infty} I_n\left(\frac{\rho_1\rho_2}{\delta^2}\right) \times C_{k-n} C_{k'-n} \rho_1^{|k-n|} \rho_2^{|k'-n|} \delta_{l,k-n}\delta_{l,k'-n} \tag{31}$$

with $1/w_d^2 = 1/(2\delta^2) + 1/w^2$. Setting $k - n = l$ leads to

$$W^t_{kk'}(\rho_1, \rho_2) = \exp\left[-\frac{\rho_1^2+\rho_2^2}{w_d^2}\right] I_{k-l}\left(\frac{\rho_1\rho_2}{\delta^2}\right) \times C_l C_{k'-k+l} \rho_1^{|l|} \rho_2^{|k'-k+l|} \delta_{l,k'-k+l}, \tag{32}$$

or, alternatively,

$$W^t_{kk'}(\rho_1, \rho_2) = \exp\left[-\frac{\rho_1^2+\rho_2^2}{w_d^2}\right] I_{k-l}\left(\frac{\rho_1\rho_2}{\delta^2}\right) C_l^2 (\rho_1\rho_2)^{|l|} \delta_{k,k'}. \tag{33}$$

Then the spectral density becomes as $S^t_{kk'}(\rho) = W^t_{kk'}(\rho, \rho)$.

Figure 2 shows the transmission of the single-element COAM matrix $S^i_{33}(\rho)$ through a diffuser with two values of δ. For larger/smaller δ [Figs. 2(a)/2(b)] the transmitted element $S^t_{33}(\rho)$ retains more/less of the incident beam's power while giving off less/more power to the other transmitted elements.

If, instead, the incident field is a superposition of two OAM modes, say l_A and l_B, then

$$U^i_m(\rho) = C_m \rho^{|m|} \exp\left[-\frac{\rho^2}{w^2}\right] (\delta_{ml_A} + \delta_{ml_B}). \tag{34}$$

Hence, the incident COAM matrix becomes

$$W^i_{mm'}(\rho_1, \rho_2) = C_m C_{m'} \rho_1^{|m|} \rho_2^{|m'|} \exp\left[-\frac{\rho_1^2+\rho_2^2}{w^2}\right] (\delta_{ml_A}\delta_{m'l_A} + \delta_{ml_A}\delta_{m'l_B} + \delta_{ml_B}\delta_{ml_A} + \delta_{ml_B}\delta_{m'l_B}). \tag{35}$$

Substituting the equation above into Eq. (32) yields,

$$W^t_{kk'}(\rho_1, \rho_2) = \exp\left[-\frac{\rho_1^2+\rho_2^2}{w_d^2}\right] \Big[I_{k-l_A}\left(\frac{\rho_1\rho_2}{\delta^2}\right) C^2_{l_A} \times (\rho_1\rho_2)^{|l_A|} \delta_{k'-k,0} + I_{k-l_A}\left(\frac{\rho_1\rho_2}{\delta^2}\right) C_{l_A} C_{l_B} \rho_1^{|l_A|} \rho_2^{|l_B|} \times \delta_{k'-k,l_B-l_A} + I_{k-l_B}\left(\frac{\rho_1\rho_2}{\delta^2}\right) C^2_{l_B} (\rho_1\rho_2)^{|l_B|} \delta_{k'-k,0} + I_{k-l_B}\left(\frac{\rho_1\rho_2}{\delta^2}\right) C_{l_A} C_{l_B} \rho_1^{|l_B|} \rho_2^{|l_A|} \delta_{k'-k,l_A-l_B} \Big]. \tag{36}$$

While for a single-mode incident field $S^t_{kk'}$ represents an infinitely dimensional diagonal matrix, for a two-mode incident field, $S^t_{kk'}$ is an infinitely dimensional three-diagonal matrix, including the major diagonal ($k = k'$) and two minor diagonals being apart from the major diagonal by $\pm(l_A - l_B)$ units.

Although the obtained laws can fully characterize the transformation of a deterministic field or the COAM matrix, it appears rather impractical to apply it as in Eq. (23) with coefficients as given in Eq. (28). Indeed, as a similar calculation in polarization optics shows [5] that even with two field and two transparency components, the handling of 16 separate correlations is required. Therefore, it appears useful to introduce an equivalent transformation that is analogous to the Stokes–Mueller calculus. Indeed, following Ref. [10], the N-dimensional COAM matrix can be expressed as a linear combination of N^2 parameters $S_{(\alpha)}(\rho_1, \rho_2)$:

$$\overleftrightarrow{W}(\rho_1, \rho_2) = \frac{1}{N}\left[\sum_{\alpha=0}^{N-1} S_{(\alpha)}(\rho_1, \rho_2)\overleftrightarrow{\sigma}_\alpha\right], \tag{37}$$

where N^2 generalized Pauli matrices $\overleftrightarrow{\sigma}_\alpha$ have forms

$$\begin{aligned} &\overleftrightarrow{\sigma}_0 = \overleftrightarrow{\sigma}^I; \quad \overleftrightarrow{\sigma}_\alpha = \overleftrightarrow{\sigma}^D_\alpha, \quad \alpha = 1, \ldots, N-1; \\ &\overleftrightarrow{\sigma}_\alpha = \overleftrightarrow{\sigma}^S{}_\alpha, \quad \alpha = N, \ldots, N + (N-1)/2; \\ &\overleftrightarrow{\sigma}_\alpha = \overleftrightarrow{\sigma}^A{}_\alpha, \quad \alpha = N + 1 + N(N-1)/2, \ldots, N^2. \end{aligned} \tag{38}$$

Here matrices with superscripts I (identity), D (diagonal), S (symmetric), and A (antisymmetric) are defined as follows. Let

$$\overleftrightarrow{E}_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{39}$$

and then form $N \times N$ matrices as

$$\begin{aligned} &\overleftrightarrow{\sigma}^I = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \quad \overleftrightarrow{\sigma}^D_\alpha = \sqrt{\frac{2}{\alpha(\alpha+1)}}\left(\sum_{j=1}^{\alpha} \overleftrightarrow{E}_{jj} - \alpha\overleftrightarrow{E}_{\alpha+1,\alpha+1}\right), \\ &\overleftrightarrow{\sigma}^S_\alpha = \overleftrightarrow{E}_{jk} + \overleftrightarrow{E}_{kj}, \quad \overleftrightarrow{\sigma}^A_\alpha = -i(\overleftrightarrow{E}_{jk} - \overleftrightarrow{E}_{kj}). \end{aligned} \tag{40}$$

Conversely,

$$S_{(\alpha)}(\rho_1, \rho_2) = Tr\left[\overleftrightarrow{\sigma}_\alpha \overleftrightarrow{W}(\rho_1, \rho_2)\right]. \tag{41}$$

This implies that one can express the transformation law between the incident and the transmitted vectors $\overrightarrow{S}^i$, $\overrightarrow{S}^t$ as

$$\overrightarrow{S}^t(\rho_1, \rho_2) = \overleftrightarrow{M}(\rho_1, \rho_2)\overrightarrow{S}^i(\rho_1, \rho_2). \tag{42}$$

Here the $N^2 \times N^2$ matrix M can be found as a linear combination of correlation matrices with elements $O_{mm'}$. Matrix $\overleftrightarrow{M}(\rho_1, \rho_2)$ can be regarded as the OAM analog of the two-point Mueller matrix of polarization optics [5].

A transformation by a cascaded system of Q devices that change OAM can be then conveniently expressed as

$$\overrightarrow{S}^t(\rho_1, \rho_2) = \prod_{q=1}^{Q} \overleftrightarrow{M}_q(\rho_1, \rho_2)\overrightarrow{S}^i(\rho_1, \rho_2), \tag{43}$$

where $\overleftrightarrow{M}_q$ is the transformation matrix of the qth device while the matrix multiplication is performed from the right to left. Of course, at coinciding vectors $\rho_1 = \rho_2 = \rho$, the two-point OAM calculus reduces to single-point, being the counterpart of the classic Stokes–Mueller calculus.

The measurement of vector $\overrightarrow{S}(\rho_1, \rho_2)$ with components expressed via those of matrix $\overleftrightarrow{W}(\rho_1, \rho_2)$ can be based on the procedure as suggested in Ref. [9]. The measurement of matrices $\overleftrightarrow{o}(\rho)$, $\overleftrightarrow{O}(\rho_1, \rho_2)$, and $\overleftrightarrow{M}(\rho_1, \rho_2)$ can be done using the same approach as in polarization optics: by generating and filtering a sufficient number of basic, mutually orthogonal OAM states and then solving the inverse problem of finding the linear maps from the pairs of entering and resulting vectors.

In this Letter, we have introduced a calculus to describe OAM-transforming elements analogous to the Jones and Mueller calculus for polarization-transforming elements, and noted similarities and significant differences between the two cases. This work can form the basis for a complete description of OAM-transforming optics using the COAM matrix formalism. In concluding, we note that, although the transmission laws are developed for the COAM matrix, they can be also deduced for the other important beam properties, such as the OAM spectrum, flux, and degree of purity [9].

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