

This article was published in Teaching Philosophy, 23(2), June 2000. Changes in the current version are documented at the end under CURRENT VERSION UPDATES



Problem Solving, Working Backwards, and Graphic Proof Representation

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Newell and Simon's seminal Human Problem Solving (1972) characterized a problem in terms of a goal state, a starting state, and a set of transition rules which define legitimate transitions from one state to another.¹ Problem solving thus becomes a process of searching through a set of alternative states (the "problem space") in an effort to find a path leading from starting state to the goal state. The search process can be guided by heuristic principles which function to reduce the problem space by judging some alternatives to be more worthy of exploration than others. This characterization of a problem and the problem solving process fits well the nature of deductive proof construction. Premise(s) and conclusion play the role of starting state and goal state, and valid rules of transformation serve as rules of legitimate transition among states. In fact, Human Problem Solving empirically investigated three particular problem solving tasks, and one of these is proof construction.... in sentential logic using an inference-replacement rule set. This empirical research identifies several strategies which facilitate problem solving, such as means-ends reasoning, difference reduction, and working backwards from goal toward starting state. As detailed below, these methods have obvious applications to proof construction as taught in logic textbooks. Newell and Simon's aim was to explain and predict the actual behavior of problem

solvers. Beyond this, however, their empirical findings have normative consequences for how problem solvers should behave if they want to be successful. Moreover, these findings can have normative pedagogical consequences for the teaching of proof construction.

Another relevant empirical study of problem solving is Richard Burton's investigation of children's subtraction errors.² Burton endeavored to identify precisely the "bugs" that plagued simple mathematical procedures. Derek Sleeman built upon this work in his investigation of basic algebra errors.³ Previously, little attention had been paid to errors themselves since they were viewed as inexplicable, random deviations from given rules. Sleeman viewed errors quite differently, as being systematically produced by rules which were, unfortunately, illegitimate versions of the correct rules. Viewed in this manner, errors were scrutinized to discover the illegitimate rule patterns ("mal-rules") which they exemplified. Hypotheses about what these patterns might be were formulated and additional errors, which might confirm or disconfirm these hypotheses, were sought. Making this determination often required formulating new problems whose attempted solutions would contribute additional evidence. The pedagogical import is that student errors which once were given scant attention now became a source of insight into the nature of student difficulties. This insight can provide a basis for designing problem sets which not only can support the diagnosis of student difficulties but which can incrementally lead students away from those difficulties.

Aspects of these cognitive approaches to problem solving are relevant to the teaching of deductive proof construction. First, there is an emphasis on empirically documenting the kinds of errors made by problem solvers. The same need is evident in respect to student learning difficulties. Second, in thinking about problem solving as a form of searching, strategies which facilitate searching are systematically explicated.

Such clarity can also serve to facilitate the learning of proof finding methods. Third, forms of problem representation are seen to influence the process of problem solving. Consequently, important questions are raised concerning the ways in which proofs and the processes by which they unfold are represented. These issues frame much of what follows; namely, reports on student errors observed during proof construction, a systematic explication of the working backwards method, and discussion of a graphic format for representing both the finished proof and the maneuvers made in discovering it.

Patterns of Student Rule Mis-applications

Newell and Simon based their empirical studies of problem solving on a particular technique of protocol collection. While subjects worked problems, an experimenter sat close by and continuously asked probing questions concerning what the subject was thinking. This approach, however, can be somewhat intrusive. In respect to the observations reported below, a number of related techniques have been employed to establish windows on student problem solving efforts. These techniques have included sitting beside students and occasionally asking questions about decisions they've made, observing student work in progress during class, giving surprise assessments, and recording problem solving efforts as students used instructional computer programs. Often, informal observations suggested types of errors that were then more rigorously sought out in data collected by instructional computer programs. What has emerged from these observations are some views about the kinds of difficulties students commonly have with sentential proof construction. Some difficulties concern rule application and others concern the use of the working backwards strategy.

The teaching of deductive proof construction has long recognized a distinction between two types of tasks: rule application and strategic thinking. Logic textbooks observe this distinction by supplying justification exercises designed to inculcate rule application proficiency prior to the presentation of full proof problems. When designing justification exercises, one question is whether some rules are more troublesome than others and thus deserve more practice. Another question is whether there are particular instantiations of rules that are more difficult to recognize and/or work with. Still another question is whether there are general patterns of rule application errors; that is, mislearned and systematically applied mal-rules of the type investigated by Sleeman. On the first score, there is one class of rules which students find particularly troublesome. These are the rules which contain negation signs in their rule forms (rules such as Disjunctive Syllogism, Modus Tollens, Implication, Transposition, Demorgans, and Double Negation). As a class, this set of rules shows a lower rate of successful application than any other connective-defined set of rules.⁴ Moreover, negation-related difficulties can affect rules not containing negations. For example, students sometimes attempt to apply Simplification to the expression ' $\sim(A \& B)$ ', usually trying to extract ' $\sim A$ ' or simply ' A '. Errors such as these raise an interesting question about what it means to have learned a rule. Students who make errors in applying Simplification can almost always write out the rule form correctly. To have learned a rule means more than merely having memorized its rule form. It also means being able to correctly apply or withhold application of the rule in a variety of particular cases.

In respect to the existence of mal-rules, the observations presented here differ from the kind of evidence marshalled by Sleeman. Sleeman studied the behavior of individuals and explicated the mal-rules that explained the errors of particular individuals. The approach here has been to look at aggregate data from students in several different logic classes working proof problems from different logic textbooks.⁵

Patterns which fit a relatively large proportion of rule application errors (both in justification problems and in full proofs) have thereby been identified, and there do exist error patterns for particular rules that occur frequently enough to be of pedagogical interest. For example, one kind of error for Modus Tollens and Disjunctive Syllogism is shown in the patterns below. These errors involve a difficulty with negations which is relatively easy to diagnose and correct.

<u>MT</u>	<u>DS</u>
p ~q	~p v q
q	p
-----	-----
~p	q

Other errors with negations are shown in the following patterns.

<u>MT</u>	<u>MT</u>	<u>DS</u>	<u>DS</u>
p q	p ~q	~p v q	p v q
~q	~q	~p	p
-----	-----	-----	-----
p	~p	q	q

Some relatively frequent error patterns for other rules are shown below.

<u>ADD</u>	<u>HS</u>	<u>IMPL</u>	<u>TRANS</u>
p	p q	p v q	p q
-----	r s	-----	-----
p & q	-----	~p q	~p ~q
	p s		

The chief pedagogical implication of the existence of frequently observed error patterns is that students should be forced to confront and demonstrate mastery over such

trouble spots. Also, students should be alerted to the most frequent errors for each rule. The error patterns shown above are put forward as a step in that direction and as an attempt to encourage others to follow in this vein. More needs to be done to empirically discover frequently occurring error patterns and to put these discoveries to pedagogical use.

The patterns described thus far concern errors made in applying rules of transformation in a working forwards (top-down) direction. When confronting a proof problem, one choice of strategy may involve the decision to work forwards, or to work backwards (bottom-up), or to work alternately in each direction with the aim of making these lines of development "meet in the middle." Newell and Simon's investigations paid particular attention to the method of working backwards, and cognitive studies continue to emphasize the importance of this mode of thinking.⁶ Nevertheless, logic texts vary greatly in the degree to which the working backwards method is emphasized. Texts which teach the Intelim rule set (one rule for introducing and one rule for eliminating each of the five sentential connectives) tend to emphasize the working backwards method more than texts which teach the inference/replacement rule set.⁷ In what follows, an explication of the working backwards method will be put forward as it applies to inference and replacement rules. In particular, some difficulties which students have in using this method will be addressed.

The Concept of Working Backwards.

The general structure of the working backwards process is as follows. Suppose I am trying to move from starting state A to goal state Z. I determine that one way of arriving at state Z is to first arrive at state Y, from which there is a clear path to state Z. I can now focus on moving from state A to state Y. Further, if I determine that one way of

arriving at state Y is to first arrive at state W, from which there is a clear path to state Y, I can now concentrate on moving from state A to state W. This is the general idea behind the working backwards strategy. The distance between the starting state A and the goal state Z is reduced by discovering intermediate states whose connection to the goal is clear. The way in which these intermediate states are discovered and the clarity of their connections to desired goals depends upon the specifics of the problem formulation and rules of transition.

In proof construction, the general nature of the working backwards method is to work from the bottom-up, from conclusion back to premises. One begins with the goal, the problem's conclusion, and determines what rule might generate it and what premise expressions are required for that rule. These premise expressions, if not already existing within the proof's premise set, become new goals (technically, 'subgoals'), and one then continues working backwards from these goals in the same way as was done for the conclusion. At each step, expressions are introduced which, if ultimately derived from the problem's premises, will lead directly to a goal or subgoal. Expressions thus introduced in the working backwards chain are not themselves justified, nor do they immediately justify any goal expression to which they lead. They merely constitute expressions which, if ultimately justified, will contribute to justifying the proof's conclusion.

The working backwards method is not merely a means for attacking a proof problem from a bottom-up direction. Used appropriately, it also serves as a heuristic device which reduces the number of alternatives to be considered. Particularly, this means that the number of rules one might consider applying in order to produce a certain expression can be reduced. More generally, it means that effective heuristics can restrict the number of branching paths that one might plausibly pursue when trying to connect premises and conclusion. To accomplish this, the method should specify in

some manner which rules are suited to what kinds of problem situations. These categories vary when considering inference (implicational) rules as opposed to considering replacement (equivalence) rules.

Working Backwards with Inference Rules

To delineate a more specific procedure for using this method, a close inspection of the implicational (inference) rules is helpful.⁸ Each of these inference rules serves either the function of extraction or construction. That is, some rules extract subexpressions which exist within larger expressions. Examples of these rules are Simplification, Modus Ponens, and Disjunctive Syllogism. In each rule form for these rules, the conclusion generated already exists as some part of the rule's premise set. Other rules serve to construct new expressions which do not already exist as part of the rule form's premise set. Examples of these rules are Modus Tollens, Conjunction, Addition, Hypothetical Syllogism, and Constructive Dilemma. The working backwards method presented here is centered around this distinction. When working backwards to produce a goal expression, the first decision is whether to extract or construct the expression. Then, the relevant rule and the particular premise expressions needed for that rule must be specified. For each goal expression, these determinations can be helpfully framed by asking three questions in sequence:

- (I)** Extract or construct?
- (II)** What rule?
- (III)** What premise(s)?

To see how this works consider the following proof problem.

1. $(A \vee B) \quad (C \& D)$
2. $A \quad / \quad C$

To work backwards on this problem, one considers the ultimate goal expression and asks whether it is to be extracted or constructed. Extraction should be considered first. In this case, the goal expression, 'C', is found in an extractable position within the first premise, so extraction is indicated. (Moreover, construction can never produce a single letter.) Next, question (II) requests that the appropriate extraction rule be selected. This is done by identifying the connective holding the goal expression, which in this case is '&'. When determining the appropriate rule, the following tables are helpful.

When EXTRACTING from this connective, use this rule:

&	SIMP
v	DS
	MP

When CONSTRUCTING this connective, use this rule:

~	MT
&	CONJ
v	ADD, CD
	HS

This chart answers question (II) in favor of Simplification, and answering question (III) requires that the premise expression for this application be specified. This can be done by instantiating the variables in the rule form for Simplification ($p \& q / p$). The instantiation for one of these variables ('p') is specified by the goal expression ('C'). Instantiating the remaining variable ('q') is based on the contents of existing expressions. In general, the intelligent instantiation of premise expressions is an important step in working backwards and often involves an estimate of what might

plausibly be derived from existing expressions. In the current case of Simplification, the given premises are scanned for a conjunction with 'C' as its left conjunct, and when found, the right conjunct serves to instantiate the variable 'q'. So, 'C & D' is postulated as the instantiated premise, and this expression becomes the new goal to be derived. In addressing question (I) for this new goal, extraction is once again indicated by the fact that this goal expression is found in an extractable position, being the consequent of the conditional in the first given premise. Since the connective holding this goal expression is '~', question (II) is answered in favor of Modus Ponens. When instantiating this rule pattern on the basis of the goal expression, 'C & D' is substituted for the variable 'q' in the rule form. So, one postulated premise must be a conditional with 'C & D' as its consequent and the other postulated premise must be identical to the antecedent of that conditional. By scanning the problem's given premises, the conditional is located as given in premise 1. This leaves its antecedent, 'A v C' to be postulated as the new goal expression. Care must be taken when addressing question (I) for this new goal. Although 'A v C' can be located within the first premise, students learn that only the consequent of a conditional can be extracted. Neither the antecedent as a whole nor any of its components can ever be extracted from a conditional expression. [Remember that we only Inference rules are used here.] This means that 'A v C' must be constructed rather than extracted, and both Addition and Constructive Dilemma are possible answers to question (II). This complication can be handled by scanning the problem's premises and, in the case of Addition, looking for an opportunity to obtain one of the goal's disjuncts, and in the case of Constructive Dilemma, looking for conditional expressions which have consequents identical to the goal's disjuncts. On these grounds, Addition looks more plausible in the present case. Since one of the goal's disjuncts, 'A', exists as a given premise, no further working backwards is required, and the search for a proof is complete.

Some proof problems will complicate the working backwards process. In such problems the three working backwards questions can have more than one answer. In respect to question (I), for example, opportunities both for extracting and constructing a goal expression might exist.⁹ The same complication can apply to question (II) and question (III). When constructing a disjunctive expression, two rules (Addition and Constructive Dilemma) may not only be possible but even plausible given the problem's premise set, and if Addition is selected, there may be two plausible alternatives depending upon which disjunct is targeted for derivation. Moreover, proof problems which have frequently repeated letters, or unnecessary premises which repeat letters existing in the necessary premises, provide a relatively high number of plausible yet dead end paths whether working forwards or backwards. It is true that proof problems in introductory logic texts do not often present problems of this complexity, but when it does occur search paths can multiply quickly, and some means for backtracking and for systematically recording one's efforts should be available. This topic will be taken up shortly.

Some Observed Student Difficulties with Working Backwards

Two types of errors stand out within students' use of the three step working backwards method for proofs requiring only inference rules. One difficulty concerns the extraction-construction judgment which students are to make in answering question (I). In one class of 28 students who constructed proofs with inference rules only, for example, 935 erroneous working backwards steps were examined closely. It turned out that in nearly half of these errors students not only selected an inappropriate rule for working backwards but in addition selected the wrong type of rule. That is, extraction was being

attempted when construction should have been, or vice versa. This difficulty should, perhaps, not be surprising given the nature of the inference/replacement rule set. The functions of these rules are not emphasized by their names, as is the case in the Intelim rule set. Rather, the rules are often categorized and strategies formulated in terms of the connectives which exist in the premise(s) and/or conclusion of the rule's form. This categorization has its use in proof construction, but it should not be the immediate focus when determining how to produce a goal expression by working backwards. In particular, students must learn that the answer to "extract or construct?" is not contained in the rules themselves. When facing this question, students sometimes will look toward their list of rules. This is a sure sign that the three step sequence is not being followed, for the choice of extraction or construction is not determined by anything in the list of rules. This choice is determined within the problem itself by comparing the goal expression with other available expressions in the proof.

Another major difficulty which emerged from examining these working backwards errors supports the view that students sometimes focus on rule forms and connectives without first considering what kind of operation is called for. One recurring error pattern involves an inversion in the matching of connectives which exist in a problem's goal expression and the rule form being considered. Correctly, the main connective in a goal expression should be matched to the main connective in a rule's conclusion. However, students sometimes match the main connective in a goal expression to the main connective in a rule form's premise expression. For example, when trying to generate the expression, 'A & B', Simplification would be selected when in fact the goal should have been constructed via Conjunction. Or, when trying to produce 'A v B', Disjunctive Syllogism would be selected when either Addition or Constructive Dilemma was appropriate. This kind of error, dubbed "connective inversion," occurred in approximately one third of the 935 working backwards mistakes examined. The

existence of connective inversion emphasizes the need for following the three step working backward sequence. Students often want to shortcut this process and to focus immediately upon rule selection without first choosing either extraction or construction. When students do faithfully follow the three step sequence, errors concerning extraction/construction judgments and connective inversions virtually disappear.¹⁰

Working Backwards with Replacement Rules

This three step working backwards procedure turns out to be extremely powerful. There are very few inference rule proof problems in introductory texts which this procedure will not quickly solve.¹¹ When replacement rules are added to the rule set, this situation changes dramatically. Question (I) must be expanded to include the possibility of transforming one expression into another. Once this question ("extract, construct, or transform?") is answered in favor of transforming some expression into the goal, questions (II) and (III) follow as before to provide the appropriate rule and premise expression. The selection of an appropriate replacement rule, however, can no longer be neatly represented in a chart as done for inference rules. Replacement rules are not neatly categorized in terms of extraction or construction. Rule selection must be guided by careful inspection of the goal expression's connective structure. With inference rules only, the three step process serves to keep the search on track and to reduce the number of alternative paths considered. With the introduction of replacement rules, the number of alternative paths increases greatly, and the working backwards procedure in itself does little to reduce the numerous alternatives spawned. Working forwards strategies now contribute more to reducing the number of plausible alternatives. For example, several texts suggest means for breaking expressions down into component parts and for recognizing when this action is possible and desirable. Another common

suggestion is to learn combinations of steps and to recognize opportunities for their application. In doing so, students can better develop a sense of what can be derived from various kinds of expressions and can better judge when an expression will or will not lead to some goal.

Once replacement rules are introduced, the search for a connection between premises and conclusion will usually involve the exploration of multiple paths. Perhaps the most fruitful technique for discovering a proof is to alternately work both backwards and forwards. Students should be able to do this with both inference and replacement rules. Over the years, however, some texts have explicitly prohibited the use of working backwards with inference rules and have obfuscated the process of working backwards with replacement rules. The source of this difficulty appears to reside in the first edition of Kahane's Logic and Philosophy (1969, p. 60). There, Kahane warned students against working backwards with inference rules because these rules are one-directional. This view is expressed in each succeeding edition of Logic an Philosophy through the sixth edition in 1990. In this edition, Kahane explained in a footnote that working backwards with inference rules is possible but that "beginners are apt to accidentally use the implication forms in the wrong direction."¹² His recommendation was for beginners to avoid working backwards with implication rules and to use the more reliable working forwards strategies. However, Kahane has persistently advocated working backwards with replacement rules. To do so, the conclusion is placed in the position of being a premise and new expressions are deduced in a series of descending steps, each of which results from applying a replacement rule to a previous line.

(Kahane, 1990)

<u>Problem</u>	<u>Working Backwards</u>	
1. A B	i. (A ∨ C) B	Conclusion
2. C B / (A ∨ C) B	ii. $\sim(A \vee C) \vee B$	i, IMPL
	iii. $(\sim A \ \& \ \sim C) \vee B$	ii, DEM
	iv. $B \vee (\sim A \ \& \ \sim C)$	iii, COMM
	v. $(B \vee \sim A) \ \& \ (B \vee \sim C)$	iv, DIST

The last expression in this working backwards sequence becomes a target for derivation from the proof's premises, and work next shifts to deriving expressions from those premises in a working forwards direction.¹³

This procedure is abandoned in recent editions of Logic and Philosophy (Kahane and Tidman,1995; Tidman and Kahane, 1999), but it has been followed closely in editions of Hurley's A Concise Introduction to Logic. In fact, Hurley repeats Kahane's warning against working backwards with inference rules. ("It is important to realize that whenever we use this strategy of working backwards from the conclusion, the rules of replacement are the only rules we may use. We may not use the rules of implication, because these rules are one-way rules."¹⁴) Hurley refers to the process of applying replacement rules to the conclusion as "deconstruction" and characterizes it as a means for discovering how the conclusion is put together. Following Kahane, the conclusion is temporarily treated as a premise to which replacement rules are applied in a top-down fashion.

(Hurley, 2000)

<u>Proof Problem</u>	<u>Deconstruction</u>	
1. K (F v B)	(F & G) v (B & G)	Conclusion
2. G & K / (F & G) v (B & G)	G & F) v (G & B)	COM, COM
	G & (F v B)	DIST

In this example, three replacement rule applications have been used starting with the conclusion to produce 'G & (F v B)'. One is now to return to the proof problem and work forwards toward this expression.¹⁵

This procedure is clearly different both from the generic concept of working backwards and its three step implementation for proofs. Consistent with those conceptions, however, is Tidman and Kahane's more recent account of working backwards. In their problem below, one considers the conclusion, '(A v C) B', and postulates the expression, '~(A v C) v B', from which the conclusion can be derived in one step, and this process continues bottom-up.¹⁶

(Kahane and Tidman, 1999)

1. A B	
2. C B / (A v C) B	
.	
.	
.	
(B v ~A) & (B v ~C)	
B v (~A & ~C)	DIST
(~A & ~C) v B	COMM
~(A v C) v B	DEM
(A v C) B	IMPL

The suggestion at this point is to work forwards from the premises toward the expression, '(B v ~A) & (B v ~C)', and then on to the conclusion filling in the appropriate line numbers along the way. Nevertheless, the process of working backwards here

could continue. The rule of Conjunction could be proposed for producing '(B \vee \sim A) & (B \vee \sim C)', and the required conjuncts could be postulated in the working backwards chain. Compare this with the Hurley (2000) example above, and note where Hurley's process of deconstruction ends and why. [We would need to use a banned Inference rule.]

Towards a Graphic Representation of Proofs

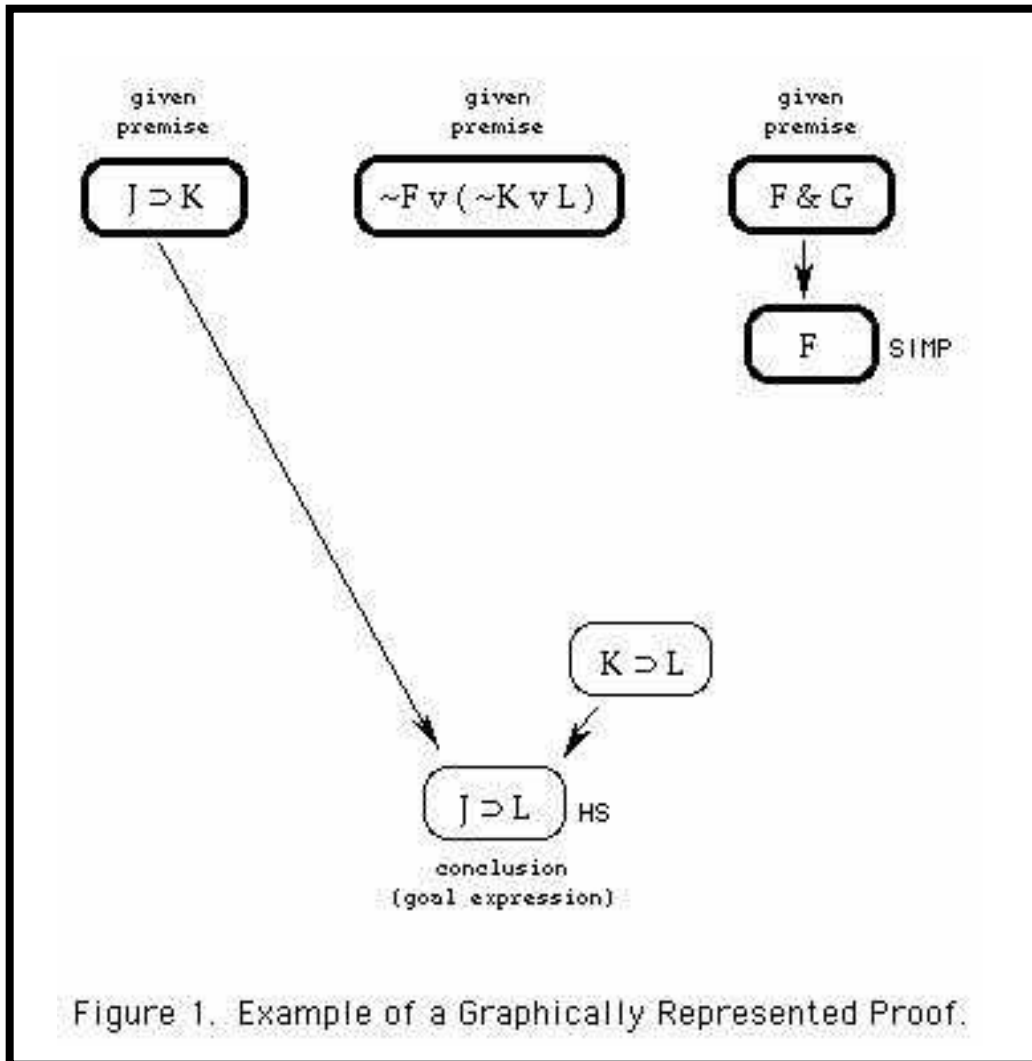
It is important to appreciate the potential difficulties of working backwards, particularly as conceived by Hurley and early Kahane. This approach raises a number of potential difficulties for students. The first consists of the additional burden of keeping clear about what rules justify what expressions in the proof. In the Kahane (1990) example, notice that the rule justifications cited in the working backwards sequence are on different lines than they will be in the proof. That is, the rule justifying line ii (IMPL) will actually justify the conclusion when the proof is written up.¹⁷ The same is true of Hurley's (2000) example. Second, the source of fear over the prospects of students applying inference rules in the wrong direction is now evident. Students taught to apply replacement rules from conclusion toward premises might well be tempted to apply inference rules in this direction as well. As Tidman and Kahane illustrate, students should not be taught to apply replacement rules in this fashion, and, once free from this dangerous procedure, students can indeed be taught to effectively work backwards with inference rules. Third, applying replacement rules from conclusion towards premises runs the risk of confusing students about the fundamental concept of justification within a proof. Students must understand that justification flows top-down from premises to conclusion. Any explication of the working backwards method must tread carefully here. The three step process delineated above and Kahane and Tidman's account also run this risk since expressions are generated against the flow of justification in the

proof. That is, one enters expressions into the proof structure which temporarily play the role of premises rather than conclusions. Students must remain absolutely clear about two points. First, the fact that a rule name has been entered for a given expression, say for the conclusion when working backwards, does not mean that the expression is justified. Second, when entering a working backwards step which will ultimately serve to justify an expression, nothing at all has been done to justify that newly entered expression. The newly entered expression has not been "generated" in the same way that a new expression has been generated as the result of applying a rule working forwards. Students can in fact learn to keep clear about these points if discussed explicitly.

Scheines and Sieg (1994) found that students who worked both backwards and forwards completed more proofs than students who worked only forwards or only backwards.¹⁸ If indeed students should acquire the ability to work both backwards and forwards, a question arises as to how this process should be carried out on paper and how its results should be represented. Some indication of how working bottom-up might occur in a linear proof format emerges from the example provided by Kahane and Tidman. An ideal environment would allow work to smoothly unfold in both a top-down and a bottom-up direction. Students should be able to explore multiple paths in either direction while staying clear about which expressions are justified and which are not. The proposal put forward here is that a graphic proof representation can be useful for these purposes. A graphic representation of a proof is shown in Figure 1.

The logical expressions which comprise the problem are written inside of oval nodes. Premise nodes are positioned across the top of the working area while the conclusion is placed near the bottom. Working forwards proceeds by applying rules to

premise expressions. The expression nodes which result from those applications are drawn below with arrows leading from premises to derived expressions.



In Figure 1, Simplification has been applied in a working forwards direction while Hypothetical Syllogism has been used in working backwards from the conclusion. Working backwards from the conclusion follows the three-step procedure. A decision to extract, construct, or transform is followed by selection of the appropriate rule. The required premise expressions are determined, and nodes for these expressions are

drawn higher in the proof structure. The premise expressions for Hypothetical Syllogism in Figure 1 have been instantiated on the basis of the conclusion and other expressions contained in the proof's premises. Also, of importance is the thickness of the node boundaries. Justified expressions have thick boundaries while unjustified expressions have thin boundaries. As unjustified expressions become justified, their boundaries are thickened.

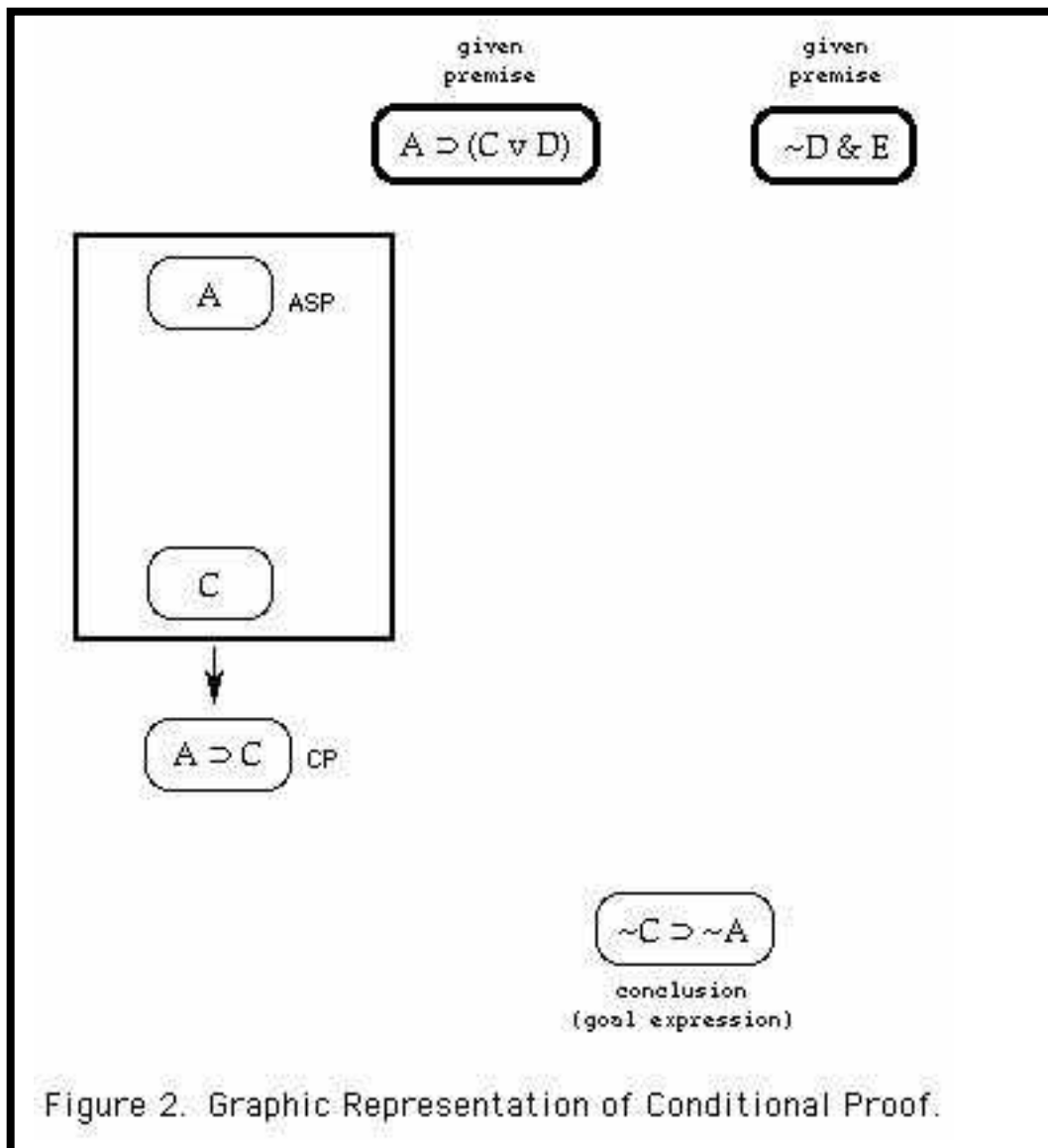
Conditional and Indirect Proof: Graphic Representation

Conditional and indirect proof are both rooted in the making and discharging of assumptions. It is important that students understand why assumptions can be made, what must be done to discharge assumptions, and the limitations placed upon the use of any expression existing within the scope of an assumption. These restrictions are of particular significance in the case of nested applications of conditional proof and indirect proof. To aid in enforcing these restrictions, many texts augment the linear proof format with some version of drawn lines or brackets.

1.	$(A \vee B) \quad (C \& D)$	/	$A \quad C$
2.	A		ASP (CP)
3.	$A \vee B$		2, ADD
4.	$C \& D$		1, 3 MP
5.	C		4, SIMP
6.	$A \quad C$		2-5, CP

The indented lines from step 2 to step 5 above are normally bracketed off to show that once the assumption is discharged, no other steps of the proof may make use of these lines.

Within a graphic format, something similar must be done to indicate the scope of an assumption and the restrictions placed upon the use of any expression derived from



it. Figure 2 shows a graphic representation of the problem above after a decision to use conditional proof has been made. To use conditional proof, one does three things. First, one draws a box within which the conditional reasoning will unfold. Next, the assumption made is drawn at the top of this box along with the rule cited, and the expression to be produced (the conditional's consequent) is drawn at the bottom of the

box. Finally, an arrow is drawn leading from the box to the conditional expression to be generated by conditional proof along with its rule abbreviation. The restriction which now governs the execution of conditional proof is as follows: Arrows leading from justified expressions may cross the boundary into the box, but no new arrows may lead out of the box from any expression. The only arrow permitted to issue from the box is that originally drawn to the ultimate object of conditional proof. This same restriction, of course, applies to nested applications of conditional proof which could involve boxes within boxes. Figure 3 provides a view of a finished conditional proof.

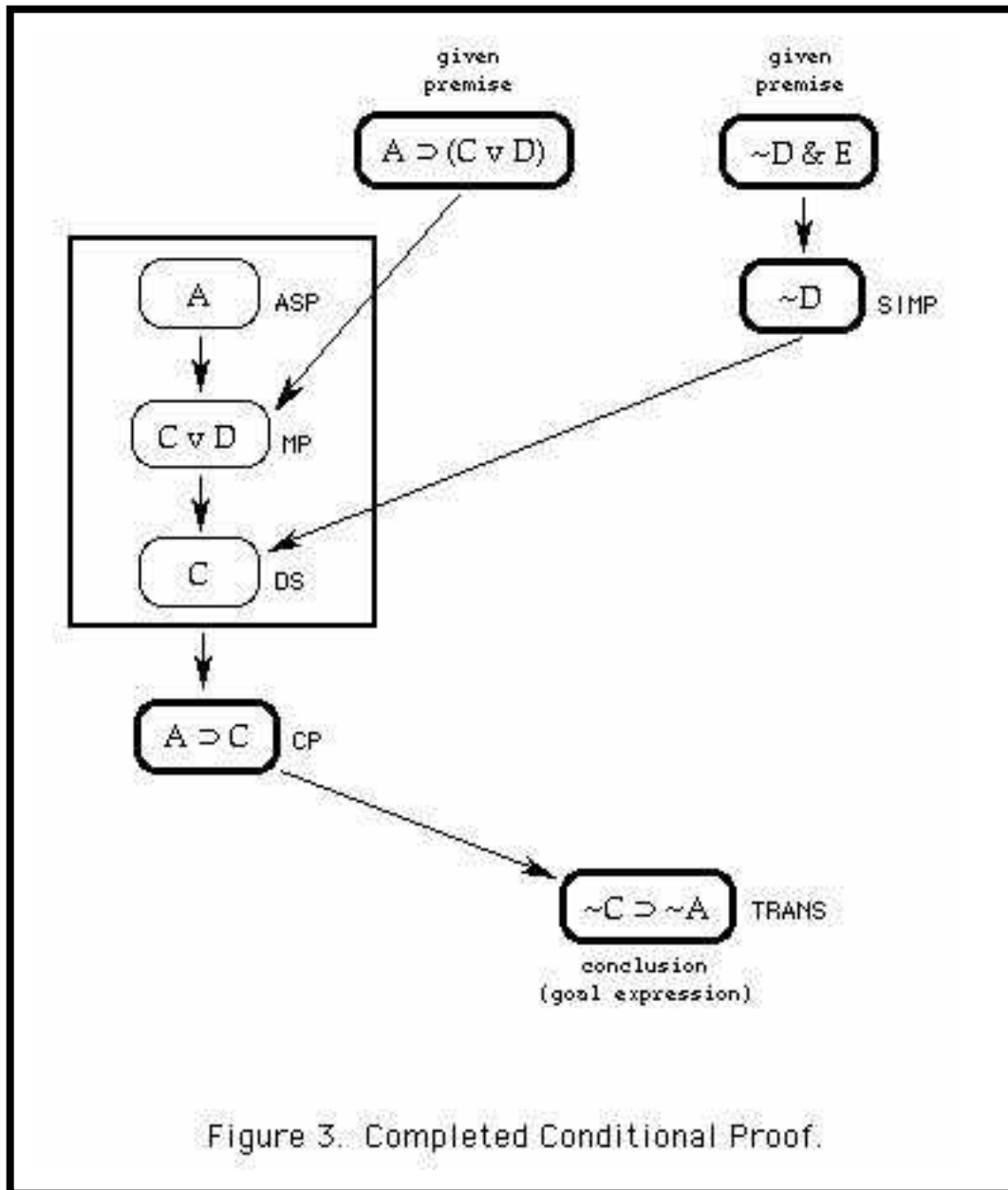


Figure 3. Completed Conditional Proof.

Whether represented graphically or linearly, indirect proof requires the same sort of restrictions that apply to conditional proof. Once an assumption is made, it must be used only to derive a contradiction within the current scope of the assumption. In a graphic format, this derivation occurs within a closed box with only one arrow exiting to the expression to be proved. In place of the conditional's consequent as the last step within the box resides a representation of a contradiction. Geometric shapes are used for this purpose, such as \blacksquare & $\sim\blacksquare$, with these squares ultimately being filled in with some expression.

There are some practical issues to be addressed when using a graphic proof representation. For instance, how large should one draw the boxed area when using conditional proof or indirect proof? How far down the page should the conclusion be drawn, i.e., how much space should be allotted for developing the proof? Judgments concerning these issues can be made at the outset of proof construction. With conditional proof, the need for nested boxes can be assessed by examining the consequent of the conditional being proved. When this consequent is or is equivalent to a conditional, the opportunity for nested applications of conditional proof exists and should be allowed for. In general, students should be encouraged to draw the conclusion at the bottom of the page and to use an entire sheet for constructing a proof. In addition, enough room should be provided for building more than one working backwards chain leading up from the conclusion, for example, where more than one rule can plausibly produce a goal expression. If one chain reaches a dead end, it can be abandoned and an alternate line can be pursued. Such backtracking can be easily carried off when alternative paths are explicitly represented in the proof. As already indicated, intelligent instantiation and simultaneous working forwards can often settle decisions concerning multiple paths before they are even drawn in the proof structure, but space for representing these paths should nevertheless be allocated.

A graphic proof representation can provide a medium for carrying out and documenting the process which generates the final product of the proof. In presenting the working backwards method, Tidman and Kahane remark that students should learn to work backwards with simple problems in their heads but may need to use a piece of scrap paper for longer proofs. This recommendation is consistent with a procedure sometimes taught by instructors, namely, that of making a diagram or other drawing off to the side when discovering the proof and then filling in the proof structure which constitutes the final product. One advantage of using a graphic proof format is that there is in principle no difference between the diagram and the completed proof. The final proof representation readily emerges from the diagram or "scrap work."

Summary

When teaching proof construction with an inference/replacement rule set, students should be taught to work backwards with both types of rules. Students should also be taught to work forwards and should be able to carry these operations out alternately when necessary. As delineated in the three step process above, the working backwards method can be explicitly formulated and can be presented in more rigor than is standardly done in texts teaching proof construction using the inference/replacement rule set. Moreover, a graphic proof representation can supply a flexible environment, sans line numbers, in which proofs can be discovered by working both backwards and forwards. Students can readily explore multiple paths, backtrack when necessary, and develop chains of steps which meet in the middle of the proof. By means of arrow direction and boundary thickness, students can also stay clear about the flow of justification within the proof. Finally, it should be mentioned that a graphic proof representation is consistent in appearance with natural language argument diagrams

and can provide an interesting basis for discussing the transition from premises to conclusions as it occurs within proofs and within natural argumentation.

Notes

I am indebted to Michael Eldridge for helpful comments on an earlier draft of this article.

1. Allan Newell and Herbert Simon, Human Problem Solving (Englewood Cliffs: Prentice Hall, 1972).
2. Richard Burton, "Diagnosing Bugs in a Simple Procedural Skill," in Derek Sleeman and John Seely Brown, eds., Intelligent Tutoring Systems (New York: Academic Press, 1982), pp 157-183.
3. Derek Sleeman, "Assessing Aspects of Competence in Basic Algebra," in Derek Sleeman and John Seely Brown, eds., Intelligent Tutoring Systems (New York: Academic Press, 1982), pp 185-199.
4. This finding is based on undergraduates working problems from Richard Butrick's Deduction and Analysis, revised edition (Washington, D.C.: University Press of America, 1981), Irving Copi's Introduction to Logic, 7th ed. (New York: Macmillan, 1986), Howard Kahane's Logic and Philosophy, 5th ed. (Belmont, CA: Wadsworth, 1986), and Virginia Klenk's Understanding Symbolic Logic, 2nd ed. Englewood Cliffs: Prentice Hall, 1989). See the reports contained in Marvin Croy, "CAI and Rule Applications for Deductive Proof Construction," Collegiate Microcomputer, VI, 1988 51-56; Marvin Croy, "The Use of CAI to Enhance Human Interaction in the Learning of Deductive Proof Construction," Computers and the Humanities, 22(4), 1988, 277-284; and Marvin Croy, "Integrating CAI Development and Empirical Research: Opportunities and Responsibilities," Computerized Logic Teaching Bulletin, IV, 1991, 2-12.
5. Marvin Croy, "Empirical Explorations of Deductive Proof Construction," Computers and Philosophy Newsletter, 4, 1989, 111-127. This and other relevant articles are presently available at: <http://>:

www.uncc.edu/colleges/arts_and_sciences/philosophy/faculty/croy.html.

6 John Anderson, Cognitive Psychology and its Implications, 3rd ed. (New York: Freeman and Company, 1990).

7. See Merrie Bergmann, James Moor, and Jack Nelson, The Logic Book, 3rd ed. (New York: McGraw-Hill, 1998) for a text which presents the intelim rule set and which emphasizes the working backward method. As for texts using the inference-replacement rule set, the working backwards method is not mentioned in Robert Johnson's A Logic Book, Fundamentals of Reasoning (Belmont, CA: Wadsworth Publishing Company, 1999). or in Steven DeHaven's The Logic Course, 2nd ed. (Peterborough, Ontario: Broadview Press, 1998). The method is mentioned with little or no development in Irving Copi and Carl Cohen's An Introduction to Logic, 10th ed. (Englewood Cliffs: Prentice Hall, 1998) and in C. Stephen Laymon's The Power of Logic (Mountain View, CA: Mayfield Publishing Company, 1999). Working backwards is described in somewhat more detail in David Kelly's The Art of Reasoning 3rd expanded ed. (New York: Norton and Company, 1998), Patrick Hurley's A Concise Introduction to Logic, 7th ed. (Belmont, CA: Wadsworth Publishing Company, 2000), and Paul Tidman and Howard Kahane's Logic and Philosophy, A Modern Introduction, 8th ed. (Belmont, CA: Wadsworth, 1999). Virginia Klenk's Understanding Symbolic Logic, 3rd ed. (Englewood Cliffs: Prentice Hall, 1994) provides the most detailed discussion of both working backwards and working forwards during proof construction.

8. As put forward in several texts, the rules included here are Modus Ponens, Modus Tollens, Disjunctive Syllogism, Simplification, Addition, Conjunction, Hypothetical Syllogism, and Constructive Dilemma. Absorption, sometimes included as an inference rule, is actually a rule of replacement given that it expresses a truth functional equivalence and will not be discussed here.

9. To illustrate this complication, consider a proof problem consisting of two premises, 'A \rightarrow (A \rightarrow B)' and '(A \rightarrow B) \rightarrow B', and the conclusion, 'A \rightarrow B'. The three step procedure would initially suggest extracting the conclusion from the first premise via Modus Ponens, and instantiating this rule leads to a fruitless search for 'A'. Once this dead end is discovered, one should backtrack to the conclusion and consider constructing it via Hypothetical Syllogism.

10. The results referred to here are part of a four semester study in which computer program was used to collect and analyze working backwards errors. A report on this analysis is forthcoming in the Journal of Computers in Mathematics and Science Teaching.

11. Working backwards runs into difficulties when confronting proof problems containing inconsistent premises. For example, given 'A' and ' \sim A' as premises and 'B' as the conclusion, it is quickly seen that the conclusion can neither be extracted nor constructed. Nevertheless, even here the standard strategy of using Addition and Disjunctive Syllogism to generate a solution can provide a basis for postulating the last two steps in the proof, thereby characterizing the types of expressions to be aimed for when working forwards.

12. Howard Kahane, Logic and Philosophy: A Modern Introduction, sixth edition, (Belmont, CA: Wadsworth Publishing Company, 1990), p. 78.

13. Ibid., p. 77.

14. Hurley, 2000, p. 399. This warning appears in all editions of A Concise Introduction to Logic.

15. Ibid, p. 388-389.

16. Tidman and Kahane, 1999, p. 99-100.

17. In early editions of Kahane's Logic and Philosophy, Arabic numerals were used for working backwards in place of the Roman numerals advocated in later editions.

Unfortunately, using Arabic numerals for working both forwards and backwards allows two different lines to have the same line number. This probably enhanced confusion over working backwards and may have contributed to Kahane's restriction of its use.

See Kahane, Logic and Philosophy, 1st edition, 1969, p. 59.

18. Richard Scheines and Wilfrid Sieg, "Computer Environments for Proof Construction," Interactive Learning Environments 4(2), 1994, 159-169.

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CURRENT VERSION UPDATES

This article has been made somewhat more accurate since I located and examined a copy of the first edition of Kahane's Logic and Philosophy (1969). The text above now contains references to this work rather than to the second edition (1973) as previously published.

Also, in the original article as published in Teaching Philosophy, references were made to Hurley (1999) which was the date on the book's page proofs from which I was working, but the final date of that edition of A Concise Introduction to Logic was 2000. I haven't updated in accordance with the 2002/2003 edition of Hurley, but could easily, since his position on working backwards hasn't changed. Hurley's position seems strange since he obviously followed Kahane (one of Hurley's examples in this respect changes only the letters in one of Kahane's proofs.), and yet when Tidman steps in to, in a sense, set Kahane straight, Hurley seems not to have noticed.