Graduate Course: Math 8071

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COURSE TITLE: ILL-POSED AND INVERSE PROBLEMS

[1] L. Beilina and M.V. Klibanov, *Approximate Global Convergence and Adaptivity for* Coefficient Inverse Problems, Springer, New York, 2012.

[2] M.V. Klibanov and J. Li, *Inverse Problems and Carleman Estimates: Global Uniqueness, Global Convergence and Experimental Data*, De Gruyter 2021.

1 Some Notations and Definitions

The theory of Ill-Posed Problems addresses the following fundamental question: *How to obtain a good approximation for the solution of an ill-posed problem in a stable way?* Roughly speaking, a numerical method, which provides a stable and accurate solution of an ill-posed problem, is called the *regularization* method for this problem.

We now briefly introduce some common notations which will be used throughout this course. We work in this course only with real valued functions. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We will always assume in our analytical derivations that its boundary $\partial\Omega \in C^3$, although we will work with piecewise smooth boundaries in numerical studies. This is one of natural discrepancies between the theory and its numerical implementation, which always exist in computations. Let $u(x), x = (x_1, ..., x_n) \in \Omega$ be a k times continuously differentiable function defined in Ω. Denote

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_n} x_n}, \quad |\alpha| = \alpha_1 + \ldots + \alpha_n$$

the partial derivative of the order $|\alpha| \leq k$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index with integers $\alpha_i \geq 0$. Denote $C^k\left(\overline{\Omega}\right)$ the Banach space of functions $u(x)$ which are continuous in the closure $\overline{\Omega}$ of the domain Ω together with their derivatives $D^\alpha u, |\alpha| \leq m$. The norm in this space is defined as

$$\|u\|_{C^k(\overline{\Omega})} = \sum_{|\alpha| \leq m} \sup_{x \in \Omega} |D^\alpha u(x)| < \infty.$$

By definition $C^0\left(\overline{\Omega}\right) = C\left(\overline{\Omega}\right)$ is the space of functions continuous in $\overline{\Omega}$ with the norm

$$\|u\|_{C(\overline{\Omega})} = \sup_{x \in \Omega} |u(x)|.$$

We also introduce Hölder spaces $C^{k+\alpha}\left(\overline{\Omega}\right)$ for any number $\alpha \in (0, 1)$. The norm in this space is defined as

$$\|u\|_{C^{k+\alpha}(\overline{\Omega})} := |u|_{k+\alpha} := \|u\|_{C^k(\overline{\Omega})} + \sup_{x,y \in \Omega,\ x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha},$$

provided that the last term in finite. It is clear that if the function $u \in C^{k+1}\left(\overline{\Omega}\right)$, then $u \in C^{k+\alpha}\left(\overline{\Omega}\right), \forall \alpha \in (0, 1)$ and

$$|u|_{k+\alpha} \leq C \|u\|_{C^{k+1}(\overline{\Omega})}, \ \forall u \in C^{k+1}\left(\overline{\Omega}\right),$$

where $C = C(\Omega, \alpha) > 0$ is a constant independent on the function u. Sometimes we also use the notion of Hölder spaces for infinite domains. Let D be such a domain. It is convenient for us to say that the function $u \in C^{k+\alpha}(D)$ if $u \in C^{k+\alpha}\left(\overline{\Omega}\right)$ for every bounded subdomain $\Omega \subset D$. Although sometimes people say that $u \in C^{k+\alpha}\left(\overline{D}\right)$ if the above Hölder norm in $\overline{D}$ is finite, we do not use this definition here.

Consider the Sobolev space $H^k(\Omega)$ of all functions with the norm defined as

$$\|u\|^2_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^2 \, dx < \infty,$$

where $D^{\alpha}u$ are weak derivatives of the function u. By the definition $H^0(\Omega) = L_2(\Omega)$. It is well known that $H^k(\Omega)$ is a Hilbert space with the inner product defined as

$$(u,v)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} \int_{\Omega} D^{\alpha}uD^{\alpha}v dx.$$

Let $T > 0$ and $\Gamma \subseteq \partial\Omega$ be a part of the boundary $\partial\Omega$ of the domain Ω. We will use the following notations throughout this book

$$Q_T = \Omega \times (0,T), S_T = \partial\Omega \times (0,T), \Gamma_T = \Gamma \times (0,T), D_T^{n+1} = \mathbb{R}^n \times (0,T).$$

The space $C^{2k,k}\left(\overline{Q}_T\right)$ is defined as the set of all functions $u(x,t)$ having derivatives $D_x^{\alpha}D_t^{\beta}u \in C\left(\overline{Q}_T\right)$ with $|\alpha| + 2\beta \leq 2k$ and with the following norm

$$\|u\|_{C^{2k,k}\left(\overline{Q}_T\right)} = \sum_{|\alpha|+2\beta\leq 2k} \max_{\overline{Q}_T} \left|D_x^{\alpha}D_t^{\beta}u(x,t)\right|.$$

The Hölder space $C^{2k+\alpha,k+\alpha/2}\left(\overline{Q}_T\right), \alpha \in (0,1)$ is defined similarly.

We now remind some definitions from the standard course of Functional Analysis.

Definition 1.1. Let B be a Banach space. The set $V \subset B$ is called *precompact* set if every sequence $\{x_n\}_{n=1}^{\infty} \subseteq V$ contains a fundamental subsequence (i.e., the Cauchy subsequence).

Although by the Cauchy criterion the subsequence of Definition 1.1.3.1 converges to a certain point, there is no guarantee that this point belongs to the set V. If we consider the closure of V, i.e. the set $\overline{V}$, then all limiting points of all convergent sequences in V would belong to V. Therefore, we arrive at

Definition 1.2. Let B be a Banach space. The set $V \subset B$ is called *compact* set if V is a closed set, $V = \overline{V}$, every sequence $\{x_n\}_{n=1}^{\infty} \subseteq V$ contains a fundamental subsequence and the limiting point of this subsequence belongs to the set V.

Definition 1.3. Let B_1 and B_2 be two Banach spaces, $U \subseteq B_1$ be a set and $A : U \to B_2$ be a continuous operator. The operator A is called a *compact operator* or *completely continuous* operator if it maps any bounded subset $U' \subseteq U$ in a precompact set in B_2. Clearly if U' is a closed set, then $A(U')$ is a compact set.

The following theorem is well known under the name of Ascoli-Archela Theorem (more general formulations of this theorem can also be found).

Theorem 1.1. *The set of functions $\mathcal{M} \subset C\left(\overline{\Omega}\right)$ is a compact set if and only if it is uniformly bounded and equicontinuous. In other words, if the following two conditions are satisfied:*

1. There exists a constant $M > 0$ such that

$$\|f\|_{C\left(\overline{\Omega}\right)} \leq M, \ \forall f \in \mathcal{M}.$$

2. For any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|f(x) - f(y)| < \varepsilon, \quad \forall x, y \in \{|x-y| < \delta\} \cap \overline{\Omega}, \ \forall f \in \mathcal{M}.$$

In particular, because of some generalizations of this theorem, any bounded set in $C^k\left(\overline{\Omega}\right)$ (or $H^k(\Omega)$), $k \geq 1$ is a compact set in $C^p\left(\overline{\Omega}\right)$ (respectively $H^p(\Omega)$) for $p \in$

$[0, k-1]$. We also remind one of the Sobolev embedding theorems for spaces $H^k(\Omega)$. Let $[n/2]$ be the least integer which does not exceed $n/2$.

Theorem 1.2. *Suppose that $k > [n/2] + m$, the domain Ω is bounded and $\partial\Omega \in C^k$. Then $H^k(\Omega) \subset C^m(\overline{\Omega})$ and $\|f\|_{C^m(\overline{\Omega})} \le C\|f\|_{H^k(\Omega)}, \forall f \in H^k(\Omega)$, where the constant $C = C(\Omega, k, m) > 0$ depends only on Ω, k, m. In addition, any bounded set in $H^k(\Omega)$ is a precompact set in $C^m(\overline{\Omega})$.*

Theorem 1.2 actually claims that the space $H^k(\Omega)$ is compactly embedded in the space $C^m(\overline{\Omega})$. "Compactly embedded" means that $\|f\|_{C^m(\overline{\Omega})} \le C\|f\|_{H^k(\Omega)}, \forall f \in H^k(\Omega)$ and any bounded set in $H^k(\Omega)$ is a precompact set in $C^m(\overline{\Omega})$. In other words, any sequence bounded in $H^k(\Omega)$ contains a subsequence, which converges in $C^m(\overline{\Omega})$, although the limit of this subsequence does not necessarily belong to $H^k(\Omega)$.

2 Some Examples of Ill-Posed Problems

Example 1 (J. Hadamard). We now describe the classical example of Hadamard. Consider the Cauchy problem for the Laplace equation for the function $u(x, y)$,

$$\Delta u = 0, \quad x \in (0, \pi), \quad y > 0, \tag{2.1}$$

$$u(x, 0) = 0, u_y(x, 0) = \alpha \sin(nx), \tag{2.2}$$

where $n > 0$ is an integer. It is well known that the Cauchy problem for a general elliptic equation with "good" variable coefficients has at most one solution (although it might not have solutions at all). The unique solution of the problem (2.1), (2.2) is

$$u(x, y) = \frac{\alpha}{n}\sinh(ny)\sin(nx). \tag{2.3}$$

Choose sufficiently small numbers $\varepsilon > 0, \alpha = \alpha(\varepsilon) > 0$ and a number $y := y_0 > 0$. Let in (2.3) $x \in (0, \pi)$. Since the function

$$\sinh(ny_0) = \frac{e^{ny_0}\left(1 + e^{-2ny_0}\right)}{2}$$

grows exponentially as $n \to \infty$, then it is clear from (2.3) that for any pair of reasonable functional spaces $C^k[0, \pi], L_2[0, \pi], H^k[0, \pi]$, etc. one can choose such two numbers $c > 0, n_0 > 0$ depending only on numbers ε, α, y_0 that

$$||\alpha \sin(nx)||_1 < \varepsilon, \quad \forall n \ge n_0,$$

$$\|u(x, y_0)\|_2 = ||\frac{\alpha}{n}\sinh(ny_0)\sin(nx)||_2 > c, \quad \forall n \ge n_0,$$

where $\|\cdot\|_1$ is the norm in one of those spaces and $\|\cdot\|_2$ is the norm in another one.

The above example demonstrates that although both the Dirichlet and Neumann boundary data are small, any reasonable norm of the solution is still large. In other words, this is a manifestation of a high instability of this problem. Based on this example, Hadamard has concluded that it makes no sense to consider unstable problems. However, his conclusion was an exaggeration. Indeed, unstable problems arise in many applications. Being inspired by applications to geophysics, Tikhonov has proposed in 1943 the fundamental concept for solving unstable problems.

Example 2. **Differentiation of a function given with a noise**. The differentiation of functions given by analytic formulas is a trivial exercise. In the reality, however, functions are often measured in experiments. Since experimental data always contain noise, then measured functions are given with a noise. Quite often it is necessary to differentiate these noisy functions. We demonstrate now that the problem of the differentiation of noisy functions is unstable. Suppose that the function $f(x), x \in [0,1]$ is given with a noise. In other words, suppose that instead of $f(x) \in C^1[0,1]$ the following function $f_\delta(x)$ is given

$$f_\delta(x) = f(x) + \delta f(x), x \in [0,1],$$

where $\delta f(x)$ is the noisy component. Let $\delta > 0$ be a small parameter characterizing the level of noise. We assume that the noisy component is small, $\|\delta f\|_{C[0,1]} \leq \delta$. The problem of calculating the derivative $f_\delta'(x)$ is unstable. Indeed, let for example

$$\delta f(x) = \frac{\sin(n^2 x)}{n},$$

where $n > 0$ is a large integer. Then the $C[0,1]$-norm of the noisy component is small,

$$\|\delta f\|_{C[0,1]} \leq \frac{1}{n}.$$

However, the difference between derivatives of noisy and exact functions

$$f_\delta'(x) - f'(x) = n \cos n^2 x$$

is not small in any reasonable norm.

We now describe a simple regularization method of stable calculation of derivatives. The idea is that the step size h in the corresponding finite difference should be connected with the level of noise δ. Thus, h cannot be made arbitrary small, as it is the case of the classic definition of the derivative. We obviously have

$$f_\delta'(x) \approx \frac{f(x+h) - f(x)}{h} + \frac{\delta f(x+h) - \delta f(x)}{h}. \tag{2.4}$$

The first term in the right hand side of (2.4) is close to the exact derivative $f'(x)$, if h is small enough. The second term, however, comes from the noise. Hence, we need to balance these two terms via an appropriate choice of $h = h(\delta)$. Obviously

$$\left| f_\delta'(x) - \frac{f(x+h) - f(x)}{h} \right| \leq \frac{2\delta}{h}.$$

Hence, we should choose $h = h(\delta)$ such that

$$\lim_{\delta \to 0} \frac{2\delta}{h(\delta)} = 0.$$

For example, let $h(\delta) = \delta^\mu$, where $\mu \in (0,1)$. Then

$$\lim_{\delta \to 0} \left| f_\delta'(x) - \frac{f(x+h) - f(x)}{h} \right| \leq \lim_{\delta \to 0} \left(2\delta^{1-\mu} \right) = 0.$$

Hence, the problem becomes stable for this choice of the grid step size $h(\delta) = \delta^{\mu}$. This means that $h(\delta)$ is the regularization parameter here. There are many practical methods in the literature designed for stable differentiation. For example, one can approximate the function $f_{\delta}(x)$ via cubic B splines and differentiate this approximation then. However, the number of these splines should not be too large: otherwise, the problem would become unstable. So, the number of cubic B splines is the regularization parameter in this case, and its intuitive meaning is the same as the meaning of the number $1/h(\delta)$. A more detailed description of regularization methods for the differentiation procedure is outside of the scope of this book.

Let $\Omega \subset \mathbb{R}^n$ is a bounded domain and the function $K(x,y) \in C\left(\overline{\Omega} \times \overline{\Omega}\right)$. Recall that the equation

$$g(x) + \int\limits_{\Omega} K(x,y)\, g(y)\, dy = p(x), x \in \Omega, \tag{2.5}$$

where $p(x)$ is a bounded function, is called *integral equation of the second kind.* These equations are considered quite often in the classic theory of PDEs. The classical Fredholm theory works for these equations. Next, let $\Omega' \subset \mathbb{R}^n$ be a bounded domain and the function $K(x,y) \in C\left(\overline{\Omega} \times \overline{\Omega}\right)$. Unlike (2.5), the equation

$$\int\limits_{\Omega} K(x,y)\, g(y)\, dy = p(x), x \in \Omega' \tag{2.6}$$

is called the integral equation of the first kind. The Fredholm theory does not work for such equations. The problem of solution of equation (2.6) is an ill-posed problem, see Example 3.

Example 3. Integral equation of the first kind. Consider equation (2.6). The function $K(x,y)$ is called *kernel* of the integral operator. Equation (2.6) can be rewritten in the form

$$Kf = p, \tag{2.7}$$

where $K : C\left(\overline{\Omega}\right) \to C\left(\overline{\Omega}'\right)$ is the integral operator in (2.6). It is well known from the standard Functional Analysis course that K is a compact operator. We now show that the problem (2.7) is ill-posed. Let $\Omega = (0,1), \Omega' = (a,b)$. Replace the function f with the function $f_n(x) = f(x) + \sin nx$. Then

$$\int\limits_{0}^{1} K(x,y)\, f_n(y)\, dy = g_n(x),\ \ x \in (0,1), \tag{2.8}$$

where $g_n(x) = p(x) + p_n(x)$ and

$$p_n(x) = \int\limits_{0}^{1} K(x,y) \sin nydy.$$

By the Lebesque lemma

$$\lim_{n\to\infty} \|p_n\|_{C[a,b]} = 0.$$

However, it is clear that

$$\|f_n(x) - f(x)\|_{C[0,1]} = \|\sin nx\|_{C[0,1]}$$

is not small for large n.

Example 4. The case of a general compact operator. We now describe an example of a general ill-posed problem. Let H_1 and H_2 be two Hilbert spaces with $\dim H_1 = \dim H_2 = \infty$. We remind that a sphere in an infinitely dimensional Hilbert space is not a compact set. Indeed, although the orthonormal basis in this space belongs to the unit sphere, it does not contain a fundamental subsequence.

Theorem 2.1. *Let $G = \left\{ \|x\|_{H_1} \leq 1 \right\} \subset H_1$. Let $A : G \to H_2$ be a compact operator and let $R(A) := A(G)$ be its range. Consider an arbitrary point $y_0 \in R(A)$. Let $\varepsilon > 0$ be a number and $U_\varepsilon(y_0) = \left\{ y \in H_2 : \|y - y_0\|_{H_2} < \varepsilon \right\}$. Then there exists a point $y \in U_\varepsilon(y_0) \diagdown R(A)$. If, in addition, the operator A is one-to-one, then the inverse operator $A^{-1} : R(A) \to G$ is not continuous. Hence, the problem of the solution of the equation*

$$A(x) = z, x \in G, z \in R(A) \tag{2.9}$$

is unstable, i.e. this is an ill-posed problem.

Proof. First, we prove the existence of a point $y \in U_\varepsilon(y_0) \diagdown R(A)$. Assume to the contrary, i.e. assume that $U_\varepsilon(y_0) \subset R(A)$. Let $\{y_n\}_{n=1}^\infty \subset H_2$ be an orthonormal basis in H_2. Then the sequence

$$\left\{ y_0 + \frac{\varepsilon}{2} y_n \right\}_{n=1}^\infty := \{z_n\}_{n=1}^\infty \subset \left\{ \|y - y_0\| = \frac{\varepsilon}{2} \right\} \subset U_\varepsilon(y_0).$$

We have

$$\|z_n - z_m\|_{H_2} = \frac{\varepsilon}{\sqrt{2}}.$$

Hence, the sequence $\{z_n\}_{n=1}^\infty$ does not contain a fundamental subsequence. Therefore, $U_\varepsilon(y_0)$ is not a precompact set in H_2. On the other hand, since G is a closed bounded set and A is a compact operator, then $R(A)$ is a compact set. Hence, $U_\varepsilon(y_0)$ is a precompact set. We got a contradiction, which proves the first assertion of this lemma.

We now prove the second assertion. Assume to the contrary: that the operator $A^{-1} : R(A) \to G$ is continuous. By the definition of the operator A we have $A^{-1}(R(A)) = G$. Since $R(A)$ is a compact set in H_2, then the continuity of A^{-1} implies that G is a compact set in H_1, which is not true. $\square$

We now summarize some conclusions which follow from Theorem 2.1. By this theorem the set $R(A)$ is not dense everywhere. Therefore, the question about the existence of the solution of either of equations (2.7) or (2.9) does not make an applied sense. Indeed, since the set $R(A)$ is not dense everywhere, then it is very hard to describe a set of values y belonging to this set. As an example, consider the case when the kernel $K(x, y) \in C([a, b] \times [0, 1])$ in equation (2.8) is an analytic function of the real variable $x \in (a, b)$. Then the right hand side $p(x)$ of equation (2.6) should also be analytic with respect to $x \in (a, b)$. However, in applications the function $p(x)$ is a result of measurements, it is given only at a number of discrete points and contains noise. Clearly it is impossible to determine from this information whether the function $p(x)$ is analytic or not. Hence, we got the following important

Conclusion. Assuming that conditions of Theorem 2.1 are satisfied, the problem of solving equation (2.9) is ill-posed in the following terms: **(a)** the proof of an existence theorem makes no applied sense, and **(b)** small fluctuations of the right hand side y can lead to large fluctuations of the solution x, i.e. the problem is unstable.

Example 5. A Coefficient Inverse Problem (CIP). Let the function $a(x) \in C^{\alpha}(\mathbb{R}^n), \alpha \in (0,1)$ and $a(x) = 0$ outside of the bounded domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^3$. Consider the following Cauchy problem

$$u_t = \Delta u + a(x)\, u, \ \ (x,t) \in D_T^{n+1}, \tag{2.10}$$

$$u(x,0) = f(x). \tag{2.11}$$

Here the function $f(x) \in C^{2+\alpha}(\mathbb{R}^n)$ has a finite support in $\mathbb{R}^n$. Although less restrictive conditions on f can also be imposed, we are not doing this here for brevity. Another option for the initial condition is

$$f(x) = \delta(x - x_0), \tag{2.12}$$

where the source position $x_0 \notin \overline{\Omega}$. Throughout the book we will always assume that the source is located outside of the domain of interest Ω. The reason of doing this is that we do not want to work with singularities, since CIPs are very complicated even without singularities. The second reason is that in the majority of applications sources are indeed located outside of domains of interest.

Statement of a Coefficient Inverse Problem. Assume that the function $a(x)$ is unknown inside of the domain Ω. Determine this function for $x \in \Omega$ assuming that the following function $g(x,t)$ is known

$$u\mid_{S_T} = g(x,t). \tag{2.13}$$

The function $g(x,t)$ is an additional boundary condition. This function can be interpreted as a result of measurements: one is measuring the function $u(x,t)$ at the boundary of the domain Ω in order to reconstruct the function $a(x)$ inside of Ω. Indeed, if the coefficient $a(x)$ would be known in the entire space $\mathbb{R}^n$, then one would uniquely determine the function $u(x,t)$ in D_T^{n+1}. But since $a(x)$ is unknown, then the function $u\mid_{S_T}$ can be determined only via measurements. Note that since $a(x) = 0$ outside of Ω, then one can uniquely solve the following initial boundary value problem outside of Ω,

$$u_t = \Delta u, (x,t) \in (\mathbb{R}^n \diagdown \Omega) \times (0,T),$$

$$u(x,0) = f(x), x \in \mathbb{R}^n \diagdown \Omega,$$

$$u\mid_{S_T} = g(x,t).$$

Hence, one can uniquely determine the Neumann boundary condition for the function u at the boundary $\partial\Omega$, and we will use this consideration throughout this book. Thus, the following function $g_1(x,t)$ is known along with the function $g(x,t)$ in (2.13)

$$\partial_n u\mid_{S_T} = g_1(x,t).$$

This CIP has direct applications in imaging of the *turbid media* using light propagation. In a turbid medium, photons of light, originated by a laser, propagate randomly in the diffuse manner. In other words, they experience many random scattering events. Two examples of turbid media are smog and flames in the air. The most popular example is the biological tissue, including human organs. Assuming that the diffusion coefficient $D = 1$, we obtain that in (2.10) the coefficient $a(x) = -\mu_a(x) \leq 0$, where $\mu_a(x)$ is the absorption coefficient of the medium. The case of smog and flames has military applications. Since $\mu_a(x) = \infty$ for any metallic target, then imaging small inhomogeneities with large values of the absorption coefficient might lead to detection of those targets. In the case of medical applications, high values of $\mu_a(x)$ usually correspond to malignant legions. Naturally one is interested to image those legions noninvasively via solving a CIP.

Thus, in both applications the main interest is in imaging of small sharp abnormalities, rather than in imaging of a slowly changing background function. Furthermore, to correctly identify those abnormalities, one needs to image with a good accuracy the value of the coefficient $\mu_a(x)$ within them. Naturally, in both applications one should use the function (2.12) as the initial condition. In this case x_0 is the location of the light source.

We now show that this CIP is an ill-posed problem. Let the function u_0 be the fundamental solution of the heat equation $u_{0t} = \Delta u_0$,

$$u_0(x,t) = \frac{1}{\left(2\sqrt{\pi t}\right)^n} \exp\left(-\frac{|x|^2}{4t}\right).$$

It is well known that the function u has the following integral representation

$$u(x,t) = \int_{\mathbb{R}^n} u_0(x-\xi,t) f(\xi)\, d\xi + \int_0^t \int_\Omega u_0(x-\xi,t-\tau)\, a(\xi)\, u(\xi,\tau)\, d\tau. \tag{2.14}$$

Because of the presence of the integral

$$\int_0^t (\cdot)\, d\tau,$$

(2.14) is a Volterra-like integral equation of the second kind. Hence, it can be solved as:

$$u(x,t) = \int_{\mathbb{R}^n} u_0(x-\xi,t) f(\xi)\, d\xi + \sum_{n=1}^{\infty} u_n(x,t), \tag{2.15}$$

$$u_n(x,t) = \int_0^t \int_\Omega u_0(x-\xi,t-\tau)\, a(\xi)\, u_{n-1}(\xi,\tau)\, d\tau.$$

One can prove that each function $u_n \in C^{2+\alpha,1+\alpha/2}\left(\overline{D}_T^{n+1}\right)$ and

$$\left|D_x^\beta D_t^k u_n(x,t)\right| \leq \frac{(Mt)^n}{n!}, \quad |\beta| + 2k \leq 2, \tag{2.16}$$

where $M = \|a\|_{C^\alpha\left(\overline{\Omega}\right)}$. In the case when $f = \delta(x-x_0)$ the first term in the right hand side of (2.15) should be replaced with $u_0(x-x_0,t)$. Let $u_0^f(x,t)$ be the first term of the right hand side of (2.15) and $v(x,t) = u(x,t) - u_0^f(x,t)$. Using (2.16), one can rewrite (2.15) as

$$v(x,t) = \int_0^t \int_\Omega u_0(x-\xi,t-\tau)\left(a(\xi)\, u_0^f(\xi,\tau) + P(a)(\xi,\tau)\right) d\xi d\tau, \tag{2.17}$$

where $P(a)$ is a nonlinear operator applied to the function a. It is clear from (2.15)-(2.17) that the operator $P : C^{\alpha}\left(\overline{\Omega}\right) \to C^{2+\alpha,1+\alpha/2}\left(\overline{Q}_T\right)$ is continuous. Setting in (2.17) $(x,t) \in S_T$, recalling (2.13) and denoting denoting $\overline{g}(x,t) = g(x,t) - u_0^f(x,t)$, we obtain a nonlinear integral equation of the first kind with respect to the unknown coefficient $a(x)$

$$\int_0^t \int_\Omega u_0(x-\xi, t-\tau)\left(u_0^f(\xi,\tau)\, a(\xi) + P(a)(\xi,\tau)\right) d\xi d\tau = \overline{g}(x,t)\,, \quad (x,t) \in S_T. \quad (2.18)$$

Let $A(a)$ be the operator in the left hand side of (2.18). Let $H_1 = L_2(\Omega)$ and $H_2 = L_2(S_T)$. Consider now the set U of functions defined as

$$U = \left\{a : a \in C^{\alpha}\left(\overline{\Omega}\right), \|a\|_{C^{\alpha}\left(\overline{\Omega}\right)} \le M\right\} \subset H_1.$$

Since the $L_2(\Omega)$ $-$norm is weaker than the $C^{\alpha}\left(\overline{\Omega}\right)$ $-$norm, then U is a bounded set in H_1. Using (2.16) and Theorem 2.1, one can prove that $A : U \to C(S_T)$ is a compact operator. Since the norm in $L_2(S_T)$ is weaker than the norm in $C(S_T)$, then $A : U \to H_2$ is also a compact operator. Hence, Theorem 2.1 implies that the problem of solution of the equation

$$A(a) = g, a \in U \subset H_1, g \in H_2$$

is ill-posed in terms of the above Conclusion.

3 The Foundational Theorem of A.N. Tikhonov

This theorem "restores" stability of unstable problems, provided that uniqueness theorems hold for such problems. The original motivation for this theorem came from the collaboration of Tikhonov with geophysicists. To his surprise, Tikhonov has learned that geophysicists successfully solve problems which are unstable from the mathematical standpoint. Naturally, Tikhonov was puzzled by this. This puzzle has prompted him to explain that "matter of fact" stability of unstable problems from the mathematical standpoint. He has observed that geophysicists have worked with rather simple models, which included only a few abnormalities. In addition, they knew very well ranges of parameters they have worked with. Also, they knew that the functions, which they have reconstructed from measured date, had only very few oscillations. In other words, they have reconstructed only rather simple media structures. On the other hand, the Ascoli-Archela Theorem basically requires *a priori* known upper bounds of both the function and its first derivatives. Clearly there is a connection between the number of oscillations per a bounded set in $\mathbb{R}^n$ and the upper bound of the modulus of the gradient of the corresponding function. These observations have made Tikhonov to believe that actually geophysicists have worked with compact sets. This was the starting point for the formulation of the foundational Tikhonov theorem (below). In particular, this means that in an ill-posed problem one should not expect to reconstruct a complicated fine structure of the medium of interest. Rather, one should expect to reconstruct rather simple features of this medium.

The key idea of Tikhonov was that to restore stability of an unstable problem, one should solve this problem on a compact set. The question is then whether it is reasonable to assume that the solution belongs to a specific compact set. The answer on this question lies in applications. Indeed, by Ascoli-Archela theorem 1.1 an example of a compact set in the space $C\left(\overline{\Omega}\right)$ is the set of all functions from $C^1\left(\overline{\Omega}\right)$ which are bounded together with the absolute values of their first derivatives by an *a priori* chosen constant. On the other hand, it is very often known in any specific application that functions of ones interest are bounded by a certain known constant. In addition, it is also known that those functions do not have too many oscillations, which is guaranteed by an a priori bound imposed on absolute values of their first derivatives. These bounds should be uniform for all functions under consideration. Similar arguments can be brought up in the case of other conventional functional spaces, like, e.g. $C^k\left(\overline{\Omega}\right), H^k\left(\Omega\right)$, etc. Another expression of these thoughts, which is often used in applications, is that the admissible range of parameters is known in advance. On the other hand, because of the compact set requirement of Theorem 3.1, the foundational Tikhonov theorem essentially requires a higher smoothness of sought for functions than one would originally expect. The latter is the true underlying reason why computed solutions of ill-posed problems usually look smoother than the original ones. In particular, sharp boundaries usually look as smooth ones.

Although the proof of Theorem 3.1 is short and simple, this result is one of only a few backbones of the entire theory of Ill-Posed Problems.

Theorem 3.1. (Tikhonov, 1943). *Let B_1 and B_2 be two Banach spaces. Let $U \subset B_1$ be a compact set and $F : U \to B_2$ be a continuous operator. Assume that the operator F is one-to-one. Let $V = F(U)$. Then the inverse operator $F^{-1} : V \to U$ is continuous.*

Proof. Assume the opposite: that the operator F^{-1} is not continuous on the set V. Then there exists a point $y_0 \in V$ and a number $\varepsilon > 0$ such that for any $\delta > 0$ there exists a point y_δ such that although $\|y_\delta - y_0\|_{B_2} < \delta$, still $\|F^{-1}(y_\delta) - F^{-1}(y_0)\|_{B_1} \geq \varepsilon$. Hence, there exists a sequence $\{\delta_n\}_{n=1}^{\infty}$, $\lim_{n\to\infty} \delta_n = 0^+$ and the corresponding sequence $\{y_n\}_{n=1}^{\infty} \subset V$ such that

$$\|y_{\delta_n} - y_0\|_{B_2} < \delta_n, \left\|F^{-1}(y_n) - F^{-1}(y_0)\right\|_{B_1} \geq \varepsilon. \tag{3.1}$$

Denote

$$x_n = F^{-1}(y_n), x_0 = F^{-1}(y_0). \tag{3.2}$$

Then

$$\|x_n - x_0\|_{B_1} \geq \varepsilon. \tag{3.3}$$

Since U is a compact set and all points $x_n \in U$, then one can extract a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subseteq \{x_n\}_{n=1}^{\infty}$ from the sequence $\{x_n\}_{n=1}^{\infty}$. Let $\lim_{k\to\infty} x_{n_k} = \overline{x}$. Then $\overline{x} \in U$. Since $F(x_{n_k}) = y_{n_k}$ and the operator F is continuous, then by (3.1) and (3.2) $F(\overline{x}) = y_0 = F(x_0)$. Since the operator F is one-to one, we should have $\overline{x} = x_0$. However, by (3.3) $\|\overline{x} - x_0\|_{B_1} \geq \varepsilon$. We got a contradiction. $\square$

4 Classical Correctness and Conditional Correctness

The notion of the classical correctness is called sometimes *Correctness by Hadamard.*

Definition 4.1. Let B_1 and B_2 be two Banach spaces. Let $G \subseteq B_1$ be an open set and $F : G \to B_2$ be an operator. Consider the equation

$$F(x) = y, \ x \in G. \tag{4.1}$$

The problem of solution of equation (4.1) is called *well-posed by Hadamard,* or simply *well-posed,* or *classically well-posed* if the following three conditions are satisfied:

1. For any $y \in B_2$ there exists a solution $x = x(y)$ of equation (4.1) (existence theorem).

2. This solution is unique (uniqueness theorem).

3. The solution $x(y)$ depends continuously on y. In other words, the operator $F^{-1} : B_2 \to B_1$ is continuous.

Thus, the well-posedness by Hadamard means the existence of the solution of the operator equation (4.1) for any right hand side y. This solution should be unique. In addition, it should depend on the data y continuously. All classical boundary value problems for PDEs, which are studied in the standard PDE course, satisfy these criteria and are, therefore, well-posed by Hadamard.

If equation (4.1) does not satisfy to at least one these three conditions, then the problem (4.1) is called *ill-posed.* The most pronounced feature of an ill-posed problem is its instability, i.e. small fluctuations of y can lead to large fluctuations of the solution x.

Since the experimental data are always given with a random noise, we need to introduce the notion of the error in the data. In practice this error is always due to that random noise as well as due to an inevitable discrepancy between the mathematical model and the reality. However, we do not assume the randomness of y in (4.1). Let $\delta > 0$ be a small number. We say that the right hand side of equation (4.1) is given with an error of the level δ if $\|y^* - y\|_{B_2} \leq \delta$, where y^* is the exact value of y, which has no error.

Definition 4.2. Let B_1 and B_2 be two Banach spaces. Let $G \subset B_1$ be an *a priori* chosen set of the form $G = \overline{G}_1$, where G_1 is an open set in B_1. Let $F : G \to B_2$ be a continuos operator. Suppose that the right hand side of equation (4.1) $y := y_\delta$ is given with an error of the level $\delta > 0$, where δ is a small number, $\|y^* - y_\delta\|_{B_2} \leq \delta$. Here y^* is the ideal noiseless data y^*. The problem (4.1) is called *conditionally well-posed on the set* G, or *well-posed by Tikhonov* on the set G if the following three conditions are satisfied:

1. It is *a priori* known that there exists an ideal solution $x^* = x^*(y^*) \in G$ of this problem for the ideal noiseless data y^*.

2. The operator $F : G \to B_2$ is one-to-one.

3. The inverse operator F^{-1} is continuous on the set $F(G)$.

Definition 4.3. The set G of Definition 4.2 is called *correctness set* for the problem (4.1).

We point out that, unlike the Classical Well-Posedness, the Conditional Well-Posedness, does not require the correctness set G to coincide with the entire Banach space B_1. Likewise, Definition 4.2 does not require a proof of an existence theorem, unlike the classical case. Indeed, it follows from Theorem 2.1 that it is hopeless to prove such a theorem for equation (2.9). In addition, such a result would not have a practical meaning. For comparison, recall that a significant part of the classical PDE theory is devoted to proofs of existence theorems, as it is required by the definition of the Classical Well-Posedness. On the other hand, in the definition of the Conditional Well-Posedness the existence is assumed *a priori.* Still, the existence is assumed not for every y in (4.1) but only for an *ideal*, noiseless $y := y^*$. The assumption of the existence of the ideal solution x^* is a very important notion of the theory of Ill-Posed Problems. Neither the ideal right hand side y^* nor the ideal solution x^* are never known in applications. This is because of the presence of the noise in any experiment. Still, this assumption is a quite reasonable one, because actually it tells one that the physical process is indeed in place and that the mathematical model, which is described by the operator F, governs this process accurately.

The second condition in Definition 4.2 means uniqueness theorem. Combined with Theorem 2.1, this condition emphasizes the importance of uniqueness theorems for the theory of Ill-Posed Problems.

The third condition in Definition 4.2 means that the solution of the problem (4.1) is stable with respect to small fluctuations of the right hand side y, as long as $x \in G$. This goes along well with Theorem 3.1. In other words, the third condition restores the most important feature: stability. The requirement that the correctness set $G \subset B_1$ is not conventionally used in the classical theory of PDEs. In other words, the requirement of x belonging to a "special" subset of B_1 is not imposed in classically well-posed problems.

Motivated by the above arguments, Tikhonov has introduced

The Fundamental Concept of Tikhonov. This concept consists of the following three conditions which should be in place when solving the ill-posed problem (4.1):

1. One should *a priori* assume that there exists an ideal exact solution x^* of equation (4.1) for an ideal noiseless data y^*.

2. The correctness set G should be chosen *a priori*, meaning that some *a priori* bounds imposed on the solution x of equation (4.1) should be imposed.

3. To construct a stable numerical method for the problem (4.1), one should assume that there exists a family $\{y_\delta\}$ of right hand sides of equation (4.1), where $\delta > 0$ is the level of the error in the data with $\|y^* - y_\delta\|_{B_2} \leq \delta$. Next, one should construct a family of approximate solutions $\{x_\delta\}$ of equation (4.1), where x_δ corresponds to y_δ. The family $\{x_\delta\}$ should be such that

$$\lim_{\delta \to 0^+} \|x_\delta - x^*\| = 0.$$

5 Quasi-Solution

The concept of quasi-solutions was originally proposed by V. K. Ivanov. It is designed to provide a rather general method for solving the ill-posed problem (4.1). This concept is actually a quite useful, as long as one is seeking a solution on a compact set. An example is when the solution is parametrized, i.e.

$$x = \sum_{i=1}^{N} a_i \varphi_i,$$

where elements $\{\varphi_i\}$ are a part of an orthonormal basis in a Hilbert space, the number N is fixed and coefficients $\{a_i\}_{n=1}^{N}$ are unknown. So, one is seeking numbers $\{a_i\}_{n=1}^{N} \subset G$, where $G \subset \mathbb{R}^N$ is *a priori* chosen closed bounded set. This set is called sometimes "the set of admissible parameters".

Since the right hand side y of equation (4.1) is given with an error, Theorem **??** implies that it is unlikely that y belongs to the range of the operator F. Therefore, the following natural question can be raised about the usefulness of Theorem 3.1: *Since the right hand side y of equation (4.1) most likely does not belong to the range $F(G)$ of the operator F, then what is the practical meaning of solving this equation on the compact set G, as required by Theorem 3.1?* The importance of the notion of quasi-solutions is that it addresses this question in a natural way.

Suppose that the problem (4.1) is conditionally well-posed and let $G \subset B_1$ be a compact set. Then the set $F(G) \subset B_2$ is also a compact set. We have $\|y - y^*\|_{B_2} \leq \delta$.

Consider the minimization problem,

$$\min_G J(x),\ \text{where } J(x) = ||F(x) - y||_{B_2}^2 \tag{5.1}$$

Since G is a compact set, then there exists a point $x = x(y_\delta) \in G$ at which the minimum in (5.1) is achieved. In fact, one can have many points $x(y_\delta)$. Nevertheless, it follows from Theorem 1.5 that they are located close to each other, as long as the number δ is sufficiently small.

Definition 5.1. Any point $x = x(y) \in G$ of the minimum of the functional $J(x)$ in (5.1) is called *quasi-solution* of equation in (4.1) on the compact set G.

A natural question is: *How far is the quasi-solution from the exact solution x^*?* Since by Theorem 3.1 the operator $F^{-1} : F(G) \to G$ is continuous and the set $F(G)$ is compact, then one of classical results of Real Analysis implies that there exists the modulus of the continuity $\omega_F(z)$ of the operator F^{-1} on the set $F(G)$. The function $\omega_F(z)$ satisfies the following four conditions:

1. $\omega_F(z)$ is defined for $z \geq 0$.
2. $\omega_F(z) > 0$ for $z > 0$, $\omega_F(0) = 0$ and $\lim_{z\to 0^+} \omega_F(z) = 0$.
3. The function $\omega_F(z)$ is monotonically increasing for $z > 0$.
4. For any two points $y_1, y_2 \in F(G)$ the following estimate holds

$$||F^{-1}(y_1) - F^{-1}(y_2)||_{B_1} \leq \omega_F(||y_1 - y_2)||_{B_2}).$$

The following theorem characterizes the accuracy of the quasi-solution.

Theorem 5.1. *Let B_1 and B_2 be two Banach spaces, $G \subset B_1$ be a compact set and $F : G \to B_2$ be a continuous one-to-one operator. Consider equation (4.1). Suppose that its right hand side $y := y_\delta$ is given with an error of the level $\delta > 0$, where δ is a small number, $\|y^* - y_\delta\|_{B_2} \leq \delta$. Here y^* is the ideal noiseless data y^*. Let $x^* \in G$ be the ideal exact solution of equation (4.1) corresponding to the ideal data y^*, i.e. $F(x^*) = y^*$. Let x_δ^q be a quasi-solution of equation (4.1), i.e.*

$$J(x_\delta^q) = \min_G ||F(x) - y_\delta||_{B_2}^2. \tag{5.2}$$

Let $\omega_F(z), z \geq 0$ be the modulus of the continuity of the operator $F^{-1} : F(G) \to G$ which exists by Theorem 3.1. Then the following error estimate holds

$$\|x_\delta^q - x^*\|_{B_1} \leq \omega_F(2\delta). \tag{5.3}$$

In other words, the problem of finding a quasi-solution is stable and two quasi-solutions are close to each other as long as the error in the data is small..

Proof. Since $\|y^* - y_\delta\|_{B_2} \leq \delta$ then

$$J(x^*) = ||F(x^*) - y_\delta||_{B_2}^2 = \|y^* - y_\delta\|_{B_2}^2 \leq \delta^2.$$

Since the minimal value of the functional $J(x^*)$ is achieved at the point x_δ^q, then

$$J(x_\delta^q) \leq J(x^*) \leq \delta^2.$$

Hence, $||F(x_\delta^q) - y_\delta||_{B_2} \leq \delta$. Hence,

$$||F(x_\delta^q) - F(x^*)||_{B_2} \leq ||F(x_\delta^q) - y_\delta||_{B_2} + \|y_\delta - F(x^*)\|_{B_2}$$

$$= ||F(x_\delta^q) - y_\delta||_{B_2} + \|y_\delta - y^*\|_{B_2} \leq 2\delta.$$

Thus, we have obtained that $||F(x_\delta^q) - F(x^*)||_{B_2} \leq 2\delta$. Therefore, the definition of the modulus of the continuity of the operator F^{-1} implies (5.3). $\square$

This theorem is very important for justifying the practical value of Theorem 3.1. Still, the notion of the quasi-solution has a drawback. This is because it is unclear how to actually find the target minimizer in practical computations. Indeed, to find it, one should minimize the functional $J(x)$ on the compact set G. The commonly acceptable minimization technique for any least squares functional is via searching points where the Fréchet derivative of that functional equals zero. However, the well known obstacle on this path is that this functional might have multiple local minima and ravines. Therefore, most likely the norm of the Fréchet derivative is sufficiently small at many points of, e.g. a ravine. Thus, it is unclear how to practically select a quasi-solution. In other words, we come back to the following question:

The Central Question of the theory of Nonlinear Ill-Posed Problems: *How to find a good approximation for the exact solution without an advanced knowledge of a small neighborhood of this solution?*

6 Regularization

To solve ill-posed problems, regularization methods should be used. In this section we present main ideas of the regularization. Note that we do not assume in Definition **??** that the operator F is defined on a compact set.

Definition 6.1. Let B_1 and B_2 be two Banach spaces and $G \subset B_1$ be a set. Let the operator $F : G \to B_2$ be one-to-one. Consider the equation

$$F(x) = y. \tag{6.1}$$

Let y^* be the ideal noiseless right hand side of equation (6.1) and x^* be the ideal noiseless solution corresponding to $y^*, F(x^*) = y^*$. Let $\delta_0 \in (0,1)$ be a sufficiently small number. For every $\delta \in (0, \delta_0)$ denote

$$K_\delta(y^*) = \left\{z \in B_2 : \|z - y^*\|_{B_2} \leq \delta\right\}.$$

Let $\alpha > 0$ be a parameter and $R_\alpha : K_\delta(y^*) \to G$ be a continuous operator depending on the parameter α.

Definition 6.2 (regularization operator and the regularization parameter). The operator R_α is called the *regularization operator* for equation (6.1) if there exists a function $\alpha(\delta)$ defined for $\delta \in (0, \delta_0)$ such that

$$\lim_{\delta \to 0} \left\|R_{\alpha(\delta)}(y_\delta) - x^*\right\|_{B_1} = 0.$$

The parameter α is called the regularization parameter. The procedure of constructing the approximate solution $x_{\alpha(\delta)} = R_{\alpha(\delta)}(y_\delta)$ is called the *regularization procedure*, or simply *regularization*.

There might be several regularization procedures for the same problem. This is a simplified notion of the regularization. In our experience, in the case of CIPs, usually $\alpha(\delta)$ is a vector of regularization parameters, such as, e.g. the number of iterations, the truncation value of the parameter of the Laplace transform, the number of finite elements, etc.. Since this vector has many coordinates, then its practical choice is usually quite time consuming. This is because one should choose a proper combination of several components of the vector $\alpha(\delta)$.

We now present an example of the regularization operator. Consider the problem of the solution of the heat equation with the reversed time. Let the function $u(x,t)$ be the solution of the following problem

$$u_t = u_{xx},\ x \in (0,\pi),\ t \in (0,T),$$

$$u(x,T) = y(x) \in L_2(0,\pi),$$
$$u(0,t) = u(\pi,t) = 0.$$

Obviously, the solution of this problem, if it exists, is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} y_n e^{n^2(T-t)} \sin nx, \qquad (6.2) \\ y_n &= \sqrt{\frac{2}{\pi}} \int_0^{\pi} y(x) \sin nx dx. \end{aligned}$$

It is clear, however that the Fourier series (6.2) converges for a narrow class of functions $y(x)$. This is because the numbers $\left\{e^{n^2(T-t)}\right\}_{n=1}^{\infty}$ grow exponentially with n.

To regularize this problem, consider the following approximation for the function $u(x,t)$,

$$u_N(x,t) = \sum_{n=1}^{N} y_n e^{n^2(T-t)} \sin nx.$$

Here $\alpha = 1/N$ is the regularization parameter.

First, we consider the following

Inverse Problem. *For each function $f \in L_2(0,\pi)$ consider the solution of the following initial boundary value problem*

$$v_t = v_{xx},\ x \in (0,\pi),\ t \in (0,T), \qquad (6.3)$$

$$v(x,0) = f(x), \qquad (6.4)$$

$$v(0,t) = v(\pi,t) = 0. \qquad (6.5)$$

Given the function $y(x) = v(x,T)$, determine the initial condition $f(x)$ in (6.4).

Regularization Operator for Problem (6.3)-(6.5).

Following the Fundamental Concept of Tikhonov, let $y^* \in L_2(0,\pi)$ be the "ideal" noiseless function y, which corresponds to the function f^* in (**??**). Let the function $y_\delta \in L_2(0,\pi)$ be such that $\|y_\delta - y^*\|_{L_2(0,\pi)} \leq \delta$. Define the regularization parameter $\alpha := 1/N$ and the regularization operator $R_\alpha(y)$ as

$$R_\alpha(y_\delta)(x) = \sum_{n=1}^{N} y_n e^{n^2(T-t)} \sin nx, \qquad (6.6)$$

$$y_n = \sqrt{\frac{2}{\pi}} \int_0^\pi y_\delta(x) \sin nx dx$$

Let $f^* \in C^1[0,\pi]$ and $f^*(0) = f^*(\pi) = 0$. The integration by parts leads to

$$f_n^* = \sqrt{\frac{2}{\pi}} \int_0^\pi f^*(x) \sin nx dx = \frac{1}{n} \sqrt{\frac{2}{\pi}} \int_0^\pi (f^*(x))' \cos nx dx.$$

Hence,

$$(f_n^*)^2 \le \frac{\left\|(f^*(x))'\right\|^2}{n^2}.$$

Hence,

$$\sum_{n=N+1}^{\infty} (f_n^*)^2 \le \frac{C\left\|(f^*(x))'\right\|_{L_2(0,\pi)}^2}{N}, \tag{6.7}$$

where $C > 0$ is a constant independent on the function f^*. Consider now the function $R_\alpha(y) - f^*$,

$$R_\alpha(y_\delta) - f^* = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{N} (y_n - y_n^*) e^{n^2 T} \sin nx - \sqrt{\frac{2}{\pi}} \sum_{n=N+1}^{\infty} f_n^* \sin nx.$$

Since functions $\left\{(2/\pi)^{1/2} \sin nx\right\}_{n=1}^{\infty}$ form an orthonormal basis in $L_2(0,\pi)$, then

$$\|R_\alpha(y) - f^*\|_{L_2(0,\pi)}^2 \le e^{2N^2T} \sum_{n=1}^{N} (y_n - y_n^*)^2 + \sum_{n=N+1}^{\infty} (f_n^*)^2.$$

This implies that

$$\|R_\alpha(y) - f^*\|_{L_2(0,\pi)}^2 \le e^{2N^2T}\delta^2 + \sum_{n=N+1}^{\infty} (f_n^*)^2. \tag{6.8}$$

The second term in the right hand side of (6.8) is independent on the level of error δ. However, it depends on the exact solution as well as on the regularization parameter $\alpha = 1/N$. So, the idea of obtaining the error estimate here is to balance these two terms via equalizing them. To do this, we need to impose an *a priori* assumption first about the maximum of a certain norm of the exact solution f^*. Hence, we assume that $\left\|(f^*)'\right\|_{L_2(0,\pi)}^2 \le M^2$, where M is *a priori* given positive constant. This means, in particular, that the resulting estimate of the accuracy of the regularized solution will hold uniformly for all functions f^* satisfying this condition. This is a typical scenario in the theory of Ill-Posed Problems.

Using (6.7), we obtain from (6.8)

$$\|R_\alpha(y_\delta) - f^*\|_{L_2(0,\pi)}^2 \le e^{2N^2T}\delta^2 + \frac{CM^2}{N}. \tag{6.9}$$

The right hand side of (6.9) contains two terms, which we need to balance by equalizing them,

$$e^{2N^2T}\delta^2 = \frac{CM^2}{N}.$$

Since $e^{2N^2T}N\left(CM^2\right)^{-1} < e^{3N^2T}$ for sufficiently large N, we set

$$e^{3N^2T} = \frac{1}{\delta^2}.$$

Hence, the regularization parameter is

$$\alpha\left(\delta\right) := \frac{1}{N\left(\delta\right)} := \left\{\left[\ln\left(\delta^{-2/3T}\right)\right]^{1/2}\right\}^{-1}.$$

Here $\{a\}$ denotes the least integer for a number $a > 0$. Thus, (6.9) implies that

$$\left\|R_\alpha\left(y_\delta\right) - f^*\right\|^2_{L_2(0,\pi)} \leq \delta^{2/3} + \frac{CM^2}{\left[\ln\left(\delta^{-2/3T}\right)\right]^{1/2}}.$$

It is clear that the right hand side of this inequality tends to zero as $\delta \to 0$. Hence, $R_{\alpha(\delta)}$ is indeed a regularization operator for the above Inverse Problem.

7 The Tikhonov Regularization Functional

A Member of Russian Academy of Science Dr. Andrey Tikhonov (Moscow State University) has constructed a general regularization functional which works for a broad class of ill-posed problems. That functional carries his name in the literature. In this section we construct this functional and study its properties. The Tikhonov functional has proven to be a very powerful tool for solving ill-posed problems.

7.1 The Tikhonov Functional

Definition 7.1. Let B and Q be two Banach spaces with norms $\|\cdot\|_B$ and $\|\cdot\|_Q$ respectively. Let $Q \subset B$ as a set. We say that the space Q is *compactly embedded* in the space B if any closed subset of Q, which is bounded in terms of the norm $\|\cdot\|_Q$, is a compact set in the space B and also

$$\|z\|_{B_1} \leq \|z\|_Q, \forall z \in Q \subset B_1,$$

i.e. the norm in B_1 is weaker than the norm in Q.

Let B_1 be a Banach space. Let the space Q be $Q \subset B_1$ as a set, $\overline{Q} = B_1$, where the closure is understood in the norm of the space B_1 and let Q be compactly embedded in B_1. Some examples of Q and B_1 are:

a. $B_1 = L_2\left(\Omega\right), Q = H^k\left(\Omega\right), \forall k \geq 1$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain.

b. $B_1 = C^m\left(\overline{\Omega}\right), Q = C^{m+k}\left(\overline{\Omega}\right), \forall m \geq 0, \forall k \geq 1$, where m and k are integers.

c. $B_1 = C^m\left(\overline{\Omega}\right), Q = H^k\left(\Omega\right), k > [n/2] + m$, assuming that $\partial\Omega \in C^k$. This follows from the Sobolev embedding theorem.

Let $G \subset B_1$ be the closure of an open set. Then $G \cap Q \neq \varnothing$,since $\overline{Q} = B_1$. Let B_2 be another Banach space. Consider a continuous one-to-one operator $F : G \to B_2$. The continuity here is in terms of the pair of spaces B_1, B_2, rather that in terms of the pair Q, B_2. We are again interested in solving the equation

$$F\left(x\right) = y_\delta, x \in G \cap Q. \tag{7.1}$$

Just as above, we assume that the right hand side of this equation is given with a small error of the level $\delta \in (0,1)$. Let y^* be the ideal noiseless right hand side corresponding to the ideal exact solution $x^* \in G \cap Q$,

$$F(x^*) = y^*, \quad \|y_\delta - y^*\|_{B_2} < \delta. \tag{7.2}$$

The Tikhonov regularization functional $J_\alpha(x)$ is

$$J_\alpha(x) = \frac{1}{2}\|F(x) - y\|_{B_2}^2 + \frac{\alpha}{2}\|x - x_0\|_Q^2, \tag{7.3}$$

$$J_\alpha : G \cap Q \to \mathbb{R}, \quad x_0 \in G \cap Q,$$

where $\alpha = \alpha(\delta) > 0$ is a small regularization parameter and the point $x_0 \in Q$. In general, the choice of the point x_0 depends on the problem at hands. Usually x_0 is a good first approximation for the exact solution x^*. Because of this, x_0 is called the *first guess* or the *first approximation*. The dependence $\alpha = \alpha(\delta)$ will be specified later. The term $\alpha\|x - x_0\|_Q^2$ is called the *Tikhonov regularization term* or simply the *regularization term*.

Thus,

$$x_0, x^* \in G \cap Q. \tag{7.4}$$

Suppose that

$$\|z\|_Q \leq A, \forall z \in G \cap Q, \tag{7.5}$$

where $A > 0$ is a constant. Then by the triangle inequality, (7.4) and (7.5) in (**??**)

$$1 + \|x_0\|_Q + \|x^* - x_0\|_Q \leq 1 + A + 2A = 3A + 1. \tag{7.6}$$

Consider the set $M \subset Q$,

$$M = \left\{ z \in Q : \|z\|_Q \leq 3A + 1 \right\}. \tag{7.7}$$

We assume that

$$M \subset (G \cap Q). \tag{7.8}$$

By (7.4)-(7.8)

$$x_0, x^* \in M \subset (G \cap Q). \tag{7.9}$$

7.2 Approximating the exact solution x^*

Consider a sequence $\{\delta_k\}_{k=1}^\infty$ such that $\delta_k > 0, \lim_{k\to\infty} \delta_k = 0$. We want to construct sequences $\{\alpha(\delta_k)\}, \left\{x_{\alpha(\delta_k)}\right\}$ such that

$$\lim_{k\to\infty} \left\|x_{\alpha(\delta_k)} - x^*\right\|_{B_1} = 0.$$

Hence, if such a sequence will be constructed, then we will approximate the exact solution x^* in a stable way, and this would correspond well with the second condition of the Fundamental Concept of Tikhonov.

Using (7.2) and (7.3), we obtain

$$J_\alpha(x^*) \leq \frac{\delta^2}{2} + \frac{\alpha}{2}\|x^* - x_0\|_Q^2 \leq \frac{\delta^2}{2} + \frac{\alpha}{2}\|x^* - x_0\|_Q^2. \tag{7.10}$$

Let

$$m_{\alpha(\delta_k)} = \inf_{G\cap Q} J_{\alpha(\delta_k)}(x).$$

By (7.10)

$$m_{\alpha(\delta_k)} \leq \frac{\delta_k^2}{2} + \frac{\alpha(\delta_k)}{2}\|x^* - x_0\|_Q^2.$$

Hence, there exists a point $x_{\alpha(\delta_k)} \in G$ such that

$$m_{\alpha(\delta_k)} \leq J_{\alpha(\delta_k)}\left(x_{\alpha(\delta_k)}\right) \leq \frac{\delta_k^2}{2} + \frac{\alpha(\delta_k)}{2}\|x^* - x_0\|_Q^2. \tag{7.11}$$

Theorem 7.1. *Suppose that the above conditions imposed on the sets G, Q and the operator F hold. Also, assume that conditions (7.4)-(7.9) are valid. Let*

$$\lim_{k\to\infty} \alpha(\delta_k) = 0 \text{ and } \frac{\delta_k^2}{\alpha(\delta_k)} \leq 1. \tag{7.12}$$

Then the sequence $\left\{x_{\alpha(\delta_k)}\right\}$ converges to the exact solution x^* of equation (7.2) in the norm of the space B_1, i.e.

$$\lim_{k\to\infty}\left\|x_{\alpha(\delta_k)} - x^*\right\|_{B_1} = 0. \tag{7.13}$$

Remarks 7.1:

1. *Note that convergence (7.13) takes place in the norm of the space B_1. This norm is weaker than the norm of the space Q in which the regularization term is taken. Still, the sequence $x_{\alpha(\delta_k)}$ converges to the point x^* which is in Q rather than in $B_1 \diagdown Q$. This is typical for ill-posed problems.*
2. *In (7.12), the inequality $\delta_k^2/\alpha(\delta_k) \leq 1$ can be replaced with $\delta_k^2/\alpha(\delta_k) \leq C$ for any fixed constant $C > 0$. We use "1" only for the convenience.*

Proof of Theorem 7.1. By (7.3) and (7.11)

$$\left\|x_{\alpha(\delta_k)} - x_0\right\|_Q^2 \leq \frac{\delta_k^2}{\alpha(\delta_k)} + \|x^* - x_0\|_Q^2. \tag{7.14}$$

We know that

$$\sqrt{a^2+b^2} \leq a + b, \forall a, b \geq 0.$$

Hence, by (7.14)

$$\left\|x_{\alpha(\delta_k)} - x_0\right\|_Q \leq \frac{\delta_k}{\sqrt{\alpha(\delta_k)}} + \|x^* - x_0\|_Q. \tag{7.15}$$

Furthermore, by the triangle inequality $\left\|x_{\alpha(\delta_k)} - x_0\right\|_Q \geq \left\|x_{\alpha(\delta_k)}\right\|_Q - \|x_0\|_Q$. Hence, (7.12), (7.6) and (7.15) imply that

$$\left\|x_{\alpha(\delta_k)}\right\|_Q \leq 1 + \|x_0\|_Q + \|x^* - x_0\|_Q \leq 3A + 1.$$

Hence, the sequence $\left\{x_{\alpha(\delta_k)}\right\} \subset Q$ is bounded in the norm of the space Q. Namely, by (7.7) and (7.8)

$$\left\{x_{\alpha(\delta_k)}\right\}_{k=1}^{\infty} \subset M \subset (G \cap Q).$$

Since Q is compactly embedded in B_1, then (7.7) implies that M is a compact set in B_1. Hence, there exists a subsequence of the sequence $\left\{x_{\alpha(\delta_k)}\right\}_{k=1}^{\infty}$, which converges in the norm of the space B_1 to a certain point $\overline{x} \in G$. For brevity and without any loss of generality we assume that the sequence $\left\{x_{\alpha(\delta_k)}\right\}_{k=1}^{\infty}$ itself converges to $\overline{x}$,

$$\lim_{k\to\infty} \left\|x_{\alpha(\delta_k)} - \overline{x}\right\|_{B_1} = 0. \tag{7.16}$$

Then (7.11) and the first equality (7.12) imply that

$$\lim_{k\to\infty} J_{\alpha(\delta_k)}\left(x_{\alpha(\delta_k)}\right) = 0. \tag{7.17}$$

On the other hand, by (7.2)

$$\lim_{k\to\infty} \left\|y_{\delta_k} - y^*\right\|_{B_2} = 0. \tag{7.18}$$

Hence, the first equality (7.12) and (7.16)-(7.18) imply that

$$\lim_{k\to\infty} J_{\alpha(\delta_k)}\left(x_{\alpha(\delta_k)}\right) = \frac{1}{2}\lim_{k\to\infty}\left[\left\|F\left(x_{\alpha(\delta_k)}\right) - y_{\delta_k}\right\|_{B_2}^2 + \alpha\left(\delta_k\right)\left\|x_{\alpha(\delta_k)} - x_0\right\|_Q^2\right] \tag{7.19}$$

$$= \frac{1}{2}\left\|F\left(\overline{x}\right) - y^*\right\|_{B_2}^2.$$

Hence, (7.16) and (7.19) imply that $\left\|F\left(\overline{x}\right) - y^*\right\|_{B_2} = 0$, which means that $F\left(\overline{x}\right) = y^*$. Since the operator F is one-to-one on the set G and since by (7.9) $x^* \in G$, then $\overline{x} = x^*$.

Thus, we have constructed the sequence of regularization parameters $\left\{\alpha\left(\delta_k\right)\right\}_{k=1}^{\infty}$ and the sequence $\left\{x_{\alpha(\delta_k)}\right\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} \left\|x_{\alpha(\delta_k)} - x^*\right\|_{B_1} = 0. \ \square$$

To guarantee the second inequality (7.12), one can choose, for example $\alpha\left(\delta_k\right) = C\delta_k^{\mu}, \mu \in (0,2]$. It is reasonable to call $\left\{x_{\alpha(\delta_k)}\right\}_{k=1}^{\infty}$ *regularizing sequence.*

We point out that the condition that the operator F is one-to-one plays a crucial role in the above construction since it guarantees that $\overline{x} = x^*$. The sequence $\left\{x_{\alpha(\delta_k)}\right\}_{k=1}^{\infty}$ is called *minimizing sequence.*

There are two inconveniences in the above construction. First, it is unclear how to find the minimizing sequence computationally. Second, the problem of multiple local minima and ravines of the functional (7.3) presents a significant complicating factor in the goal of the construction of such a sequence.

7.3 Regularized solution and accuracy estimate in a finite dimensional space

The construction of section 7.2 does not guarantee that the functional $J_\alpha(x)$ indeed achieves it minimal value. Suppose now that the functional $J_\alpha(x)$ does achieve its minimal value,

$$J_\alpha\left(x_\alpha\right) = \min_{G\cap Q} J_\alpha(x), \alpha = \alpha(\delta).$$

Then $x_{\alpha(\delta)}$ is called a *regularized solution* of equation (7.1) for this specific value $\alpha = \alpha(\delta)$ of the regularization parameter. It is important to prove convergence of regularized solutions to the true one x^*. This is done in the current subsection.

Consider now the case when $B_1 = \mathbb{R}^n$, i.e. the finite dimensional space. In this case any closed bounded set is a compact set. The norm in this set is denoted $\|x\|$.

Remark 7.2. *The importance of a finite dimensional space is due to the fact that real computations are always done in finite dimensional spaces.*

Let $G \subset \mathbb{R}^n$ be a closed bounded set. Let B_2 be another Banach space. Consider a continuous one-to-one operator $F : G \rightarrow B_2$. We are again interested in solving the equation

$$F(x) = y_\delta, x \in G. \tag{7.20}$$

Let the number $\delta \in (0,1)$ be the level of a small error in the data. Let y^* be the ideal noiseless right hand side corresponding to the ideal exact solution x^*,

$$F(x^*) = y^*, \quad \|y_\delta - y^*\|_{B_2} < \delta. \tag{7.21}$$

Consider the Tikhonov regularization functional $J_\alpha(x) : G \rightarrow \mathbb{R}$

$$J_\alpha(x) = \frac{1}{2}\|F(x) - y_\delta\|_{B_2}^2 + \frac{\alpha}{2}\|x - x_0\|^2. \tag{7.22}$$

Let $A = const. > 0$. We impose the following analogs of conditions (7.4)-(7.9): Let

$$M = \{z \in \mathbb{R}^n : \|z\| \leq 3A + 1\}, \tag{7.23}$$

We assume that

$$M \subseteq G \tag{7.24}$$

and that

$$\|x_0\|, \|x^*\| \leq A. \tag{7.25}$$

Hence, by (7.23)-(7.25)

$$x_0, x^* \in M \subseteq G. \tag{7.26}$$

Let $R(M)$ be the range of the operator F while it acts only on elements of the set M. By the fundamental theorem of Tikhonov the operator

$$F^{-1} : R(M) \rightarrow M \tag{7.27}$$

is continuous. Therefore, as it is well known from the Analysis course, that there exists the modulus of the continuity of the operator in (7.27). In other words, there exists a function $\omega_{F,M} : (0,\infty) \rightarrow (0,\infty)$ such that:

1. $\omega_{F,M}(z_1) \leq \omega_{F,M}(z_2)$ if $z_1 \leq z_2$.
2. $\lim_{z\rightarrow 0^+} \omega_{F,M}(z) = 0$.
3. The following estimate holds:

$$\|x_1 - x_2\| \leq \omega_{F,M}\left(\|F(x_1) - F(x_2)\|_{B_2}\right), \forall x_1, x_2 \in M. \tag{7.28}$$

Theorem 7.2. *Suppose that conditions of this subsection imposed on the set $G \subset R^n$ and the operator $F : G \rightarrow B_2$ hold. Consider the problem of the solution of equation (7.20). Assume that conditions (7.21)-(7.27) are valid. Then the functional F achieves its minimal value on the set M at a point $x_{\delta,\alpha} \in M$. Furthermore, the following accuracy estimate for the regularized solution $x_{\delta,\alpha}$ is valid*

$$\|x_{\delta,\alpha} - x^*\| \leq \omega_{F,M}\left(2\left(\delta + A\sqrt{\alpha}\right)\right). \tag{7.29}$$

In particular, if $\alpha = \alpha(\delta) = \delta^2$, *then*

$$\left\| x_{\delta,\alpha(\delta)} - x^* \right\| \leq \omega_{F,M}\left(2\delta\left(A+1\right)\right). \tag{7.30}$$

Remark 7.3. *Just as in section 6, let*

$$K_\delta\left(y^*\right) = \left\{z \in B_2 : \left\|z - y^*\right\|_{B_2} \leq \delta\right\}.$$

Assume that the regularization parameter $\alpha = \alpha(\delta) = \delta^2$. *Consider the operator* $R_{\alpha(\delta)}$: $K_\delta(y^*) \to M$ *defined as* $R_{\alpha(\delta)}(y_\delta) = x_{\delta,\alpha(\delta)}$, *where* $x_{\delta,\alpha(\delta)}$ *is defined in Theorem 7.2. Then* $R_{\alpha(\delta)}(y_\delta)$ *is the regularization operator for the problem (7.20).*

Proof of Theorem 7.2. Since the closed ball M is a compact set in $\mathbb{R}^n$, then by the Weierstrass theorem, the functional $J_\alpha(x)$ achieves its minimal value on M at a point $x_{\delta,\alpha} \in M$. Since by (7.26) $x^* \in M$, then

$$\begin{aligned} J_\alpha\left(x_{\delta,\alpha}\right) \leq J_\alpha\left(x^*\right) &= \frac{1}{2}\left\|F(x^*) - y_\delta\right\|_{B_2}^2 + \frac{\alpha}{2}\left\|x^* - x_0\right\|^2 \\ &= \frac{1}{2}\left\|y^* - y_\delta\right\|_{B_2}^2 + \frac{\alpha}{2}\left\|x^* - x_0\right\|^2. \end{aligned} \tag{7.31}$$

By (7.21) $\left\|y^* - y_\delta\right\|_{B_2}^2 \leq \delta^2$. Hence, (7.22) and (7.31) imply that

$$\left\|F\left(x_{\delta,\alpha}\right) - y_\delta\right\|_{B_2} \leq \frac{1}{\sqrt{2}}\left(\delta + \sqrt{\alpha}\left\|x^* - x_0\right\|\right),$$

This, (7.25) and the triangle inequality imply

$$\left\|F\left(x_{\delta,\alpha}\right) - y_\delta\right\|_{B_2} \leq \frac{1}{\sqrt{2}}\left(\delta + 2A\sqrt{\alpha}\right) \leq \delta + 2A\sqrt{\alpha}. \tag{7.32}$$

Next, using the triangle inequality and (7.21),

$$\begin{aligned} \left\|F\left(x_{\delta,\alpha}\right) - y_\delta\right\|_{B_2} &= \left\|\left(F\left(x_{\delta,\alpha}\right) - F\left(x^*\right)\right) + \left(F\left(x^*\right) - y_\delta\right)\right\|_{B_2} \\ &\geq \left\|F\left(x_{\delta,\alpha}\right) - F\left(x^*\right)\right\|_{B_2} - \left\|F\left(x^*\right) - y_\delta\right\|_{B_2} \\ &\geq \left\|F\left(x_{\delta,\alpha}\right) - F\left(x^*\right)\right\|_{B_2} - \delta. \end{aligned}$$

Thus,

$$\left\|F\left(x_{\delta,\alpha}\right) - y_\delta\right\|_{B_2} \geq \left\|F\left(x_{\delta,\alpha}\right) - F\left(x^*\right)\right\|_{B_2} - \delta.$$

Substituting this in (7.32), we obtain

$$\left\|F\left(x_{\delta,\alpha}\right) - F\left(x^*\right)\right\|_{B_2} \leq 2\left(\delta + A\sqrt{\alpha}\right). \tag{7.33}$$

Estimates (7.29) and (7.30) follow immediately from (7.33) and (7.28). $\square$

8 Uniqueness of a Coefficient Inverse Problem for a 1-Dimensional Wave-Like Equation

A Coefficient Inverse Problem (CIP) is the problem of finding a coefficient of a Partial Differential Equation (PDE) from boundary measurements. The topic of CIPs is the single most challenging and single most applied one among all other topics of the field of Inverse Problems. The unknown coefficient characterizes a material property of a constituent material of a medium of interest. Therefore CIP is the problem of finding the unknown property of the interior using boundary measurements only. In other words, this is a non invasive technique! Most exiting medical examples of CIPs are ultrasound devices and X-ray tomography.

I have dedicated my entire career to CIPs. Initially I was focusing on uniqueness questions only. But starting at about 2001, I became increasingly more interested in globally convergent numerical methods for them.

All CIPs are nonlinear and all ill-posed. The reason of their nonlinearity is that solution of a PDE depends nonlinearly on its coefficients. For example, consider a trivial ODE

$$y' = ay, y(0) = 1,$$

where $a = const$. Then

$$y(t,a) = e^{at}.$$

Hence, the function $y(t,a)$ depends nonlinearly on the coefficient a.

8.1 Statements of forward and inverse problems

Consider the wave-like equation with the potential function $q(x) \in C(\mathbb{R})$,

$$u_{tt} = u_{xx} + q(x)u, x \in \mathbb{R}, t \in (0,T), \tag{8.1}$$

$$u(x,0) = 0, u_t(x,0) = \delta(x), \tag{8.2}$$

The Cauchy problem (8.1), (8.2) is our *forward problem.*

Here $\delta(x)$ is Dirac delta function. Actually this is a functional (by Sobolev) which acts on “good functions" as

$$(\delta(x), f(x)) = f(0). \tag{8.3}$$

This is often **CONDITIONALLY** written as

$$\int_{\mathbb{R}} \delta(x) f(x)\, dx = f(0),$$

although this is not a true integral. Paul Dirac (a French physicist and a Nobel laureate) has defined his function as:

$$\delta(x) = \begin{cases} 0, x \neq 0, \\ \infty, x = 0 \end{cases}$$

and

$$\int_{\mathbb{R}} \delta(x)\, dx = 1.$$

This is of course not a rigorous definition. So, a Russian mathematician Sergey Sobolev in 1930-ies has introduced a rigorous definition of the delta function as in (8.3). He has introduced the so-called "generaliz0ed" functions, which are functionals on a linear space of $C^{\infty}-$ functions.

In Physics, $\delta(x)$ corresponds to the point source at $\{0\}$. We can also consider $\delta(x-x_0)$ which means the point source at $\{x_0\}$. Hence,

$$(\delta(x-x_0), f(x)) = f(x_0),$$

or, conditionally,

$$\int_{\mathbb{R}} \delta(x-x_0) f(x)\, dx = f(x_0).$$

Therefore, the physical meaning of the solution $u(x,t)$ of problem (8.1), (8.2) is that the wave propagation process was generated by a point source located at (0) at the moment of time $t=0$. And $u(x,t)$ is the amplitude of the wave at the point $x \in \mathbb{R}$ and at the moment of time $t \in (0,T)$. $q(x)$ is the scattering potential, i.e. this function causes waves to scatter.

Definition 8.1. *If the coefficient $q(x)$ is known, then the problem of finding the function $u(x,t)$ from conditions (8.1), (8.2) is called the "forward problem".*

Definition 8.2. *In fact, the problem (8.1), (8.2) is called the "Cauchy problem" for the hyperbolic equation (8.1).*

Coefficient Inverse Problem 8.1. *Suppose that the following two functions $f_0(t), f_1(t)$ are given:*

$$f_0(t) = u(0,t), f_1(t) = u_x(0,t), t \in (0,T). \tag{8.4}$$

Find the function $q(x)$ for $x \in (-a,a)$, where the number $a>0$ will be defined later.

So, we measure the scattered wave at the point $\{x=0\}$ and want to find the scatterer $q(x)$ from these measurements.

In practice usually only the function $f_0(t)$ is measured, i.e. the amplitude of the wave at $\{x=0\}$. Assuming that the function $q(x)=0$ for $x<0$, one can easily find the function $u(x,t)$ for $x<0, t \in (0,T)$ as the solution of the following initial boundary value problem:

$$u_{tt} = u_{xx}, x \in (-\infty,0), t \in (0,T),$$

$$u(x,0) = 0, u_t(x,0) = \delta(x),$$

$$u(0,t) = f_0(t).$$

This is well known from the first part of the PDE course. Therefore, we can uniquely determine the function $u_x(0,t) = f_1(t)$.

8.2 The form of the solution of the forward problem

Consider just the Cauchy problem for the wave equation

$$u_{tt} = u_{xx} + f(x,t), x \in \mathbb{R}, t \in (0,T), \tag{8.5}$$

$$u(x,0) = 0, u_t(x,0) = \psi(x), \tag{8.6}$$

where $f(x,t) \in C(\mathbb{R} \times [0,T])$ is a given function. Then by the D'Alembert formula

$$u(x,t) = \frac{1}{2}\int\limits_{x-t}^{x+t} \psi(\xi)\, d\xi + \frac{1}{2}\int\limits_{0}^{t}\int\limits_{x-\tau}^{x+\tau} f(\xi,\tau)\, d\xi d\tau.$$

Let now $\psi(x) = \delta(x)$. Denoting $g(x) \equiv 1$, we obtain

$$\int\limits_{x-t}^{x+t} \psi(\xi)\, d\xi = \int\limits_{x-t}^{x+t} \delta(\xi)\cdot g(\xi)\, d\xi = \begin{cases} g(0) = 1 \text{ if } x+t>0 \text{ and } x-t<0, \\ 0 \text{ otherwise.} \end{cases}$$

$x+t>0$ and $x-t<0$ means $-t<x<t$, i.e.

$$t > |x|.$$

Consider the Heaviside function

$$H(z) = \begin{cases} 1, z>0, \\ 0, z<0. \end{cases}$$

Hence,

$$\int\limits_{x-t}^{x+t} \delta(\xi)\cdot g(\xi)\, d\xi = H(t-|x|).$$

Hence, if $\psi(x) = \delta(x)$, then the solution of the problem (8.5), (8.6) is

$$u(x,t) = \frac{1}{2}H(t-|x|) + \frac{1}{2}\int\limits_{0}^{t}\int\limits_{x-\tau}^{x+\tau} f(\xi,\tau)\, d\xi d\tau. \tag{8.7}$$

If $f \equiv 0$, then (8.7) means that the wave amplitude $u(x,t) = 0$ at the observation point x for $t < |x|$, i.e. for those times t, for which wave did not yet reach the point x. In particular, this means that the speed of waves propagation is just 1. But as soon as the wave reaches the point x at the moment of time $t = |x|$, it stands unchanged there for all follow up times.

Denoting temporary $f(x,t) = q(x)u(x,t)$, we obtain that the solution of problem (8.1), (8.2) has the form:

$$u(x,t) = \frac{1}{2}H(t-|x|) + \frac{1}{2}\int\limits_{0}^{t}\int\limits_{x-\tau}^{x+\tau} q(\xi)\, u(\xi,\tau)\, d\xi d\tau. \tag{8.8}$$

Lemma 8.1. *In (8.8)* $u(x,t) = 0$ *for* $t<|x|$.

Proof. If $t<|x|$, then by (8.8)

$$u(x,t) = \frac{1}{2}\int\limits_{0}^{t}\int\limits_{x-\tau}^{x+\tau} q(\xi)\, u(\xi,\tau)\, d\xi d\tau,\ \ t<|x|. \tag{8.9}$$

The integration is carried out over the triangle with the vertex at (x,t) and its sides intercept $x-$axis at points $x-t$ and $x+t$. Hence, this triangle is located below the graph of the function $t=|x|$. Hence in integral (8.9) $\tau<|\xi|$,

Hence, (8.9) is a homogeneous Volterra integral equation. Hence, (8.9) implies that $u(x,t)=0$ for $t<|x|$. $\square$

Recall that the Volterra integral equation is

$$y(t)=\int_0^t K(t,\tau)\,y(\tau)\,d\tau+f(t),$$

If $f(t)\equiv 0$, then $y(t)\equiv 0$, see my previous PDE course.

Because of Lemma 1, we can rewrite equation (8.8) in the form

$$u(x,t)=\frac{1}{2}+\frac{1}{2}\int_{(x-t)/2}^{(x+t)/2} q(\xi)\,d\xi\int_{|\xi|}^{t-|x-\xi|} u(\xi,\tau)\,d\tau,t>|x|. \tag{8.10}$$

The integration in (8.10) is carried out over the domain D in the (x,t) plane,

$$D=\{(\xi,\tau):|\xi|<\tau<t-|x-\xi|\}. \tag{8.11}$$

The domain D is the intersection of $\{t>|x|\}$ with the triangle whose vertex is at the point (x,t) and sides intercept the $x-$axis at points $x-t$ and $x+t$.

Exercise 1. *Using this geometrical description, prove that limits of integration in (8.10) are correct ones.*

8.3 Uniqueness of the Coefficient Inverse Problem

I will prove uniqueness theorem first. And will formulate it then.

The following formula is well known

$$\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} g(t,\tau)\,d\tau=g(t,\beta(t))\,\beta'(t)-g(t,\alpha(t))\,\alpha'(t)+\int_{\alpha(t)}^{\beta(t)} g_t(t,\tau)\,d\tau, \tag{8.12}$$

for appropriate functions involved here. Also,

$$|x|'=sign(x)=\begin{cases}1,x>0,\\-1,x<0.\end{cases} \tag{8.13}$$

Consider

$$u(t,t)=\lim_{t\to|x|^+} u(x,t),x>0.$$

By (8.10)

$$u(t,t)=\frac{1}{2}+\frac{1}{2}\int_0^t q(\xi)\,d\xi\int_{|\xi|}^{t-|t-\xi|} u(\xi,\tau)\,d\tau. \tag{8.14}$$

Since by (8.11) $t > |\xi|$, then

$$t - |t - \xi| = \xi.$$

If $\xi > 0$, then $\xi = |\xi|$. Hence, upper and lower limits of the interior integral in (8.14) equal to each other in this case. Hence, the integral in (8.14) equals zero in this case.

If $\xi < 0$, then $t - |t - \xi| < |\xi|$. Hence, by (8.11) the integral in (8.14) again equals zero. Thus,

$$u(t,t) = \lim_{t \to |x|^+} u(x,t) = \frac{1}{2}, x > 0. \tag{8.15}$$

Exercise 2. Prove that

$$u(-t,t) = \lim_{t \to |x|^+} u(x,t) = \frac{1}{2}, x < 0. \tag{8.16}$$

Using (8.10), (8.12), (8.13),

$$u_x(x,t) = \frac{1}{2} \int\limits_{(x-t)/2}^{(x+t)/2} q(\xi)\, u(\xi, t - |x - \xi|)\, sgn(\xi - x)\, d\xi. \tag{8.17}$$

Hence, by (8.4)

$$\int\limits_{-t/2}^{t/2} q(\xi)\, u(\xi, t - |\xi|)\, sgn(\xi)\, d\xi = 2f_1(t).$$

Hence, by (8.15) and (8.16)

$$2f_1'(t) = \frac{1}{2} q\left(\frac{t}{2}\right) + \frac{1}{2} q\left(-\frac{t}{2}\right) + \int\limits_{-t/2}^{t/2} q(\xi)\, u_t(\xi, t - |\xi|)\, sgn(\xi)\, d\xi. \tag{8.18}$$

Next, since we have u_t in (8.18), then we have to have an equation for u_t. By (8.10), (8.12), (8.13)

$$u_t(x,t) = \frac{1}{2} \int\limits_{(x-t)/2}^{(x+t)/2} q(\xi)\, u(\xi, t - |x - \xi|)\, d\xi, t > |x|. \tag{8.19}$$

Next,

$$u_t(0,t) = f_0'(t) = \frac{1}{2} \int\limits_{-t/2}^{t/2} q(\xi)\, u(\xi, t - |\xi|)\, d\xi.$$

Hence,

$$2f_0'(t) = \frac{1}{2} q\left(\frac{t}{2}\right) - \frac{1}{2} q\left(-\frac{t}{2}\right) + \int\limits_{-t/2}^{t/2} q(\xi)\, u_t(\xi, t - |\xi|)\, d\xi. \tag{8.20}$$

Sum up (8.18) and (8.20),

$$q\left(\frac{t}{2}\right) + \int\limits_{-t/2}^{t/2} q(\xi)\, u_t(\xi, t - |\xi|)\, (sign(\xi) + 1)\, d\xi = 2\left(f_0'(t) + f_1'(t)\right). \tag{8.21}$$

We now subtract (8.20) from (8.18),

$$q\left(-\frac{t}{2}\right)+\int\limits_{-t/2}^{t/2} q(\xi)\,u_t(\xi,t-|\xi|)\,(sign(\xi)-1)\,d\xi=2\left(f_1'(t)-f_0'(t)\right) \tag{8.22}$$

Denote

$$p(t)=q(t)\,,r(-t)=q(-t)\,,t>0. \tag{8.23}$$

Then

$$q(x)=H(x)\,p(x)+H(-x)\,r(-x)\,. \tag{8.24}$$

Then (8.21) and (8.22) can be rewritten as:

$$p\left(\frac{t}{2}\right)+2\int\limits_0^{t/2} p(\xi)\,u_t(\xi,t-|\xi|)\,d\xi=2\left(f_0'(t)+f_1'(t)\right), \tag{8.25}$$

$$r\left(\frac{t}{2}\right)-2\int\limits_{-t/2}^{0} r(\xi)\,u_t(\xi,t-|\xi|)\,d\xi+=2\left(f_1'(t)-f_0'(t)\right). \tag{8.26}$$

Assume now that we have two solutions of our inverse problem (q_1,u_1) and (q_2,u_2). Then by (8.24) and (8.25)

$$p_j(t)=q_j(t)\,,r_j(t)=q_j(-t)\,,t>0,j=1,2,$$

$$q_j(x)=H(x)\,p_j(x)+H(-x)\,r_j(x)\,,j=1,2.$$

Denote

$$\widetilde{p}(x)=p_1(x)-p_2(x)\,,\widetilde{r}(-x)=r_1(-x)-r_2(-x)\,,x>0,$$

$$\widetilde{u}(x,t)=u_1(x,t)-u_2(x,t)\,.$$

Hence, to prove uniqueness for our inverse problem, we should prove that

$$\widetilde{p}(x)=\widetilde{r}(-x)=0,x>0,$$

$$\widetilde{u}(x,t)=0 \text{ for all } x \text{ and all } t>0.$$

Note that for any numbers a_1,b_1,a_2,b_2

$$a_1b_1-a_2b_2=(a_1b_1-a_2b_1)+(a_2b_1-a_2b_2) \tag{8.27}$$

$$=(a_1-a_2)\,b_1+a_2\,(b_1-b_2)\,.$$

Hence, substituting in (8.25) and (8.26) (p_1,r_1,u_1) and (p_2,r_2,u_2) and then subtracting, we obtain

$$\widetilde{p}\left(\frac{t}{2}\right)+2\int\limits_0^{t/2}\widetilde{p}(\xi)\,u_{1t}(\xi,t-|\xi|)\,d\xi+2\int\limits_0^{t/2}p_2(\xi)\,\widetilde{u}_t(\xi,t-|\xi|)\,d\xi=0, \tag{8.28}$$

$$\widetilde{r}\left(-\frac{t}{2}\right)+2\int\limits_{-t/2}^{0}\widetilde{r}\left(-\xi\right)u_{1t}\left(\xi,t-\left|\xi\right|\right)d\xi+2\int\limits_{-t/2}^{0}r_{2}\left(-\xi\right)\widetilde{u}_{t}\left(\xi,t-\left|\xi\right|\right)d\xi=0. \tag{8.29}$$

In addition, by (8.19) and (8.24)

$$\widetilde{u}_{t}\left(x,t\right)=\frac{1}{2}\int\limits_{(x-t)/2}^{(x+t)/2}\left(H\left(\xi\right)\widetilde{p}\left(\xi\right)+H\left(-\xi\right)\widetilde{r}\left(-\xi\right)\right)u_{1}\left(\xi,t-\left|x-\xi\right|\right)d\xi \tag{8.30}$$

$$+\frac{1}{2}\int\limits_{(x-t)/2}^{(x+t)/2}q_{2}\left(\xi\right)\widetilde{u}\left(\xi,t-\left|x-\xi\right|\right)d\xi,t>\left|x\right|.$$

Also, by (8.10)

$$\widetilde{u}\left(x,t\right)=\frac{1}{2}\int\limits_{(x-t)/2}^{(x+t)/2}\left(H\left(\xi\right)\widetilde{p}\left(\xi\right)+H\left(-\xi\right)\widetilde{r}\left(-\xi\right)\right)\int\limits_{\left|\xi\right|}^{t-\left|x-\xi\right|}u_{1}\left(\xi,\tau\right)d\tau d\xi$$

$$+\frac{1}{2}\int\limits_{(x-t)/2}^{(x+t)/2}q_{2}\left(\xi\right)\int\limits_{\left|\xi\right|}^{t-\left|x-\xi\right|}\widetilde{u}\left(\xi,\tau\right)d\tau d\xi,t>\left|x\right|. \tag{8.31}$$

Integral equations (8.28)-(8.31) represent a homogeneous system of Volterra integral equations with respect to the vector function

$$\left(\widetilde{p}\left(\frac{t}{2}\right),\widetilde{r}\left(-\frac{t}{2}\right),\widetilde{u}_{t}\left(x,t\right),\widetilde{u}\left(x,t\right)\right)^{T}.$$

Recall that the data $f_0\left(t\right), f_1\left(t\right)$ for the inverse problem are given for $t\in\left(0,T\right)$. Therefore,

$$\widetilde{p}\left(x\right)=\widetilde{r}\left(-x\right)\text{ for }x\in\left(0,T/2\right),$$

$$\widetilde{u}\left(x,t\right)=0\text{ for }x\in\left(0,T/2\right),t\in\left(0,T\right).$$

Thus, we have proven Theorem 8.1:

Theorem 8.1 (uniqueness). *The Coefficient Inverse Problem 8.1 has at most one solution* $q\left(x\right)\in C\left[-T/2,T/2\right]$.

Also, in fact, it follows from an insignificant modification of the above discussion that Theorem 8.2 is also valid:

Theorem 8.2. *Assume that the function* $q\left(x\right)\in C\left[-T/2,T/2\right]$. *Then there exists unique solution* $u\left(x,t\right)\in C^{2}\ \left(\left|x\right|\leq t\leq T\right)$ *of the forward problem (8.1), (8.2).*

Exercise 3. Prove Theorem 8.2.

9 Introduction to the Bukhgeim-Klibanov Method

9.1 Preliminaries

In this section 9 we will see how to prove an analog of Theorem 8.1. But instead of integral equations of section 8, we will use the Bukhgeim-Klibanov method, which is based on Carleman estimates. This method, so as the apparatus of Carleman estimates, was first introduced in the field of Coefficient Inverse Problem by A. L. Bukhgeim and M. V. Klibanov in their paper

"Uniqueness in the large of a class of multidimensional inverse problems", *Soviet Mathematics Doklady*, 17, 244–247, 1981. "Doklady" means "reports" in Russian. This journal publishes only short communications of either members/corresponding members of the Russian Academy of Science (RAS), or those works, which are communicated by members of RAS. As of 10/27/2021, this paper has 601 citations, https://scholar.google.com/citations?user=pE

Here is the copy of the first paragraph of this paper "Uniqueness theorems for multidimensional inverse problems have at present been obtained mainly in classes of piecewise analytic functions and similar classes or locally (see [1] -[7] , [15] and the literature cited there). Moreover, the technique of investigating these problems has, as a rule, depended in an essential way on the type of the differential equation. In this note a new method of investigating inverse problems is proposed that is based on weighted a priori estimates. This method makes it possible to consider in a unified way a broad class of inverse problems for those equations $Pu = f$ for which the solution of the Cauchy problem admits a Carleman estimate of the type considered in [8] -[11] .The theorems of §1 were proved by M. V. Klibanov and those of §2 by A. L. Bukhgeim. They were obtained simultaneously and independently."

We will prove both uniqueness and stability theorems for the CIP 8.1. "Stability theorem" means an estimate of the modulus of the continuity of the inverse operator.

We remind the material of section 4. Let B_1 and B_2 be two Banach spaces. Let $G \subseteq B_1$ be compact set and $F : G \to B_2$ be a continuos one-to-one operator. Then by the Tikhonov foundational theorem the operator $F^{-1} : F(G) \to G$ is continuous. Since F is continuous, then the $F(G)$ is a compact set. Hence, one of classical results of Real Analysis implies that there exists the modulus of the continuity $\omega_F(z)$ of the operator F^{-1} on the set $F(G)$. The function $\omega_F(z)$ satisfies the following four conditions:

1. $\omega_F(z)$ is defined for $z \geq 0$.
2. $\omega_F(z) > 0$ for $z > 0$, $\omega_F(0) = 0$ and $\lim_{z\to 0^+} \omega_F(z) = 0$.
3. The function $\omega_F(z)$ is monotonically increasing for $z > 0$.
4. For any two points $y_1, y_2 \in F(G)$ the following estimate holds

$$\left\| F^{-1}(y_2) - F^{-1}(y_1) \right\|_{B_1} \leq \omega_F\left(\|y_2 - y_1\|_{B_2} \right), \forall y_1, y_2 \in F(G). \tag{9.1}$$

Definition. *"Stability estimate" of the operator $F^{-1} : F(G) \to G$ is either estimate (9.1) or any estimate of the right hand side of (9.1) as $\omega_F(z) \leq \omega_F^{(1)}(z)$ for all $z \geq 0$, where $\omega_F^{(1)}(z)$ is another function satisfying conditions 1-3. This means that*

$$\left\| F^{-1}(y_2) - F^{-1}(y_1) \right\|_{B_1} \leq \omega_F\left(\|y_2 - y_1\|_{B_2} \right) \leq \omega_F^{(1)}\left(\|y_2 - y_1\|_{B_2} \right),$$

$$\forall y_1, y_2 \in F(G).$$

9.2 The forward and inverse problems

We recall the statement of the CIP. But unlike section 8, we assume now that

$$q(x) \in C(\mathbb{R}),\ q(x) = \begin{cases} \geq 0 \text{ for } x \in (0,1), \\ = 0 \text{ for } x \notin (0,1). \end{cases} \tag{9.2}$$

The forward problem:

$$u_{tt} = u_{xx} + q(x) u, x \in \mathbb{R}, t \in (0,T), \tag{9.3}$$

$$u(x,0) = 0, u_t(x,0) = \delta(x). \tag{9.4}$$

Coefficient Inverse Problem 9.1. *Suppose that the following two functions $f_0(t), f_1(t)$ are given:*

$$u(0,t) = f_0(t), u_x(0,t) = f_1(t), t \in (0, T+1). \tag{9.5}$$

Find the function $q(x)$ satisfying conditions (9.2).

Therefore, we should find $q(x)$ only for $x \in (0,1)$. As we have derived in Lemma 8.1 and in (8.10),

$$u(x,t) = \begin{cases} \frac{1}{2} + \frac{1}{2} \int\limits_{(x-t)/2}^{(x+t)/2} q(\xi)\, d\xi \int\limits_{|\xi|}^{t-|x-\xi|} u(\xi,\tau)\, d\tau,\ t > |x|, \\ 0, t < |x|. \end{cases} \tag{9.6}$$

Lemma 9.1 (absorbing boundary conditions). *For every two numbers $A \geq 1$ and $B > 0$, the function $u(x,t)$ satisfies the absorbing boundary conditions:*

$$u_x(A,t) + u_t(A,t) = 0,\ u_x(-B,t) - u_t(-B,t) = 0,\ \forall t \in (0,T). \tag{9.7}$$

Proof. Using (9.2), we can rewrite (9.6) as

$$u(x,t) = \frac{1}{2} H(t-|x|) + \frac{1}{2} H(t-|x|) \int\limits_{0}^{(x+t)/2} q(\xi)\, d\xi \int\limits_{|\xi|}^{t-|x-\xi|} u(\xi,\tau)\, d\tau. \tag{9.8}$$

The presence of the function $H(t-|x|)$ in the second term of the right hand side of (9.8) is due to Lemma 8.1, which claims that $u(x,t) = 0$ for $t < |x|$. Denote

$$u_0(x,t) = \frac{1}{2} H(t-|x|).$$

Clearly the function $u_0(x,t)$ satisfies (9.7). Denote $\overline{u}(x,t) = u(x,t) - u_0(x,t)$. Differentiating (9.8), we obtain similarly with (8.17)

$$\overline{u}_x(x,t) = -\frac{1}{2} \int\limits_{0}^{(x+t)/2} sign(x-\xi)\, q(\xi)\, u(\xi, t-|x-\xi|)\, d\xi,\ t > |x|, \tag{9.9}$$

$$\overline{u}_t(x,t) = \frac{1}{2} \int\limits_{0}^{(x+t)/2} q(\xi)\, u(\xi, t-|x-\xi|)\, d\xi,\ t \geq |x|. \tag{9.10}$$

If $x \geq 1$, then in (9.9) $(x - \xi) = 1$, since $q(\xi) = 0$ for $\xi \geq 1$. Hence, by (9.9) and (9.10)

$$\overline{u}_x(x,t) = -\overline{u}_t(x,t), x > 1,$$

which implies the first equality (9.7).

Next, if $x \leq 0$, then in (9.9) $(x - \xi) = -1$ since $q(\xi) = 0$ for $\xi \leq 0$. Hence, (9.9) and (9.10) imply in this case

$$\overline{u}_x(x,t) = \overline{u}_t(x,t), x \leq 0,$$

which implies the second equality (9.7). $\square$

Lemma 9.2. *We have:*

$$u(x,t) \geq \frac{1}{2}, t \geq |x|, \tag{9.11}$$

$$u(t,t) = \lim_{t \to x^+} u(x,t) = \frac{1}{2}, x \geq 0. \tag{9.12}$$

Proof. Since (9.8) is the Volterra integral equation, then we can solve it by successive approximations as:

$$u_0(x,t) = \frac{1}{2}H(t - |x|), \tag{9.13}$$

$$u_n(x,t) = \frac{1}{2}H(t - |x|) \int\limits_0^{(x+t)/2} q(\xi)\,d\xi \int\limits_{|\xi|}^{t-|x-\xi|} u_{n-1}(\xi,\tau)\,d\tau, n = 1, ..., \tag{9.14}$$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{9.15}$$

Suppose that $x \in [a,b]$ for an arbitrary finite closed interval $[a,b]$. Then the function $u_n(x,t)$ can be estimated as:

$$|u_n(x,t)| \leq \frac{(Ct)^n}{n!}, x \in [a,b], \tag{9.16}$$

where the positive constant $C = C\left(a, b, \|q\|_{C[0,1]}\right) > 0$ depends only on listed parameters. Hence, (9.16) implies that series (9.15) converges uniformly and absolutely for $(x,t) \in [a,b] \times [0,T]$. Next, since by (9.2) $q(x) \geq 0$ for all x, then (9.14) implies that all functions $u_n(x,t) \geq 0$. This, (9.13), (9.15) and (9.16) imply (9.11). Equality (9.12) follows from (8.15). $\square$

9.3 Integro-differential equation

Consider the function $u(x,t)$ for $x > 0$ above the characteristic line $\{t = x\}$ and change the variables as

$$v(x,t) = u(x,t+x), \text{for } x,t > 0. \tag{9.17}$$

Substituting in (9.3) and using Lemma 9.2, we obtain

$$v_{xx} - 2v_{xt} + q(x)\,v = 0, \text{ for } x,t > 0, \tag{9.18}$$

$$v(x,0) = \frac{1}{2}, \text{ for } x > 0, \tag{9.19}$$

$$v\left(0,t\right)=f_{0}\left(t\right),v_{x}\left(0,t\right)=f_{0}^{\prime}\left(t\right)+f_{1}\left(t\right), \tag{9.20}$$

$$v(x,t)\geq\frac{1}{2},\text{ for }x,t>0. \tag{9.21}$$

It follows from (9.21) that we can consider the function $p(x,t)$,

$$p(x,t)=\ln v(x,t). \tag{9.22}$$

Using (9.18)-(9.22), we obtain a nonlinear equation,

$$p_{xx}-2p_{xt}+p_{x}^{2}-2p_{x}p_{t}=-q\left(x\right),\text{ for }\left(x,t\right)\in\left(0,1\right)\times\left(0,T\right), \tag{9.23}$$

$$p(x,t)=-\ln 2, \tag{9.24}$$

$$p\left(0,t\right)=\ln f_{0}\left(t\right),\ p_{x}\left(0,t\right)=\frac{f_{0}^{\prime}\left(t\right)+f_{1}\left(t\right)}{f_{0}\left(t\right)}. \tag{9.25}$$

The absorbing boundary condition at $x=1$ implies

$$p_{x}\left(1,t\right)=0,t\in\left(0,T\right). \tag{9.26}$$

Hence, the target coefficient is

$$q\left(x\right)=2p_{xt}\left(x,0\right). \tag{9.27}$$

Equation (9.23) has two unknown functions, $p(x,t)$ and $q(x)$, which is inconvenient. On the other hand, the function $q(x)$ stands alone is in (9.23), and it is independent on t. Therefore, we follow the first step of the method of Bukhgeim-Klibanov method. More precisely, we differentiate both sides of equation (9.23) with respect to t. Thus, we eliminate the unknown coefficient $q(x)$ from this equation and obtain an integro-differential equation this way.

Let

$$w(x,t)=p_{t}(x,t). \tag{9.28}$$

Then (9.24) and (9.28) imply

$$p(x,t)=\int\limits_{0}^{t}w(x,\tau)d\tau-\ln 2. \tag{9.29}$$

Also, by (9.27) and (9.28)

$$q\left(x\right)=2w_{x}\left(x,0\right). \tag{9.30}$$

Define the quasilinear integro-differential operator L as

$$L(w)=w_{xx}-2w_{xt}+2w_{x}\int\limits_{0}^{t}w_{x}(x,\tau)d\tau-2w_{x}w-2w_{t}\int\limits_{0}^{t}w_{x}(x,\tau)d\tau. \tag{9.31}$$

Hence, (9.25)-(9.31) imply

$$L(w)=0,\ \ \left(x,t\right)\in\left(0,1\right)\times\left(0,T\right), \tag{9.32}$$

$$w\left(0,t\right)=g_{0}\left(t\right),\ w_{x}\left(0,t\right)=g_{1}\left(t\right), \tag{9.33}$$

where

$$g_0(t) = f_0'(t)/f_0(t), \;\; g_1(t) = \frac{d}{dt}[(f_0'(t) + f_1(t))/f_0(t)].$$

Assume that we have two solutions of problem (9.31)-(9.33), w_1, w_2. Then by (9.30) we also have two coefficients q_1, q_2. Denote

$$\widetilde{w} = w_1 - w_2. \tag{9.34}$$

Use formula (**??**)

$$a_1 b_1 - a_2 b_2 = (a_1 - a_2)\, b_1 + a_2\, (b_1 - b_2)\,, \forall a_1, b_1, a_2, b_2 \in \mathbb{R}.$$

Denote

$$R = \{(x,t) \in (0,1) \times (0,T)\}\,.$$

Then (9.31)-(9.33) imply

$$\widetilde{w}_{xx} - 2\widetilde{w}_{xt} = -2\widetilde{w}_x \int\limits_0^t w_{1x}(x,\tau)d\tau - 2w_{2x}\int\limits_0^t \widetilde{w}_x(x,\tau)d\tau \tag{9.35}$$

$$+2\widetilde{w}_x w_1 + 2w_{2x}\widetilde{w} + 2\widetilde{w}_t \int\limits_0^t w_{1x}(x,\tau)d\tau + 2w_{2t}\int\limits_0^t \widetilde{w}_x(x,\tau)d\tau, (x,t) \in R.$$

$$\widetilde{w}\,(0,t) = \widetilde{w}_x\,(0,t) = 0, t \in (0,T)\,, \tag{9.36}$$

where

$$R = (0,1) \times (0,T)\,.$$

We need to prove that (9.35) and (9.36) imply that

$$\widetilde{w}\,(x,t) \equiv 0 \text{ for } \;(x,t) \in R.$$

9.4 Carleman estimate for the operator $\partial_{xx}^2 - 2\partial_x\partial_t$

Introduce the function $\varphi_\lambda(x,t)$,

$$\varphi_\lambda(x,t) = \exp\left(-2\lambda(x+\alpha t)\right), \alpha \in \left(0, \frac{1}{2}\right), \tag{9.37}$$

where $\lambda \geq 1$ is a parameter, and α is another parameter. Define the space

$$H_0^2(R) = \left\{u \in H^2(R) : u\,(0,t) = u_x\,(0,t) = 0\right\}. \tag{9.38}$$

Theorem 9.1 (Carleman estimate). *There exist constants $C = C(\alpha) > 0$ and $\lambda_0 = \lambda_0(\alpha) \geq 1$ depending only on α such that for all functions $u \in H_0^2(R)$ and for all $\lambda \geq \lambda_0$*

the following Carleman estimate is valid:

$$\int\limits_R (u_{xx}-2u_{xt})^2\varphi_\lambda dxdt \geq C\lambda \int\limits_R (u_x^2+u_t^2)\varphi_\lambda dxdt + C\lambda^3 \int\limits_R u^2\varphi_\lambda dxdt$$
$$+ C\lambda \int\limits_0^1 u_x^2(x,0)e^{-2\lambda x}dx + C\lambda^3 \int\limits_0^1 u^2(x,0)e^{-2\lambda x}dx - C\lambda e^{-2\lambda\alpha T}\int\limits_0^1 u_x^2(x,T)dx \tag{9.39}$$
$$- C\lambda^3 e^{-2\lambda\alpha T}\int\limits_0^1 u^2(x,T)dx.$$

Remarks 9.1:

1. Note that integrals over $\{x\in(0,1), t=0\}$ are non-negative ones in (9.39). On the other hand, integrals over $\{x\in(0,1), t=T\}$ are non positive ones, which is inconvenient. So, we will handle the latter ones in the proof of Theorem 9.2 in subsection 9.6.

2. Another interesting observation here is that an integral over $\{x=1, t\in(0,T)\}$ is not present in (9.39). This means that we do not need to add a boundary condition at $\{x=1\}$ in (9.36) when proving (**??**).

3. Since the function $\varphi_\lambda(x,t)$ is involved as the weight function in Carleman estimate (9.39), then is called the "Carleman Weight Function" for the operator $\partial_x^2-2\partial_x\partial_t$ in the rectangle R.

Proof of Theorem 9.1. In this proof $C=C(\alpha)>0$ denotes different constants depending only on α. We assume in this proof that the function $u\in C^2(\overline{R})\cap H_0^2(R)$. The more general case $u\in H_0^2(R)$ can be obtained from this one via density arguments. Introduce a new function

$$v(x,t)=u(x,t)e^{-\lambda(x+\alpha t)} \tag{9.40}$$

and express $u_{xx}-2u_{xt}$ via derivatives of the function $v(x,t)$. We obtain:

$$u=ve^{\lambda(x+\alpha t)},\quad u_x=(v_x+\lambda v)e^{\lambda(x+\alpha t)},\quad u_t=(v_t+\lambda\alpha v)e^{\lambda(x+\alpha t)},$$
$$u_{xx}=\left(v_{xx}+2\lambda v_x+\lambda^2 v\right)e^{\lambda(x+\alpha t)},\quad u_{xt}=\left(v_{xt}+\lambda\alpha v_x+\lambda v_t+\lambda^2\alpha v\right)e^{\lambda(x+\alpha t)},$$
$$(u_{xx}-2u_{xt})^2e^{-2\lambda(x+\alpha t)}=\left[\left(v_{xx}-2v_{xt}+\lambda^2(1-2\alpha)v\right)+(2\lambda(1-\alpha)v_x-2\lambda v_t)\right]^2.$$

Hence,

$$(u_{xx}-2u_{xt})^2e^{-2\lambda(x+\alpha t)} \geq \frac{(u_{xx}-2u_{xt})^2e^{-2\lambda(x+\alpha t)}}{x+1} \geq \tag{9.41}$$
$$\frac{(4\lambda(1-\alpha)v_x-4\lambda v_t)\left(v_{xx}-2v_{xt}+\lambda^2(1-2\alpha)v\right)}{x+1}.$$

We estimate from below in two steps two products in the second line of (9.41) involving v_x and v_t.

Step 1. Estimate

$$\frac{4\lambda(1-\alpha)v_x\left(v_{xx}-2v_{xt}+\lambda^2(1-2\alpha)v\right)}{x+1}=\left(\frac{2\lambda(1-\alpha)v_x^2}{x+1}\right)_x+\frac{2\lambda(1-\alpha)v_x^2}{(x+1)^2}+$$
$$\left(-\frac{4\lambda(1-\alpha)v_x^2}{x+1}\right)_t+\left(\frac{2\lambda^3(1-\alpha)(1-2\alpha)v^2}{x+1}\right)_x+\frac{2\lambda^3(1-\alpha)(1-2\alpha)v^2}{(x+1)^2}.$$

Thus, we have obtained on the first step:

$$\frac{4\lambda(1-\alpha)v_x\left(v_{xx}-2v_{xt}+\lambda^2(1-2\alpha)v\right)}{x+1}=\frac{2\lambda(1-\alpha)v_x^2}{(x+1)^2}+\frac{2\lambda^3(1-\alpha)(1-2\alpha)v^2}{(x+1)^2}+$$
$$\left(\frac{2\lambda(1-\alpha)v_x^2}{x+1}+\frac{2\lambda^3(1-\alpha)(1-2\alpha)v^2}{x+1}\right)_x+\left(-\frac{4\lambda(1-\alpha)v_x^2}{x+1}\right)_t. \tag{9.42}$$

Step 2. Estimate

$$-\frac{4\lambda v_t\left(v_{xx}-2v_{xt}+\lambda^2(1-2\alpha)v\right)}{x+1}=\left(-\frac{4\lambda v_tv_x}{x+1}\right)_x+\frac{4\lambda v_{xt}v_x}{x+1}-\frac{4\lambda v_tv_x}{(x+1)^2}+ \tag{9.43}$$
$$\left(\frac{4\lambda v_t^2}{x+1}\right)_x+\frac{4\lambda v_t^2}{(x+1)^2}+\left(-\frac{2\lambda^3(1-2\alpha)v^2}{x+1}\right)_t=\frac{4\lambda v_t^2-4\lambda v_tv_x}{(x+1)^2}+$$
$$\left(\frac{2\lambda v_x^2-2\lambda^3(1-2\alpha)v^2}{x+1}\right)_t+\left(\frac{4\lambda v_t^2-4\lambda v_tv_x}{x+1}\right)_x.$$

Thus,

$$-\frac{4\lambda v_t\left(v_{xx}-2v_{xt}+\lambda^2(1-2\alpha)v\right)}{x+1}=\frac{4\lambda v_t^2}{(x+1)^2}-\frac{4\lambda v_tv_x}{(x+1)^2}$$
$$\left(\frac{2\lambda v_x^2-2\lambda^3(1-2\alpha)v^2}{x+1}\right)_t+\left(\frac{4\lambda v_t^2-4\lambda v_tv_x}{x+1}\right)_x. \tag{9.44}$$

Summing up (9.42) with (9.44), we obtain

$$(u_{xx}-2u_{xt})^2e^{-2\lambda(x+\alpha t)}\geq\frac{2\lambda}{(x+1)^2}\left[(1-\alpha)v_x^2-2v_xv_t+2v_t^2\right]+$$
$$\frac{2\lambda^3(1-\alpha)(1-2\alpha)v^2}{(x+1)^2}+\left(\frac{-2(1-2\alpha)\left(\lambda v_x^2+\lambda^3v^2\right)}{x+1}\right)_t \tag{9.45}$$
$$+\left(\frac{2\lambda(1-\alpha)v_x^2-4\lambda v_tv_x+4\lambda v_t^2}{x+1}+\frac{2\lambda^3(1-\alpha)(1-2\alpha)v^2}{x+1}\right)_x$$

The Cauchy-Schwarz inequality with ε :

$$2ab\geq-\varepsilon a^2-\frac{1}{\varepsilon}b^2,\forall a,b\in\mathbb{R}.$$

Hence,

$$2\lambda(1-\alpha)v_x^2-4\lambda v_tv_x+4\lambda v_t^2\geq2\lambda\left[(1-\alpha-\varepsilon)v_x^2+\left(2-\frac{1}{\varepsilon}\right)v_t^2\right]. \tag{9.46}$$

Thus, in order to ensure the positivity of both terms in the right hand side of (9.46), we should have $1/2 < \varepsilon < 1 - \alpha$. We take ε as the average of lower and upper bounds of these two inequalities,

$$\varepsilon = \frac{1}{2}\left(\frac{1}{2} + (1-\alpha)\right) = \frac{3-2\alpha}{4}.$$

Hence, (9.46) becomes

$$2\lambda (1-\alpha) v_x^2 - 4\lambda v_t v_x + 4\lambda v_t^2 \geq \frac{\lambda (1-2\alpha)}{2} v_x^2 + \frac{4\lambda (1-2\alpha)}{3-2\alpha} v_t^2. \tag{9.47}$$

Note that since $u \in C^2\left(\overline{R}\right) \cap H_0^2(R)$, then $v(0,t) = v_x(0,t) = 0$. Hence, (9.47) implies that the integral of the last line of (9.45) is

$$\int\limits_R \left(\frac{2\lambda (1-\alpha) v_x^2 - 4\lambda v_t v_x + 4\lambda v_t^2}{x+1} + \frac{2\lambda^3 (1-\alpha)(1-2\alpha) v^2}{x+1} \right)_x dxdt$$

$$= \int\limits_0^T \left(\frac{2\lambda (1-\alpha) v_x^2 - 4\lambda v_t v_x + 4\lambda v_t^2}{x+1} + \frac{2\lambda^3 (1-\alpha)(1-2\alpha) v^2}{x+1} \right)(1,t)\, dt$$

$$- \int\limits_0^T \left(\frac{2\lambda (1-\alpha) v_x^2 - 4\lambda v_t v_x + 4\lambda v_t^2}{x+1} + \frac{2\lambda^3 (1-\alpha)(1-2\alpha) v^2}{x+1} \right)(0,t)\, dt \tag{9.48}$$

$$= \int\limits_0^T \left(\frac{2\lambda (1-\alpha) v_x^2 - 4\lambda v_t v_x + 4\lambda v_t^2}{x+1} + \frac{2\lambda^3 (1-\alpha)(1-2\alpha) v^2}{x+1} \right)(1,t)\, dt \geq 0.$$

We have used the fact here that the third line of (9.48) equals zero.

Hence, integrating (9.45) over R and taking into account (9.48), we obtain

$$\int\limits_R (u_{xx} - 2u_{xt})^2 e^{-2\lambda(x+\alpha t)} \geq C\lambda \int\limits_R \left(v_x^2 + v_t^2\right) dxdt + C\lambda^3 \int\limits_R v^2 dxdt$$

$$+ C\lambda \int\limits_0^1 v_x^2(x,0)\, dx + C\lambda^3 \int\limits_0^1 v^2(x,0)\, dx - C\lambda \int\limits_0^1 v_x^2(x,T)\, dx - C\lambda^3 \int\limits_0^1 v^2(x,T)\, dx. \tag{9.49}$$

We now replace in (9.49) the function v with the function u via (9.40). We have

$$\lambda v_x^2 = \lambda\left(u_x^2 - 2\lambda u_x u + \lambda^2 u^2\right) e^{-2\lambda(x+\alpha t)} \geq \left(\frac{\lambda}{2} u_x^2 - \lambda^3 u^2\right) e^{-2\lambda(x+\alpha t)},$$

$$\lambda v_t^2 = \lambda\left(u_t^2 - 2\lambda \alpha u_t u + \lambda^2 \alpha^2 u^2\right) e^{-2\lambda(x+\alpha t)} \geq \left(\frac{\lambda}{2} u_t^2 - \lambda^3 \alpha^2 u^2\right) e^{-2\lambda(x+\alpha t)}.$$

Thus,

$$C\lambda\left(v_x^2 + v_t^2\right) \geq \frac{C}{4}\lambda\left(v_x^2 + v_t^2\right) \geq \left(\frac{C}{8}\lambda\left(u_x^2 + u_t^2\right) - \frac{C}{2}\lambda^3 u^2\right) e^{-2\lambda(x+\alpha t)}.$$

Hence, (9.49) implies the following estimate, which is equivalent with (9.39):

$$\int\limits_R (u_{xx} - 2u_{xt})^2 e^{-2\lambda(x+\alpha t)} \geq \frac{C}{8}\lambda \int\limits_R \left(u_x^2 + u_t^2\right) e^{-2\lambda(x+\alpha t)} dxdt$$

$$+ \frac{C}{2}\lambda^3 \int\limits_R u^2 e^{-2\lambda(x+\alpha t)} dxdt + \frac{C}{8}\lambda \int\limits_0^1 u_x^2(x,0) e^{-2\lambda x} dx$$

$$+ \frac{C}{2}\lambda^3 \int\limits_0^1 u^2(x,0) e^{-2\lambda x} dx - C\lambda e^{-2\lambda\alpha T} \int\limits_0^1 u_x^2(x,T) dx - C\lambda^3 e^{-2\lambda\alpha T} \int\limits_0^1 u^2(x,T) dx.$$

□

9.5 Estimating a Volterra-like integral

Lemma 9.3 is very important for the Bukhgeim-Klibanov method.

Lemma 9.3. *For any two numbers $\lambda, \alpha > 0$ and for any function $g \in L_2(R)$ the following estimate is valid:*

$$\int\limits_R \left(\int\limits_0^t g(x,\tau)d\tau\right)^2 \varphi_\lambda dxdt \leq \frac{1}{\alpha^2\lambda^2} \int\limits_R g^2 \varphi_\lambda dxdt. \tag{9.50}$$

Proof. Using (9.37), integration by parts and the Cauchy-Schwarz inequality, we obtain

$$I = \int\limits_R \left(\int\limits_0^t g(x,\tau)d\tau\right)^2 \varphi_\lambda dxdt = \int\limits_0^1 e^{-2\lambda x} \int\limits_0^T e^{-2\lambda\alpha t} \left(\int\limits_0^t g(x,\tau)d\tau\right)^2 dtdx =$$

$$\int\limits_0^1 e^{-2\lambda x} \int\limits_0^T \frac{d}{dt}\left(-\frac{e^{-2\lambda\alpha t}}{2\lambda\alpha}\right) \left(\int\limits_0^t g(x,\tau)d\tau\right)^2 dtdx =$$

$$-\int\limits_0^1 e^{-2\lambda x} \frac{e^{-2\lambda\alpha T}}{2\lambda\alpha} \left(\int\limits_0^T g(x,\tau)d\tau\right)^2 dx$$

$$+ \frac{1}{\lambda\alpha} \int\limits_R e^{-2\lambda x} e^{-2\lambda\alpha t} g(x,t) \left(\int\limits_0^t g(x,\tau)d\tau\right) dtdx \leq$$

$$\frac{1}{\lambda\alpha} \left[\int\limits_R g^2\varphi_\lambda dxdt\right]^{1/2} \left[\int\limits_R \left(\int\limits_0^t g(x,\tau)d\tau\right)^2 \varphi_\lambda dxdt\right]^{1/2}.$$

Here, we have used the fact that the term in the third line of the above is negative. Hence, we have obtained that

$$I \leq \frac{1}{\lambda\alpha} \left(\int\limits_R g^2\varphi_\lambda dxdt\right)^{1/2} \sqrt{I}. \tag{9.51}$$

Dividing both sides of (9.51) by $\sqrt{I}$ and squaring both sides of the resulting inequality, we obtain (9.50). $\square$

In a different form, inequality (9.50) can be expressed as follows: For $\lambda, \alpha > 0$, consider the weighted L_2-space

$$L_{2,\varphi_\lambda}(R) = \left\{ u(x,t) : \|u\|^2_{L_{2,\varphi}(R)} = \int_R u^2 \varphi_\lambda dxdt < \infty \right\}.$$

Then (9.50) means that

$$\|u\|^2_{L_{2,\varphi_\lambda}(R)} \leq \frac{1}{\alpha^2\lambda^2} \left\| \int_0^t u(x,\tau)\, d\tau \right\|^2_{L_{2,\varphi_\lambda}(R)}, \forall u \in L_2(R).$$

9.6 Uniqueness theorem for Coefficient Inverse Problem 9.1

Theorem 9.2. *Assume that two functions $q_1(x)$ and $q_2(x)$ satisfy conditions (9.2). Let $u_1(x,t)$ and $u_2(x,t)$ be two corresponding solutions of the Cauchy problem (9.1)-(9.3). Suppose that both functions $u_1(x,t)$ and $u_2(x,t)$ satisfy the same boundary conditions (9.5) at $x = 0$. Assume also that*

$$T > 2. \tag{9.52}$$

Then $q_1(x) \equiv q_2(x)$, and $u_1(x,t) \equiv u_2(x,t)$ for $(x,t) \in R$.

Proof. Denote

$$\widetilde{q}(x) = q_1(x) - q_2(x).$$

By (9.30) and (9.34)

$$\widetilde{q}(x) = 2\widetilde{w}_x(x,0).$$

Hence, it is sufficient to prove that

$$\widetilde{w}(x,0) = 0, x \in (0,1). \tag{9.53}$$

To do this, we combine the Carleman estimate of Theorem 9.1 and Lemma 9.3.

Rewrite equation (9.35) in a more general form as an integro-differential inequality rather than just an equation,

$$|\widetilde{w}_{xx} - 2\widetilde{w}_{xt}| \leq B \left(|\widetilde{w}_x| + |\widetilde{w}_t| + |\widetilde{w}| + \int_0^t |\widetilde{w}_x(x,\tau)|\, d\tau \right), (x,t) \in R. \tag{9.54}$$

And also by (9.36)

$$\widetilde{w}(0,t) = \widetilde{w}_x(0,t) = 0, t \in (0,T). \tag{9.55}$$

Here and below $B > 0$ denotes *different* positive constants depending only on maximal values of functions $|w_{1x}|$, $|w_{2x}|$,$|w_1|$, $|w_{2t}|$ over $\overline{R}$ as well as of R. In addition, below B will also depend on the parameter $\alpha \in (0,1/2)$, see (9.37).

Square both sides of (9.54) and apply Cauchy-Schwarz inequality. We obtain

$$(\widetilde{w}_{xx} - 2\widetilde{w}_{xt})^2 \leq B \left[\widetilde{w}_x^2 + \widetilde{w}_t^2 + \widetilde{w}^2 + \left(\int_0^t |\widetilde{w}_x(x,\tau)|\, d\tau \right)^2 \right], (x,t) \in R.$$

Multiply both sides of this inequality by the Carleman Weight Function $\varphi_\lambda(x,t)$ in (9.37) and integrate over R. We obtain

$$B\int\limits_R \left[\widetilde{w}_x^2+\widetilde{w}_t^2+\widetilde{w}^2+\left(\int\limits_0^t \left|\widetilde{w}_x(x,\tau)\right|d\tau\right)^2\right]\varphi_\lambda dxdt$$

$$\geq \int\limits_R (\widetilde{w}_{xx}-2\widetilde{w}_{xt})^2\,\varphi_\lambda dxdt. \tag{9.56}$$

By (9.38) and (9.55) the function $\widetilde{w}\in H_0^2(R)$. Hence, Theorem 9.1 is applicable to this function. Applying (9.39), we obtain from (9.56) for $\lambda\geq\lambda_0\geq 1$

$$B\int\limits_R \left[\widetilde{w}_x^2+\widetilde{w}_t^2+\widetilde{w}^2+\left(\int\limits_0^t \left|\widetilde{w}_x(x,\tau)\right|d\tau\right)^2\right]\varphi_\lambda dxdt$$

$$\geq C\lambda\int\limits_R \left(\widetilde{w}_x^2+\widetilde{w}_t^2+\widetilde{w}^2\right)\varphi_\lambda dxdt + C\lambda\int\limits_0^1 \left(\widetilde{w}_x^2+\widetilde{w}^2\right)(x,0)\,e^{-2\lambda x}dx \tag{9.57}$$

$$-C\lambda^3 e^{-2\lambda\alpha T}\int\limits_0^1 \left(\widetilde{w}_x^2+\widetilde{w}^2\right)(x,T)\,dx.$$

Using Lemma 9.3, we make estimate (9.57) stronger via absorbing the term with the Volterra integral in the left hand side of (9.57) for $\lambda\geq\lambda_0\geq 1$,

$$B\int\limits_R \left(\widetilde{w}_x^2+\widetilde{w}_t^2+\widetilde{w}^2\right)\varphi_\lambda dxdt$$

$$\geq C\lambda\int\limits_R \left(\widetilde{w}_x^2+\widetilde{w}_t^2+\widetilde{w}^2\right)\varphi_\lambda dxdt + C\lambda\int\limits_0^1 \left(\widetilde{w}_x^2+\widetilde{w}^2\right)(x,0)\,e^{-2\lambda x}dx$$

$$-C\lambda^3 e^{-2\lambda\alpha T}\int\limits_0^1 \left(\widetilde{w}_x^2+\widetilde{w}^2\right)(x,T)\,dx.$$

Choose $\lambda_1\geq\lambda_0$ so large that

$$B<\frac{C\lambda_1}{2}.$$

Then for all $\lambda\geq\lambda_1$

$$0\geq B\lambda\int\limits_R \left(\widetilde{w}_x^2+\widetilde{w}_t^2+\widetilde{w}^2\right)\varphi_\lambda dxdt + B\lambda\int\limits_0^1 \left(\widetilde{w}_x^2+\widetilde{w}^2\right)(x,0)\,e^{-2\lambda x}dx$$

$$-\lambda^2 e^{-2\lambda\alpha T}\int\limits_0^1 \left(\widetilde{w}_x^2+\widetilde{w}^2\right)(x,T)\,dx.$$

Or, making the latter inequality stronger,

$$0 \geq \int_0^1 \left(\widetilde{w}_x^2 + \widetilde{w}^2\right)(x,0)\, e^{-2\lambda x} dx - B\lambda^2 e^{-2\lambda\alpha T} \int_0^1 \left(\widetilde{w}_x^2 + \widetilde{w}^2\right)(x,T)\, dx. \tag{9.58}$$

Obviously $e^{-2\lambda x} \geq e^{-2\lambda}$ for $x \in (0,1)$. Hence, dividing both sides of (9.58) by $e^{-2\lambda}$, we obtain for all $\lambda \geq \lambda_1$

$$\int_0^1 \left(\widetilde{w}_x^2 + \widetilde{w}^2\right)(x,0)\, dx \leq B\lambda^2 \exp\left(-2\lambda\left(\alpha T - 1\right)\right) \int_0^1 \left(\widetilde{w}_x^2 + \widetilde{w}^2\right)(x,T)\, dx. \tag{9.59}$$

Since by (9.52) $T > 2$, then there exists a number $\alpha \in (0, 1/2)$ such that $\alpha T - 1 > 0$. Hence, for this α

$$\lim_{\lambda\to\infty} \left[\lambda^3 \exp\left(-2\lambda\left(\alpha T - 1\right)\right)\right] = 0.$$

Hence, setting in (9.59) $\lambda \to \infty$, we obtain

$$\int_0^1 \left(\widetilde{w}_x^2 + \widetilde{w}^2\right)(x,0)\, dx = 0,$$

which proves (9.53). $\square$

9.7 The scheme of the Bukhgeim-Klibanov method

Here is a brief scheme of the Bukhgeim-Klibanov method. I give it here only for proofs of uniqueness theorems of Coefficient Inverse Problems. This method can also be applied for proofs of conditional stability results for these problems. In addition, it can be applied for the numerical side of this theory. In these cases, the scheme is modified, see the above cited books of Beilina and Klibanov and Klibanov and Li.

The equation with the unknown coefficient is inconvenient to investigate. Indeed, it is just one equation containing two unknown functions: solution u of this equation and the unknown coefficient q. Therefore, it is desirable to eliminate somehow q from the equation. However, a price for this must be paid. This price is usually the elimination of one of the original initial conditions for the resulting function $\widehat{u}$.

1. **Step 1**. Isolate the unknown coefficient q as in (9.23). You obtain a nonlinear partial differential equation the new function $p(x,t)$ as in (9.23).

2. **Step 2.** Differentiate the equation for p with respect to such a variable from which q does not depend. In our specific case, this parameter is t.

3. **Step 3.** Let $w = p_t$. Express p via w as in (9.29)

$$p(x,t) = \int_0^t w(x,\tau)d\tau + p(x,0).$$

   You obtain a nonlinear integro-differential equation with respect to w with Volterra integrals as in (9.31) with the boundary conditions as in (9.33).

4. **Step 4.** Assuming the existence of two possible solutions of the original Coefficient Inverse Problem, consider the difference $\widetilde{w} = w_1 - w_2$ and obtain a linear integro-differential equation for $\widetilde{w}$ with zero boundary conditions as in (9.35), (9.36).

5. **Step 5.** Replace the boundary value problem of Step 4 with a more general problem via replacing the equation with the inequality as in (9.54), (9.55).

6. **Step 6.** Apply the Carleman estimate to the problem of Step 5. In doing so, it is absolutely necessary to estimate those Volterra integrals, similarly with Lemma 9.3.

7. **Step 7.** In the end of the proof set the parameter of the Carleman Weight Function $\lambda \to \infty$ to prove that $\widetilde{w} = 0$ in the desired subdomain of the original domain of interest.