ENDPOINT-HOMOGENEOUS FANS

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Abstract. A fan F is *endpoint-homogeneous* if for any two endpoints e, e' of F, there is a homeomorphism $h : F \to F$ such that $h(e) = e'$.

We prove there are uncountably many distinct homeomorphism types of endpoint-homogeneous smooth fans. To do this, we associate to each such fan F a topological invariant, in the form of an equivalence class $EPG(F)$ of subsets of $[0, 1]$ describing how the endpoints of F limit onto any given blade of F. We describe precisely all the uncountably many different classes that can arise as $EPG(F)$ for some endpoint-homogeneous smooth fan F. We also prove the existence of $\frac{1}{n}$-homogeneous smooth fans for all $n \geq 5$.

1. Introduction

A topological space X is *homogeneous* if for any $x, y \in X$, there is a homeomorphism $h : X \to X$ such that $h(x) = y$. This property expresses that X has an especially rich group of self-homeomorphisms.

Many continua fail to be homogeneous, but still have very rich self-homeomorphism groups. The class of *fans*, the main object of study in this paper, contains many examples of this kind. A fan cannot be homogeneous. It has one "top" point, many endpoints, and many points that are neither. These three classes of points are topologically defined, and consequently are invariant under homeomorphisms. Nevertheless, some fans, such as the Cantor fan and the Lelek fan, have extremely rich self-homeomorphism groups (see [2], [5]).

A fan F is called *endpoint-homogeneous* if for any two endpoints e, e' of F, there is a homeomorphism $h : F \to F$ such that $h(e) = e'$. Equivalently, F is endpoint-homogeneous if for any two "blades" of the fan (arcs connecting the top of the fan to some endpoint), there is a self-homeomorphism of F that restricts to a homeomorphism between these two blades.

Let F be an endpoint-homogeneous fan and E its set of endpoints. Because any blade of F can be mapped to any other blade with a homeomorphism of F, there is a set $X \subseteq [0, 1]$ such that for every blade B of F, there is a homeomorphism $B \to [0, 1]$ mapping $B \cap \overline{E \setminus B}$ onto X. We define $EPG(F)$ to be the class of all such X. Thus $EPG(F)$ is a topological invariant of F describing how the endpoints of F limit onto any given blade of F.

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In this paper we work towards a classification of endpoint-homogeneous smooth fans. Our main results in this direction are:

(1) A set $X \subseteq [0,1]$ is a member of the invariant $EPG(F)$ of some endpoint-homogeneous smooth fan F if and only if $X = [0,1]$, $X = \emptyset$, $X = \{1\}$ or X is a closed subset of $[0,1]$ such that $0 \in X$ and 1 is not a cluster point of X.
(2) In particular, there are uncountably many such $X \subseteq [0,1]$ up to homeomorphism, while the members of $EPG(F)$ for some fixed F are all pairwise homeomorphic. It follows that there are uncountably many homeomorphism types of endpoint-homogeneous smooth fans.

The proof of (1) spans Sections 3 and 5. Section 3 is devoted to proving the "only if" part of (1): i.e., that if $X \in EPG(F)$ for some smooth fan F then X must have one of the types listed above. This is stated as Theorem 3.8 at the end of Section 3. The results leading up to Theorem 3.8 give us some idea of what F must look like, depending on $EPG(F)$:

(3) Suppose F is a smooth endpoint-homogeneous fan and $X \in EPG(F)$.
    - If $X = \emptyset$ then F is an n-od, if $X = \{1\}$ then F is the Cantor fan, and if $X = [0,1]$ then F is the Lelek fan (Proposition 4.1).
    - If $1 \notin X$ then $E(F)$ is relatively discrete (Proposition 3.1 and its proof).
    - If $1 \in X$ then $E(F)$ is uncountable (Proposition 3.4) and is dense in itself.

Section 5 deals with the "if" part of (1): i.e., it gives constructions of smooth fans F with $X \in EPG(F)$ whenever X has one of the types listed above. The case with $1 \notin X$ is dealt with in Theorem 5.1, and the case with $1 \in X$ is dealt with in Theorem 5.2. Result (2), stated as Theorem 5.4 below, follows relatively easily from (1).

We conjecture that if F and G are smooth fans and $EPG(F) = EPG(G) \neq \{\emptyset\}$, then F and G are homeomorphic. The case $EPG(F) = \{\emptyset\}$ is something of a degenerate case; there are countably many homeomorphism types of smooth fans for which the endpoints do not accumulate onto any blades (namely those fans with finitely many blades, also known as the n-ods, with $n \geq 3$). If true, this conjecture, combined with statement (1) above, would give an exact classification of all endpoint-homogeneous smooth fans.

We do not say much about non-smooth fans, but we do provide a few examples of non-smooth endpoint-homogeneous fans. Some of these examples contrast with our results for smooth fans, showing that the smoothness hypotheses cannot be removed. For instance:

(4) There is a fan F with $[1/2, 1] \in EPG(F)$; in particular, for a non-smooth fan F, the members of $EPG(F)$ do not have to be among the sets described in statement (1) above (Example 7.4).

A natural weakening of homogeneity is $\frac{1}{n}$-homogeneity. A space X is $\frac{1}{n}$-*homogeneous* if there is a partition of X into n sets such that $x, y \in X$ lie in the same partition piece if and only if there is a homeomorphism $h : X \to X$

with $h(x) = y$. Equivalently, X is $\frac{1}{n}$-homogeneous if the homeomorphism group of X has exactly n distinct orbits with respect to its natural action on X.

As mentioned above, fans have 3 distinct classes of points that are invariant under homeomorphisms: the top, the endpoints, and everything else (usually called the *ordinary points*). Thus $\frac{1}{3}$-homogeneity is essentially the strongest weakening of homogeneity that can apply to fans. But it is perhaps too strong. In [2], Acosta, Hoehn, and Juárez completely classify all $\frac{1}{3}$-homogeneous smooth fans. Other than the simple n-ods (again something of a degenerate case), the $\frac{1}{3}$-homogeneous smooth fans come in precisely 4 homeomorphism types: the Cantor fan, the Lelek fan, and two others. Interestingly, they also prove that no smooth fan is $\frac{1}{4}$-homogeneous. Complementing this result of theirs, we show:

(5) For every $n \geq 5$, there is an endpoint-homogeneous, $\frac{1}{n}$-homogeneous smooth fan.

The construction for odd n is found in Theorem 6.1, and the more difficult construction for even n in Theorem 6.2.

Note that endpoint-homogeneity is a strict weakening of $\frac{1}{3}$-homogeneity: we require only that the set of endpoints (but not its complement) be an orbit of the homeomorphism group. This way of weakening $\frac{1}{3}$-homogeneity brings us from having only 4 nontrivial examples to having uncountably many.

It is worth mentioning that $\frac{1}{n}$-homogeneity is a broadly studied topic in continuum theory, not just with $n = 3$ and not just with fans. One notable example is [3], which provides a complete classification of all $\frac{1}{3}$-homogeneous dendrites. The introduction of [3] contains an extensive list of papers on the topic of $\frac{1}{n}$-homogeneity.

It is also worth mentioning that the opposite of endpoint-homogeneous fans was studied by Hernández-Gutiérrez and Hoehn in [13]. They define a fan F to be *endpoint-rigid* if every self-homeomorphism of F restricts to the identity map on the set of endpoints of F. They construct many examples of such fans, including several where the set of endpoints of the fan (with the rest of the fan removed) is a homogeneous space.

The characteristic set $EPG(F)$, as described above, is not well-defined for all smooth fans, but only those for which no two blades can be distinguished by looking at limits of endpoints. (We do not know whether this latter class is actually larger than the class of smooth endpoint-homogeneous fans.) The fans with this property were first defined in [12], where the fans F such that $X \in EPG(F)$ are called the *X-endpoint-generated* fans (below also X-EPG fans). These arise in a classification of the Mahavier products obtained from a pair of lines meeting at the origin. (In all but a few cases, it is an X-EPG fan.) This classification is the main result of [12] and [6]. The work in the present paper arose from the desire to classify those X for which smooth X-EPG fans exist.

2. Definitions and preliminaries

In this section we give some definitions, notation, and preliminary results that are used throughout the paper. All undefined notions can be found in [14]. In what follows, we usually write I to denote the unit interval $[0,1]$.

Definition 2.1. A *continuum* is a non-empty compact connected metric space. A *dendroid* is an arcwise connected continuum such that the intersection of any two of its subcontinua is connected. A *fan* is a dendroid with exactly one ramification point t, which is called the *top* of the fan. ⊣

Observe that dendroids generally, and fans in particular, are *uniquely arcwise connected*, which means that if X is a dendroid and $x, y \in X$ with $x \neq y$, then there is exactly one arc in X whose endpoints are x and y.

Definition 2.2. Let F be a fan. A point $e \in F$ is an *endpoint of* F, if it is an endpoint of every arc in F that contains it. The set of all endpoints of F is denoted $E(F)$. A fan F is *endpoint-homogeneous* if for any $e, e' \in E(F)$, there is a homeomorphism $h : F \to F$ such that $h(e) = e'$. ⊣

As examples of endpoint-homogeneous fans, let us mention the Lelek fan and the Cantor fan described in Section 4.

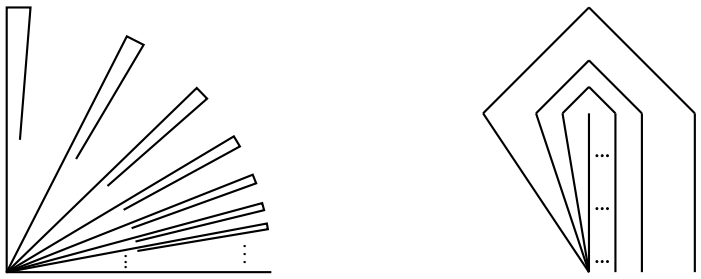
Definition 2.3. Let X be a dendroid. Given $x, y \in X$ with $x \neq y$, let $[x,y]$ denote the unique arc in X with endpoints x and y. If $x = y$, let $[x,y] = \{x\}$.

If t is the top of a fan F and $e \in E(F)$, then $[t,e]$ is a *blade* of F. The set of all blades of F is denoted $\mathcal{B}(F)$. ⊣

Definition 2.4. A fan F with top t is *smooth* if for any convergent sequence $\langle x_n : n \in \mathbb{N} \rangle$ of points in F,

$$\lim_{n\to\infty} [t, x_n] = [t, \lim_{n\to\infty} x_n]$$

where the limit on the left is taken with respect to the Hausdorff metric. ⊣



Two non-smooth fans

The following definition first appeared in [12].

Definition 2.5. Let F be a fan with top t. Given $X \subseteq I$, we say F is an *X-endpoint generated* fan (or an *X-EPG* fan) if for every $B \in \mathcal{B}(F)$, there is a homeomorphism $\phi : I \to B$ such that $\phi(0) = t$ and $\phi[X] = B \cap \overline{E(F) \setminus B}$. ⊣

For example, n-ods are $\emptyset$-EPG fans, the Cantor fan is a $\{1\}$-EPG fan, and the Lelek fan is a I-EPG fan. In fact, Charatonik's characterization of the Lelek fan shows it is the only I-EPG smooth fan.

Observation 2.6. *If F is an endpoint-homogeneous fan, then there is some $X \subseteq I$ such that F is an X-EPG fan.*

Hence, to understand endpoint-homogeneous fans, one must understand X-EPG fans. This raises several questions:

- For which $X \subseteq I$ is there an X-EPG fan? An X-EPG smooth fan?
- For which $X \subseteq I$ can such a fan be endpoint-homogeneous?
- If there is an X-EPG fan for some nonempty $X \subseteq I$, is it unique up to homeomorphism?

As mentioned in the introduction, a significant part of this paper is devoted to fully answering the first two of these questions. We record for now two basic observations in this direction.

Observation 2.7. *If there is an X-EPG fan, then X is closed.*

Observation 2.8. *Let X and Y be closed subsets of I. The following statements are equivalent:*

(1) There is an order-preserving homeomorphism $h : I \to I$ such that $h[X] = Y$.
(2) Some X-EPG fan is a Y-EPG fan.
(3) Every X-EPG fan is a Y-EPG fan.

In light of this observation, we define the following topological invariant.

Definition 2.9. Two subsets of I are *equivalently embedded* if there is an order-preserving homeomorphism $I \to I$ mapping one onto the other. This is an equivalence relation on subsets of I.

If $X \subseteq I$ and F is an X-EPG fan, we define $EPG(F)$ to be the equivalence class of X under the "equivalently embedded" relation. ⊣

Note that $X = \{0\}$, $Y = \{1/2\}$, and $Z = \{1\}$ are all homeomorphic to each other, but not equivalently embedded in I. As we shall see, there are X-EPG and Z-EPG smooth fans, but they look very different (the former have countably many blades and the latter uncountably many, for example); and there are no Y-EPG smooth fans at all.

We end this section with describing a general technique of constructing continua of interest to us. This technique does not originate here. It is used, for example, by Aarts and Oversteegen to study the Lelek fan in [1], or by Hernández-Gutiérrez to study endpoint rigid fans in [13]. At any rate, the technique essentially amounts to just a clever use of polar coordinates. We present the basic idea here, in order to keep our paper reasonably self-contained.

Definition 2.10. Let π_1 and π_2 denote the natural coordinate projection maps $I \times I \to I$; i.e., $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$ whenever $(x,y) \in I^2$. If $C \subseteq I^2$ and $y \in I$, let $C_y = \{x\colon (x,y) \in C\} = \pi_1(\pi_2^{-1}(y) \cap C)$.

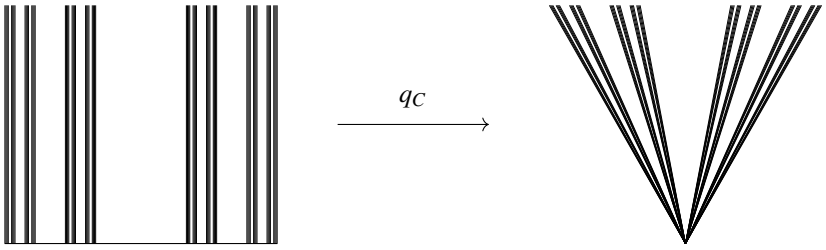
Let $K \subseteq I$ denote the Cantor set. A *comb over K* (or, in context, simply a *comb*) is a closed subset $C \subseteq I^2$ such that

(1) $C_0 = I$,
(2) C_y is a subset of the Cantor set K for all $y \in (0,1]$,
(3) if $y_1 < y_2$ then $C_{y_1} \supseteq C_{y_2}$, and
(4) $|C_y| > 2$ for some $y \in (0,1]$.

Given a comb C, let F_C denote the space obtained by collapsing $I \times \{0\}$ to a point. This space is called the *fan of the comb C*. Similarly, if F is a fan and C is a comb such that $F_C \approx F$, then C is called *a comb of F*. ⊣

Formally, we take $F_C = \{t\} \cup C \setminus I \times \{0\}$. In other words, the points of F_C are literally equal to the points of C with the X-axis removed, plus one extra point t, the top of the fan. The natural quotient map is

$$q(z) = \begin{cases} z & \text{if } \pi_2(z) > 0, \\ t & \text{if } \pi_2(z) = 0. \end{cases}$$



Observation 2.11. *There is a largest comb, namely*

$$C_K = I \times \{0\} \cup K \times (0,1],$$

and the fan of this comb is (homeomorphic to) the Cantor fan.

A proof of the following fact can be found in [9, Theorem 9], [8, Proposition 4], or [11, Corollary 4].

Proposition 2.12. *A space is a smooth fan if and only if it is homeomorphic to a subcontinuum of the Cantor fan and has a ramification point.*

A version of the following theorem can be found as Lemma 2.1 in [13].

Theorem 2.13. *If C is a comb, then F_C is a smooth fan. Conversely, if F is a smooth fan then there is a comb $C \subseteq I^2$ such that $F_C \approx F$.*

Proof. Let C be a comb. As a closed and connected subset of C_K, its quotient F_C is therefore a subcontinuum of the Cantor fan. By requirement (4) in the definition of a comb, it is clear that F_C contains a triod. As a subcontinuum of the Cantor fan, this means that F_C is necessarily a smooth fan itself (by Proposition 2.12).

For the second assertion of the theorem, suppose F is a smooth fan. By Proposition 2.12, F is a subcontinuum of the Cantor fan, which we may (and do) view as the fan of the comb C_K described in Observation 2.11. Let $q : C_K \to F_{C_K}$ denote the quotient mapping from the Cantor comb C_K onto the Cantor fan F_{C_K}, and let $C = q^{-1}(F)$. It is not difficult to check that C is a comb over K, and the fan of this comb is F. □

Observe that combs are dendroids. In particular, we may (and do) use the notation $[z, z']$ to denote the unique arc in a comb with z and z' as endpoints.

Definition 2.14. Let C be a comb, and suppose that (x, y) is an endpoint of C with $y > 0$. Then $[(x, 0), (x, y)]$ is called a *blade* of C, specifically the *blade of C at x*, and (x, y) is called the *tip* of the blade. If C has a blade at some $x \in K$, then the blade at x is denoted B_x. The set of all blades of C is denoted $\mathcal{B}(C)$, and the set of all points that are the tip of some blade is denoted $\mathcal{E}(C)$. ⊣

Observe that $|\mathcal{B}(C)| \geq 3$, for any comb C, by part (4) of the definition of a comb. Of course combs have teeth, not blades, but the following observation shows why we have chosen this terminology.

Observation 2.15. *Let $C \subseteq I^2$ be a comb. Then*

(1) If $B \in \mathcal{B}(C)$ then $q_C[B] \in \mathcal{B}(F_C)$, and if $B' \in \mathcal{B}(F_C)$ then there is some $B \in \mathcal{B}(C)$ with $q_C[B] = B'$.
(2) $\mathcal{E}(C) = E(F_C)$.

In other words, blades of C correspond naturally to blades of F_C, and the tips of those blades correspond naturally (in fact, are equal to) the endpoints of F_C. Note that $\mathcal{E}(C)$ and $E(C)$ may not exactly be equal, but they are nearly so: $\mathcal{E}(C) \subseteq E(C) \subseteq \mathcal{E}(C) \cup \{(0, 0), (1, 0)\}$. In other words, it is possible that $(0, 0)$ and $(1, 0)$ are endpoints of C, but these two points are the only possible difference between $E(C)$ and $\mathcal{E}(C)$. Whether these points are in $E(C)$ depends on whether C has blades at 0 or 1.

We close this section with an observation concerning what a comb of an X-EPG fan must look like.

Observation 2.16. *Let $C \subseteq I^2$ be a comb and let $X \subset I$.*

(1) F_C is an X-EPG fan if and only if for each for each $B \in \mathcal{B}(C)$ there exists a homeomorphism $\phi_B : I \to B$ such that $\phi_B(0) \in I \times \{0\}$ and $\phi_B(X) = B \cap \overline{\mathcal{E}(C) \setminus B}$.
(2) If F_C is an X-EPG fan then $(I \times \{0\}) \cap \overline{\mathcal{E}(C)} \neq \emptyset$ if and only if $0 \in X$.

3. Sets that do not work for $EPG(F)$

This section is devoted to restrictions on X, i.e., showing that certain $X \subseteq I$ can never be $EPG(F)$ for a smooth fan F, giving the proof of one direction of the first main theorem listed in the introduction. Along the way, we also find that the features of X can yield information about what F would need

to look like if $X = EPG(F)$. In the next section, we construct examples of X-EPG fans for every X that is not ruled out in this section.

Proposition 3.1. *If F is an X-EPG fan and $1 \notin X$, then $E(F)$ is relatively discrete and hence countable.*

Proof. Because F is X-EPG and $1 \notin X$, no member of $E(F)$ is a limit point of $E(F)$. □

Lemma 3.2. *Let C be a comb, and suppose F_C is an X-EPG fan. If $0 \notin X$ and $X \neq \emptyset$, then $\pi_1[\mathcal{E}(C)]$ is homeomorphic to the Cantor space.*

Proof. Observe that every blade of C contains exactly one member of $\mathcal{E}(C)$. Consequently, π_1 restricts to an injection on $\mathcal{E}(C)$. We claim that $\pi_1[\mathcal{E}(C)]$ is closed and has no isolated points.

To see that $\pi_1[\mathcal{E}(C)]$ is closed, note that if $0 \notin X$ then $y_0 = \inf \pi_2[\mathcal{E}(C)] > 0$ by Observation 2.16. But then $\pi_1[\mathcal{E}(C)] = C_{y_0}$, which is compact.

To see that $\pi_1[\mathcal{E}(C)]$ has no isolated points, let $x \in \mathcal{E}(C)$. If $X \neq \emptyset$, then $B_{\pi_1(x)}$ contains a cluster point of $\mathcal{E}(C) \setminus B_{\pi_1(x)}$. Since $\pi_1^{-1}(\pi_1(x)) = B_{\pi_1(x)}$, this shows that $\pi_1(x)$ is a cluster point of $\pi_1[\mathcal{E}(C)]$.

Thus $\pi_1[\mathcal{E}(C)]$ is a compact, non-empty, metrizable space with no isolated points, and hence is homeomorphic to the Cantor space by the well-known characterization of Brouwer's (see, e.g., [14, Theorem 7.14, page 109]). □

Corollary 3.3. *Suppose F is an X-EPG smooth fan. If $X \neq \emptyset$ and $0 \notin X$, then $E(F)$ is uncountable.*

Proof. Using Theorem 2.13, let C be a comb such that $F \approx F_C$. By the previous lemma, $\pi_1[\mathcal{E}(C)]$ is homeomorphic to the Cantor space, which means $\mathcal{E}(C)$ is uncountable. But $E(F_C) = \mathcal{E}(C)$, so $E(F)$ is uncountable. □

Contrast the following proposition with Proposition 3.1.

Proposition 3.4. *Let F be an X-EPG smooth fan with $1 \in X$. Then $E(F)$ is uncountable.*

Proof. Applying Theorem 2.13, let C be a comb such that $F_C \approx F$. We wish to show $\mathcal{E}(C) = E(F_C)$ is uncountable. Fix $y_0 > 0$ such that some $e \in \mathcal{E}(C)$ has $\pi_2(e) > y_0$, and let

$$\mathcal{E} = \{e \in \mathcal{E}(C) \colon \pi_2(e) > y_0\} = \mathcal{E}(C) \cap (I \times (y_0, 1]).$$

Because $1 \in X$ and F is X-EPG, $\mathcal{E}(C)$ has no isolated points. Because $\mathcal{E}$ is a (relatively) open subset of $\mathcal{E}(C)$, $\mathcal{E}$ also has no isolated points.

Arguing exactly as in the third paragraph of the proof of Lemma 3.2, but with $\overline{\mathcal{E}}$ in the place of $\mathcal{E}(C)$, one can show that $\pi_1[\overline{\mathcal{E}}]$ has no isolated points and hence is uncountable.

If $x \in \pi_1[\overline{\mathcal{E}}]$, then there is some $y \geq y_0$ such that $(x, y) \in \overline{\mathcal{E}}$. Because C is closed and $\mathcal{E}(C) \subseteq C$, this means $(x, y) \in C$. Because $y \geq y_0 > 0$, this implies C has a blade at x. Thus C has a blade over every member of $\pi_1[\overline{\mathcal{E}}]$.

Combined with the previous paragraph, this shows C has uncountably many blades, which means $\mathcal{E}(C)$ is uncountable. □

The following is the main lemma of this section. It shows how the Baire Category Theorem restricts the shape of combs, and consequently of the smooth fans obtained from those combs. Note that the lemma applies to all combs over the Cantor set, not just those giving rise to EPG fans.

Lemma 3.5. *Let C be a comb, and let (x_0, y_0) be a point of $C \setminus \overline{\mathcal{E}(C)}$ with $y_0 > 0$. Then there is an $x \in \pi_1[\mathcal{E}(C)]$ such that $\{x\} \times (y_0, 1]$ contains exactly one point of $\overline{\mathcal{E}(C)}$, namely the endpoint of the blade over x.*

Proof. Because $y_0 > 0$, the horizontal section C_{y_0} is a subspace of the Cantor space K, which is 0-dimensional. This implies that there is a basic clopen set $V \subseteq K$ with $x_0 \in V$ such that $(V \times \{y_0\}) \cap \overline{\mathcal{E}(C)} = \emptyset$. Let

$$E = \mathcal{E}(C) \cap (V \times (y_0, 1]) = \{(x, y) \in \mathcal{E}(C) \colon x \in V \text{ and } y > y_0\}.$$

For each $x \in \pi_1[\overline{E}]$, let $[m_x, e_x]$ denote the minimal segment in I that contains $(y_0, 1] \cap \pi_2[B_x \cap \overline{\mathcal{E}(C)}]$. Note that the endpoint of B_x is in $B_x \cap \overline{\mathcal{E}(C)}$, so e_x is simply the Y-coordinate of the endpoint of B_x.

Because $x_0 \in V$ and $(x_0, y_0) \in C \setminus \mathcal{E}(C)$, the endpoint of the blade B_{x_0} is a member of E. Hence $E \neq \emptyset$, and $\overline{E}$ is a nonempty compact set. Furthermore, if $(x, y) \in \overline{E}$ then $y \geq y_0$, so there is a blade over X. Thus $\pi_1[\overline{E}] \subseteq \{x \colon \text{there is a blade over } x\} = \pi_1[\mathcal{E}(C)]$.

If $m_x = e_x$ for some $x \in \pi_1[\overline{E}]$, then $x \in \pi_1[\mathcal{E}(C)]$ and $\{x\} \times (y_0, 1]$ contains exactly one point of $\overline{\mathcal{E}(C)}$, namely the endpoint of B_x. In other words, if $m_x = e_x$ then the conclusion of the lemma holds at x. To complete the proof, we show it is impossible to have $m_x < e_x$ for all $x \in \pi_1[\overline{E}]$.

Aiming for a contradiction, suppose $m_x < e_x$ for all $x \in \pi_1[\overline{E}]$. Define

$$L_k = \left\{x \in \pi_1[\overline{E}] \colon e_x - m_x \geq 1/k\right\}$$

for each $k \in \mathbb{N}$. Because $m_x < e_x$ for all $x \in \pi_1[\overline{E}]$, we have $\pi_1[\overline{E}] = \bigcup_{k \in \mathbb{N}} L_k$.

Suppose $\langle x_n \colon n \in \mathbb{N} \rangle$ is a sequence of points in $\pi_1[\overline{E}]$ converging to some point x. Then $\langle (x_n, e_{x_n}) \colon n \in \mathbb{N} \rangle$ and $\langle (x_n, m_{x_n}) \colon n \in \mathbb{N} \rangle$ are both sequences in $\overline{E}$, and because $\overline{E}$ is compact, all the subsequential limits of these sequences are in $\overline{E}$. In particular $(x, \limsup_{n\to\infty} e_{x_n}), (x, \liminf_{n\to\infty} m_{x_n}) \in \overline{E}$. It follows that $\limsup_{n\to\infty} e_{x_n} \leq e_x$ and $\liminf_{n\to\infty} m_{x_n} \geq m_x$. This shows that each of the L_k is a closed subset of $\pi_1[\overline{E}]$.

Thus $\pi_1[\overline{E}]$ is a nonempty compact set, and the union of a countable sequence $L_1, L_2, L_3, \ldots$ of closed sets. By the Baire Category Theorem, there is some k such that L_k contains a nonempty, relatively open subset U of $\pi_1[\overline{E}]$.

Set $m = \inf\{m_x : x \in U\}$, and choose $x \in U$ so that $m_x < m + \frac{1}{2k}$. Because $(x, m_x) \in \overline{E}$ there exists a point $(x', e_{x'}) \in E \cap U$ at distance less than $\frac{1}{2k}$ from (x, m_x). Then $e_{x'} < m_x + \frac{1}{2k} < m + \frac{1}{k}$. Since $m \leq m_{x'} < e_{x'}$ it follows that $e_{x'} - m_{x'} < \frac{1}{k}$. As $x' \in U \subseteq L_k$, this is a contradiction. □

Proposition 3.6. *Suppose $X \in EPG(F)$ for some smooth fan F. If $0 \notin X \neq \emptyset$, then $X = \{1\}$.*

Proof. Let C be a comb for F. Because $0 \notin X$, Observation 2.16(2) states $(I \times \{0\}) \cap \overline{\mathcal{E}(C)} = \emptyset$. Because I is compact, there means that there is some $y_0 > 0$ with $\overline{\mathcal{E}(C)} \subseteq I \times (y_0, 1]$.

Fix any x_0 such that C has a blade at x_0. The endpoint of B_{x_0} is in $\mathcal{E}(C)$, so its Y-coordinate is $> y_0$. In particular, $(x_0, y_0) \in C \setminus \overline{\mathcal{E}(C)}$. Applying Lemma 3.5, there is some $x \in \pi_1[\mathcal{E}(C)]$ such that $\{x\} \times (y_0, 1]$ contains exactly one point of $\overline{\mathcal{E}(C)}$, namely the endpoint of B_x. By our choice of y_0 in the previous paragraph, this implies $B_x \cap \overline{\mathcal{E}(C)}$ contains only the endpoint of B_x. By Observation 2.16(1) and the fact that $X \neq \emptyset$, this means $X = \{1\}$. □

Proposition 3.7. *Suppose $X = EPG(F)$ for some smooth fan F, and $X \neq I$. Then 1 is not a cluster point of X.*

Proof. Let C be a comb for F. Fix any x_0 such that C has a blade at x_0. Because $X \neq I$, Observation 2.16(1) implies there is some $y_0 > 0$ such that $(x_0, y_0) \in C \setminus \overline{\mathcal{E}(C)}$. Applying Lemma 3.5, there is some $x \in \pi_1[\mathcal{E}(C)]$ such that $\{x\} \times (y_0, 1]$ contains exactly one point of $\overline{\mathcal{E}(C)}$, namely the endpoint of B_x. Applying Observation 2.16(1) again, this shows there is some nonempty open interval $(z, 1] \subseteq I$ such that $X \cap (z, 1]$ contains only one point. Therefore 1 is not a cluster point of X. □

This finishes the proof of the main results in this section, which are collected together and summarized in the following theorem:

Theorem 3.8. *Suppose $X \subseteq I$ and there is an X-EPG smooth fan. Then $X = I$, $X = \emptyset$, $X = \{1\}$, or X is a closed subset of I such that $0 \in X$ and 1 is not a cluster point of X.*

Let us point out that Example 7.4, the non-smooth fan mentioned in the introduction that violates the conclusion of the main theorem of this section, also violates the conclusion of Propositions 3.6 and 3.7, and Lemma 3.5. Thus the "smooth" condition cannot be removed from any of these results. We do not know if the Baire Category argument from Lemma 3.5 can be extended somehow to tell us something about non-smooth fans.

4. Examples and first consequences

In this section we present a few examples of endpoint-homogeneous smooth fans. Many familiar smooth fans are endpoint-homogeneous. Interestingly, these familiar examples all have the property of being the unique X-EPG smooth fan for some X, except in the degenerate case $X = \emptyset$.

- The simple n-ods are the only $\emptyset$-EPG fans. This is because, if F is a $\emptyset$-EPG fan (smooth or not), then $E(F)$ has no accumulation points, and as F is compact, this means $E(F)$ is finite.
- The Lelek fan is the only I-EPG smooth fan. This is immediate from Charatonik's well-known characterization of the Lelek fan.

- The star (drawn below) is the only $\{0\}$-EPG smooth fan (see [3, Theorem 4.2]).
- The shrinking Cantor fan (also drawn below) is the only $\{0,1\}$-EPG smooth fan (implicit in [2, Theorem 2.2(2)]).

The following proposition adds one more item to the list.

Proposition 4.1. *Up to homeomorphism, the Cantor fan is the only* $\{1\}$*-EPG smooth fan.*

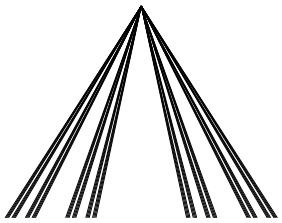
Proof. If F is a $\{1\}$-EPG smooth fan, then $E(F)$ is a closed subset of F with no isolated points, and containing no arcs. By Brouwer's characterization of the Cantor space, this implies $E(F)$ is homeomorphic to the Cantor space. By [2, Theorem 2.2], the Cantor fan is, up to homeomorphism, the unique smooth fan in which $E(F)$ is homeomorphic to the Cantor space. □

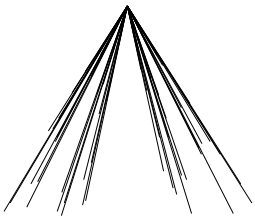
This gives a characterization of the smooth X-EPG fans for $X = \emptyset$, $\{0\}$, $\{1\}$, $\{0,1\}$, and I.

simple n-ods

Cantor fan

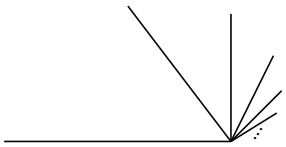
Lelek fan

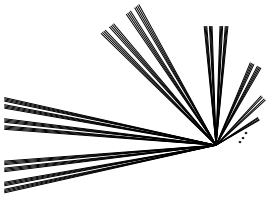




the $\{0\}$-EPG smooth fan

the $\{0,1\}$-EPG smooth fan





5. Constructions of endpoint-homogeneous fans

In the Section 3 we proved that for certain $X \subseteq I$ there can be no X-EPG smooth fan. In this section we construct smooth X-EPG fans for every $X \subseteq I$ not ruled out by the results of Section 3. Furthermore, all the examples we construct are endpoint-homogeneous.

If $X = \emptyset$, $\{1\}$, or I, then we have examples of X-EPG fans already: the Cantor fan is $\{1\}$-EPG, the Lelek fan is I-EPG, and the n-ods are all $\emptyset$-EPG.

All of these examples are endpoint-homogeneous. Furthermore, these are the only examples up to homeomorphism, as discussed in Section 4.

It remains to find X-EPG, endpoint-homogeneous fans for the other types of sets listed in Theorem 3.8. Recall from Proposition 3.1 and Corollary 3.3 that for any X-EPG smooth fan, the number of blades is dependent on whether $1 \in X$ or not. Because of this we first focus on those examples of X where $1 \notin X$.

Theorem 5.1. *If X is a closed subset of I with $0 \in X$ and $1 \notin X$, then there is an X-EPG, endpoint-homogeneous smooth fan.*

Proof. If $X = \emptyset$ or $X = \{0\}$, then we already have examples of X-EPG smooth fans: the simple n-ods for $X = \emptyset$, and the star for $X = \{0\}$. Furthermore, it is clear that these examples are endpoint-homogeneous. Thus, for the remainder of the proof, fix some closed $X \subseteq I$ such that $0 \in X$, $1 \notin X$, and $X \cap (0,1) \neq \emptyset$.

Let $D = \{y_n : n \in \mathbb{N}\}$ be a dense subset of $X \setminus \{0\}$ (which is nonempty, by the assumption of the previous paragraph), and assume without loss of generality that for each $d \in D$ there are infinitely many n with $y_n = d$.

We now describe the construction of a comb over K such that the corresponding fan will be X-EPG and endpoint-homogeneous. To begin, let $L^0 = \{0\} \times I$. Let $M = \max(X)$, and note that $M < 1$. Then:

(1) For each $n_1 \in \mathbb{N}$, let

$$L^1_{n_1} = \left\{ \frac{2}{3^{n_1}} \right\} \times [0, y_{n_1}].$$

(2) For each $n_1, n_2 \in \mathbb{N}$, let

$$L^2_{n_1,n_2} = \left\{ \frac{2}{3^{n_1}} + \frac{2}{3^{n_1+n_2}} \right\} \times [0, M^{n_1} y_{n_1} y_{n_2}].$$

(3) For each $n_1, n_2, n_3 \in \mathbb{N}$, let

$$L^3_{n_1,n_2,n_3} = \left\{ \frac{2}{3^{n_1}} + \frac{2}{3^{n_1+n_2}} + \frac{2}{3^{n_1+n_2+n_3}} \right\} \times [0, M^{n_1+n_2} y_{n_1} y_{n_2} y_{n_3}].$$

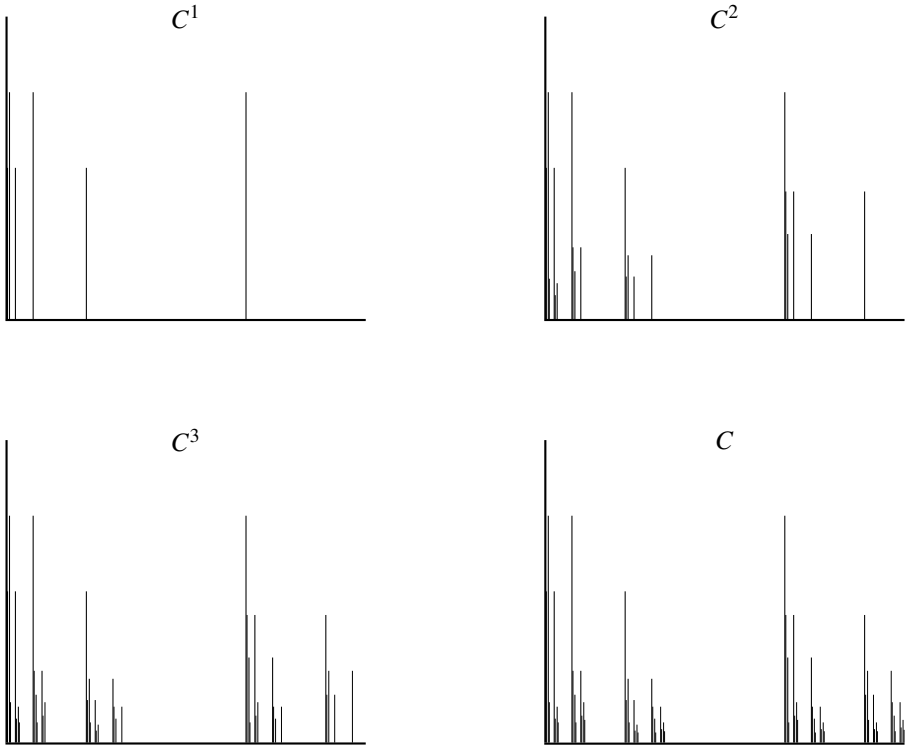
(4) More generally, for any given $k \in \mathbb{N}$ and for $n_1, n_2, \ldots, n_k \in \mathbb{N}$, let

$$L^k_{n_1,n_2,\ldots,n_k} = \left\{ 2 \cdot \sum_{i=1}^{k} \left(\frac{1}{3} \right)^{\sum_{j=1}^{i} n_j} \right\} \times \left[0, M^{\sum_{i=1}^{k-1} n_i} \cdot \prod_{i=1}^{k} y_{n_i} \right].$$

For each $N \in \mathbb{N}$, let C^N be the result of the first N stages of this construction, together with $I \times \{0\}$: i.e.,

$$C^N = I \times \{0\} \cup L^0 \cup \bigcup \left\{ L^k_{n_1,n_2,\ldots,n_k} : k \leq N \text{ and } (n_1, n_2, \ldots, n_k) \in \mathbb{N}^k \right\}.$$

Finally, let $C = \bigcup_{N \in \mathbb{N}} C^N$. This is illustrated below for $X = \{0, 1/2, 3/4\}$.



Observe that the set of all $x \in I$ where this construction places a vertical arc is exactly

$$\left\{\sum_{i\in F} \tfrac{2}{3^i} : F \text{ is a finite subset of } \mathbb{N}\right\}.$$

To put it more simply, this is just the set of all left-hand endpoints of the Cantor set K. This is a countable dense subset of K.

Let us check that C is closed. Let $\langle z_n : n \in \mathbb{N}\rangle$ be a sequence of points in C; we wish to find a convergent subsequence. If $\liminf_{n\to\infty} \pi_2(z_n) = 0$, then it is not difficult to see that some subsequence converges to a point of $I \times \{0\}$. So let us assume $\liminf_{n\to\infty} \pi_2(z_n) = \varepsilon > 0$. The exponent on M in the definition of $L^k_{n_1,n_2,\ldots,n_k}$ is at least $k-1$. Because $M < 1$, this means there is some $N \in \mathbb{N}$ such that $M^N < \varepsilon$, which implies $\{z_n : n \in \mathbb{N}\} \subseteq C^N$. But it is not difficult to see that C^N is compact. So in this case, some subsequence of the z_n converges to a point of C^N.

Thus C is closed, and it is easy to see that C satisfies every other part of Definition 2.10. Hence C is a comb, and collapsing the bottom line of C to a point yields a smooth fan F_C. To finish the proof, we must show that this fan is an endpoint-homogeneous X-EPG fan.

For each vertical arc $L^k_{n_1,\ldots,n_k}$ produced by this construction, let

$$x^k_{n_1,\ldots,n_k} = 2 \cdot \sum_{i=1}^{k} \left(\tfrac{1}{3}\right)^{\sum_{j=1}^{i} n_j},$$

i.e., $x^k_{n_1,\ldots,n_k}$ is the X-coordinate of the vertical arc $L^k_{n_1,\ldots,n_k}$. Let

$$e^k_{n_1,\ldots,n_k} = M^{\sum_{i=1}^{k-1} n_i} \cdot \prod_{i=1}^{k} y_{n_i},$$

or in other words, $e^k_{n_1,\dots,n_k}$ is the Y-coordinate of the tip of the blade at $x^k_{n_1,\dots,n_k}$. Note that $(x^k_{n_1,\dots,n_k}, e^k_{n_1,\dots,n_k}) \in \mathcal{E}(C)$. Let

$$U^k_{n_1,\dots,n_k} = \left[x^k_{n_1,\dots,n_k},\, x^k_{n_1,\dots,n_k} + \left(\tfrac{1}{3}\right)^{\sum_{i=1}^k n_i}\right] \cap K.$$

Because $x^k_{n_1,\dots,n_k}$ is the left-hand endpoint of an interval in the Cantor set K, this is simply the basic clopen subset of K with left-hand endpoint $x^k_{n_1,\dots,n_k}$ and with width $\left(\frac{1}{3}\right)^{\sum_{i=1}^k n_i}$. This particular width for $U^k_{n_1,\dots,n_k}$ is chosen precisely so as to make the following observation true.

Observation: Let $x^k_{n_1,\dots,n_k}$ and $x^\ell_{m_1,\dots,m_\ell}$ be two points at which C has a blade. Then $x^\ell_{m_1,\dots,m_\ell} \in U^k_{n_1,\dots,n_k}$ if and only if $\ell \geq k$ and $(n_1, n_2, \dots, n_k) = (m_1, m_2, \dots, m_k)$ (but $m_{k+1}, \dots, m_\ell$ may be any natural numbers).

Let $C_{>0} = \bigcup_{y>0} C_y = C \cap (0,1]$. For each point $x^k_{n_1,\dots,n_k}$ at which C has a blade, let $V^k_{n_1,\dots,n_k} = (U^k_{n_1,\dots,n_k} \times (0,1]) \cap C_{>0}$.

Claim: Let $x^k_{n_1,\dots,n_k}$ be a point at which C has a blade. There is a homeomorphism $h^k_{n_1,\dots,n_k} : V^k_{n_1,\dots,n_k} \to C_{>0}$ mapping the tip of $B_{x^k_{n_1,\dots,n_k}}$ to $(0,1)$.

Proof of Claim: This claim is proved by exploiting the fractal-like self-similarities of $C_{>0}$. The idea is simply that $V^k_{n_1,\dots,n_k}$ is just a translated, dilated copy of $C_{>0}$, but where the dilation is not quite even: the left-most blade of $V^k_{n_1,\dots,n_k}$ is dilated by a larger factor than every other blade.

To make this precise, define

$$m^k_{n_1,\dots,n_k} = \max\left\{y\colon (x^k_{n_1,\dots,n_k}, y) \in \overline{\mathcal{E}(C) \setminus B_{x^k_{n_1,\dots,n_k}}}\right\} = M^{\sum_{i=1}^k n_i} \cdot M \cdot \prod_{i=1}^k y_{n_i}.$$

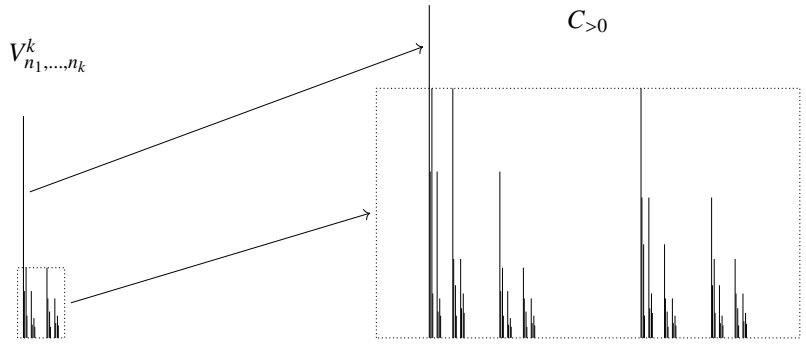
In other words, this is the Y-coordinate of the highest point on $B_{x^k_{n_1,\dots,n_k}}$ (the left-most blade in $V^k_{n_1,\dots,n_k}$) that is in the closure of the other blades. Define

$$\begin{aligned} h^k_{n_1,\dots,n_k}(x,y) &= \left(3^{\sum_{i=1}^k n_i}\left(x - x^k_{n_1,\dots,n_k}\right),\, \frac{1}{M^{\sum_{i=1}^k n_i} \cdot \prod_{i=1}^k y_{n_i}} y\right) \\ &= \left(3^{\sum_{i=1}^k n_i}\left(x - x^k_{n_1,\dots,n_k}\right),\, \frac{M}{m^k_{n_1,\dots,n_k}} y\right) \end{aligned}$$

for all $(x,y) \in V^k_{n_1,\dots,n_k}$ such that $y \leq m^k_{n_1,\dots,n_k}$. This condition, that $y \leq m^k_{n_1,\dots,n_k}$, is true of every point of $V^k_{n_1,\dots,n_k}$ except for those in the topmost part of the blade $B_{x^k_{n_1,\dots,n_k}}$, specifically the points in $\{x^k_{n_1,\dots,n_k}\} \times (m^k_{n_1,\dots,n_k}, e^k_{n_1,\dots,n_k}]$. Define $h^k_{n_1,\dots,n_k}$ on this part of this blade simply by mapping it linearly onto $\{0\} \times (M, 1]$. That is, define

$$h^k_{n_1,\dots,n_k}(x,y) = \left(0,\, M + \frac{1-M}{e^k_{n_1,\dots,n_k} - m^k_{n_1,\dots,n_k}}\left(y - m^k_{n_1,\dots,n_k}\right)\right)$$

if $x = x^k_{n_1,\dots,n_k}$ and $y > m^k_{n_1,\dots,n_k}$.



Observe that $h_{x^k_{n_1,\dots,n_k}}$ maps the leftmost blade in in $V^k_{n_1,\dots,n_k}$, namely $B_{x^k_{n_1,\dots,n_k}}$, homeomorphically onto $\{0\}\times I$ via a piecewise-linear homeomorphism.

If $B_{x^\ell_{m_1,\dots,m_\ell}}$ is any other blade in $V^k_{n_1,\dots,n_k}$, then as observed above, $k \le \ell$ and $(n_1, n_2, \dots, n_k) = (m_1, m_2, \dots, m_k)$. But then

$$\begin{aligned}
h^k_{n_1,\dots,n_k}(x^\ell_{m_1,\dots,m_\ell}, e^\ell_{m_1,\dots,m_\ell}) &= \left(3^{\sum_{i=1}^k n_i}\left(x^\ell_{m_1,\dots,m_\ell} - x^k_{n_1,\dots,n_k}\right),\ \frac{M}{m^k_{n_1,\dots,n_k}} e^\ell_{m_1,\dots,m_k}\right) \\
&= \left(3^{\sum_{i=1}^k n_i}\left(2\cdot\textstyle\sum_{i=1}^\ell \left(\frac{1}{3}\right)^{\sum_{j=1}^i m_j} - 2\cdot\sum_{i=1}^k \left(\frac{1}{3}\right)^{\sum_{j=1}^i n_j}\right),\right. \\
&\qquad \left.\frac{M}{M^{\sum_{i=1}^k n_i}\cdot M\cdot\prod_{i=1}^k y_{n_i}} M^{\sum_{i=1}^{\ell-1} m_i}\cdot\textstyle\prod_{i=1}^\ell y_{m_i}\right) \\
&= \left(2\cdot\textstyle\sum_{i=k+1}^\ell \left(\frac{1}{3}\right)^{\sum_{j=k+1}^i m_j},\ M^{\sum_{i=k+1}^{\ell-1} m_i}\cdot\prod_{i=k+1}^\ell y_{m_i}\right) \\
&= \left(x^{\ell-k}_{m_{k+1},\dots,m_\ell}, e^{\ell-k}_{m_{k+1},\dots,m_\ell}\right).
\end{aligned}$$

In other words, $h^k_{n_1,\dots,n_k}$ maps the tip of the blade at $x^\ell_{m_1,\dots,m_\ell}$ to the tip of the blade at $x^{\ell-k}_{m_{k+1},\dots,m_\ell}$. Because $h^k_{n_1,\dots,n_k}$ is just a scaling in the Y-coordinate, this means that $h^k_{n_1,\dots,n_k}$ maps the blade $B_{x^\ell_{m_1,\dots,m_\ell}}$ in $V^k_{n_1,\dots,n_k}$ to the blade $B_{x^{\ell-k}_{m_{k+1},\dots,m_\ell}}$ in $C_{>0}$. Thus every blade of $V^k_{n_1,\dots,n_k}$ maps to some blade of C, no two blades of $V^k_{n_1,\dots,n_k}$ map to the same blade of C, and every blade of C has some blade of $V^k_{n_1,\dots,n_k}$ mapping onto it. Therefore $h^k_{n_1,\dots,n_k}$ is a bijection. And $h^k_{n_1,\dots,n_k}$ is clearly continuous in both directions, so $h^k_{n_1,\dots,n_k}$ is a homeomorphism as claimed. Furthermore, $h^k_{n_1,\dots,n_k}$ maps the endpoint of $B_{x^k_{n_1,\dots,n_k}}$ to $(0,1)$. ⊣

Using this claim, we now show F_C is endpoint-homogeneous. Suppose $\bar{x}_1 = x^k_{n_1,\dots,n_k}$ and $\bar{x}_2 = x^\ell_{m_1,\dots,m_\ell}$ are two different points at which C has a blade. By the claim above, there is a homeomorphism

$$h = (h^\ell_{m_1,\dots,m_\ell})^{-1} \circ h^k_{n_1,\dots,n_k} : V^k_{n_1,\dots,n_k} \to V^\ell_{m_1,\dots,m_\ell}$$

that maps the tip of $B_{\bar{x}_1}$ to the tip of $B_{\bar{x}_2}$. Because h is a homeomorphism and $\bar{x}_1 \neq \bar{x}_2$, there is a clopen neighborhood W of $B_{\bar{x}_1}$ such that $W \cap h[W] = \emptyset$. Shrinking W if necessary, we may (and do) assume that W has the form

$V \times (0,1]$ for some basic clopen $V \subseteq K$. This implies $h[W]$ is also a clopen subset of K. Now define a homeomorphism $H : F_C \to F_C$ as follows:

$$H(z) = \begin{cases} h(z) & \text{if } z \in W, \\ h^{-1}(z) & \text{if } z \in h[W], \\ z & \text{otherwise.} \end{cases}$$

Because W and $h[W]$ are both relatively clopen subsets of $F_C \setminus \{t\}$, it is clear that $H \restriction (F_C \setminus \{t\})$ is a homeomorphism $F_C \setminus \{t\} \to F_C \setminus \{t\}$. But then this homeomorphism extends (uniquely) to the one-point compactification of $F_C \setminus \{t\}$, namely F_C, by mapping t to t; this extension is H. So H is a homeomorphism $F_C \to F_C$ mapping the tip of the blade at $\bar{x}_1$ to the tip of the blade at $\bar{x}_2$. Thus, given two arbitrary points $\bar{x}_1$ and $\bar{x}_2$ at which C has a blade, we have found a homeomorphism $F_C \to F_C$ mapping the corresponding endpoints to one another. Hence F_C is endpoint-homogeneous.

It remains only to check that F_C is X-EPG. Because we already know F_C is endpoint-homogeneous, it suffices to show for a single $B \in \mathcal{B}(F_C)$ that there is a homeomorphism $\phi : I \to B$ such that $\phi(0) = t$ and $\phi[X] = \overline{E(F_C) \setminus B}$. Let $B = \{0\} \times I$, the leftmost blade of C, and let ϕ be the natural homeomorphism $\phi(z) = (0, z)$.

Because the tip of the blade at $x^k_{n_1,\ldots,n_k}$ has height $\leq M^k$, it is clear that $\overline{\mathcal{E}(C)}$ contains points of $I \times \{0\}$ (in C), which means $t \in \overline{E(F_c) \setminus B}$ (in F_C). Hence $\phi(0) = t \in \overline{E(F_c) \setminus B}$.

Next let $x \in X \setminus \{0\}$. By our choice of the sequence $\langle y_n : n \in \mathbb{N} \rangle$ in the construction, there is a subsequence $\left\langle y_{n_k} : k \in \mathbb{N} \right\rangle$ such that $\lim_{k\to\infty} y_{n_k} = x$. But then $\left\langle (x^1_{n_k}, e^1_{n_k}) : k \in \mathbb{N} \right\rangle = \left\langle (x^1_{n_k}, y_{n_k}) : k \in \mathbb{N} \right\rangle$ is a sequence of points in $\mathcal{E}(C) \setminus B_0$ converging to $(0, x) = \phi(x)$. Combined with the previous paragraph, this shows $\phi[X] \subseteq \overline{E(F_C) \setminus B}$.

For the reverse inclusion, fix some $z \in B \cap \overline{E(F_C) \setminus B}$. Either $z = t$, or else $z = (0, y_z)$ for some $y_z \in I$. Now let $\langle (\bar{x}_k, \bar{e}_k) : k \in \mathbb{N} \rangle$ be a sequence in $E(F_C) \setminus B = \mathcal{E}(C) \setminus B$ converging to z. This implies $\lim_{k\to\infty} \bar{x}_k = 0$, and we would like to show $\lim_{k\to\infty} \bar{e}_k \in X$, since this would then imply that $z = (0, \lim_{k\to\infty} \bar{e}_k) \in \phi[X]$. We consider two cases.

For the first case, suppose there are infinitely many k such that $\bar{x}_k = x^1_{n_k}$ for some n_k. Then by construction, $(\bar{x}_k, \bar{e}_k) = (x^1_{n_k}, y_{n_k})$ for infinitely many k. But then $\lim_{k\to\infty} \bar{y}_n \in \overline{\{y_n : n \in \mathbb{N}\}} = X$.

For the second case, suppose there are only finitely many k such that $\bar{x}_k = x^1_{n_k}$. So, for all sufficiently large k, $\bar{x}_k = x^\ell_{n^k_1, n^k_2, \ldots, n^k_\ell}$ for some $\ell > 1$ and some $n^k_1, n^k_2, \ldots, n^k_\ell \in \mathbb{N}$. But $\bar{x}_k = x^\ell_{n^k_1, n^k_2, \ldots, n^k_\ell} \geq \frac{2}{3^{n^k_1}}$ by construction. Because $\lim_{k\to\infty} \bar{x}_k = 0$, this implies $\lim_{k\to\infty} n^k_1 = \infty$. But also $\bar{e}_k \leq M^{n^1_k}$ by construction, so this implies $\lim_{k\to\infty} \bar{e}_k = 0 \in X$.

In either case $z \in \phi[X]$, so $\phi[X] \supseteq B \cap \overline{E(F_C) \setminus B}$ as claimed. Hence $\phi[X] = B \cap \overline{E(F_C) \setminus B}$, and F_C is an X-EPG fan. □

It remains to show that the same is true for the remaining sets X from Theorem (1). As it turns out, this is (thankfully) a relatively easy consequence of the previous theorem.

Theorem 5.2. *Suppose X is a closed subset of I with $0, 1 \in X$, and 1 is an isolated point of X. Then there is an endpoint-homogeneous, X-EPG smooth fan.*

Proof. Because 1 is an isolated point of X, $Y = X \setminus \{1\}$ is a closed subset of I with $1 \notin Y$. By the previous theorem, there is an endpoint-homogeneous, Y-EPG smooth fan F_Y. Let t_0 denote the top of F_Y.

Let F denote the quotient space obtained from $F_Y \times K$ (where K denotes the Cantor set) by collapsing $\{t_0\} \times K$ to a single point. Equivalently, one may view F as the one-point compactification of $(F_Y \setminus \{t_0\}) \times K$. Let t denote the top of F. For convenience, we identify $F \setminus \{t\}$ with $(F_Y \setminus \{t_0\}) \times K$.

Because F_Y is a smooth fan, it is a subcontinuum of the Cantor fan F_K (with top t_K). This means F is a subset of the quotient space obtained from $F_K \times K$ by collapsing $\{t_K\} \times K$ to a single point (or, equivalently, the one-point compactification of $(F_K \setminus \{t_K\}) \times K$). This latter space is homeomorphic to F_K, because $K \times K \approx K$. Thus F is a subset of a Cantor fan.

It is straightforward to check that F is path-connected, compact, and contains a copy of F_Y. Combined with the previous paragraph, this means F is a smooth fan. Furthermore, it is clear that $E(F) = E(F_Y) \times K$.

Let $e, e' \in E(F)$. Then $e = (e_1, e_2)$ and $e' = (e'_1, e'_2)$, where $e_1, e'_1 \in E(F_Y)$ and $e_2, e'_2 \in K$. Because F_Y is endpoint-homogeneous, there is a homeomorphism $h_1 : F_Y \to F_Y$ with $h_1(e_1) = e'_1$. Because K is homogeneous, there is a homeomorphism $h_2 : K \to K$ with $h_2(e_2) = e'_2$. But then

$$H(z) = \begin{cases} (h_1(x), h_2(y)) & \text{if } z = (x, y) \in (F_Y \setminus \{t_0\}) \times K, \\ t & \text{if } z = t \end{cases}$$

is a homeomorphism $F \to F$ with $H(e) = e'$. So F is endpoint-homogeneous.

Lastly, we must show that F is an X-EPG fan. Fix $B \in \mathcal{B}(F)$. Observe that $B = B_0 \times \{c\}$ for some $B_0 \in \mathcal{B}(F_Y)$ and $c \in K$, provided that we are willing to abuse notation slightly and identify (t_0, c) with t. Because F_Y is Y-EPG, there is a homeomorphism $\phi : I \to B$ with $\phi(0) = t$ such that $\phi[Y] = B \cap \overline{E(F_Y) \setminus B}$. We claim that $\phi[Y] = B \cap \overline{E(F) \setminus B}$.

On the one hand, $F_Y \times \{c\}$ is a homeomorphic copy of F_Y sitting inside of F, and the endpoints of this copy of F_Y are all endpoints of F. Therefore

$$B \cap \overline{E(F) \setminus B} \supseteq B \cap \overline{E(F_Y \times \{c\}) \setminus B} = \phi[Y].$$

But also $\phi(1)$, the endpoint of B, is in $\overline{E(F) \setminus B}$, because it is a limit point of $\{\phi(1)\} \times K \subseteq E(F)$. Hence $B \cap \overline{E(F) \setminus B} \supseteq \phi[Y] \cup \{\phi(1)\} = \phi[X]$.

On the other hand, suppose $x \in B$ and $x \notin \phi[X]$. Fix $x_0 \in B_0$ such that $x = (x_0, c)$. Then $x_0 \notin \phi[Y]$, and x_0 is not the endpoint of B_0 (because $x \notin \phi[X]$), so there is an open neighborhood $U \subseteq F_Y$ with $x_0 \in U$ such that

$U \cap E(F_Y) = \emptyset$. But then $U \times K$ is an open neighborhood of (x_0, c) in F such that $(U \times K) \cap E(F) = (U \times K) \cap (E(F_Y) \times K) = \emptyset$. Hence $x \notin \overline{E(F) \setminus B}$. □

Putting the preceding two theorems together with the work in Section 3 yields the following theorem, a slight strengthening of Theorem (1) from the introduction, followed by Theorem (2) from the introduction:

Theorem 5.3. *Let $X \subseteq I$. The following are equivalent:*

(1) There is an X-EPG smooth fan.
(2) There is an endpoint-homogeneous X-EPG smooth fan.
(3) X satisfies one of the conditions listed in Theorem 3.8.

Theorem 5.4. *There are uncountably many distinct homeomorphism types of endpoint-homogeneous smooth fans.*

Proof. In light of the previous theorem and Observation 2.8, it suffices to show that there is an uncountable family of subsets of I such that every member of the family satisfies the three statements in Theorem 5.3, but no two of them are equivalently embedded in I.

For each $n \in \mathbb{N}$, fix some a_n, b_n such that $\frac{1}{n+1} < a_n < b_n < \frac{1}{n}$. Then, for each $A \subseteq \mathbb{N}$, let

$$X_A = \{0\} \cup \left\{\tfrac{1}{n} : n \in \mathbb{N}\right\} \cup \left\{[a_n, b_n] : n \in A\right\}.$$

The family of sets $\{X_A : A \subseteq \mathbb{N}\}$ has the required properties. □

6. $\frac{1}{n}$-HOMOGENEITY

Recall from the introduction that a space X is *$\frac{1}{n}$-homogeneous* if there is a partition of X into n sets, but no fewer, such that if $x, y \in X$ lie in the same partition piece, then there is a homeomorphism $h : X \to X$ with $h(x) = y$.

In [2], Acosta, Hoehn, and Juárez completely classify all $\frac{1}{3}$-homogeneous smooth fans: they are the simple n-ods, the Cantor fan, the Lelek fan, and the unique $\{0\}$-EPG and $\{0, 1\}$-EPG fans, see Section 4.

Acosta, Hoehn, and Juárez also prove in [2] (see Theorem 4.1) that no smooth fan is $\frac{1}{4}$-homogeneous. This section is devoted to the proof of Theorem (5) which states that for every $n \geq 5$, there is an endpoint-homogeneous, $\frac{1}{n}$-homogeneous smooth fan. No new fans are constructed though: rather, we show that for certain choices of X (depending on n), the X-EPG fans constructed in the proof of Theorem 5.1 are $\frac{1}{n}$-homogeneous. The sets X used for odd n and for even n are rather different. The odd case is simpler, and we handle it first.

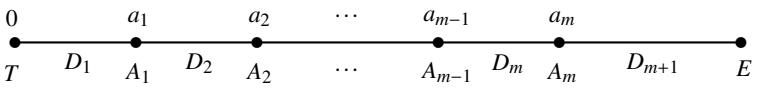
Theorem 6.1. *There is an endpoint-homogeneous, $\frac{1}{n}$-homogeneous smooth fan for all odd $n \geq 5$.*

Proof. Let n be an odd number ≥ 5, and let $m = \frac{n-3}{2}$. Fix some $a_1, a_2, \ldots, a_m$ such that $0 < a_1 < \cdots < a_m < 1$, and let $X = \{0, a_1, \ldots, a_m\}$. We claim that the X-EPG fan constructed in the proof of Theorem 5.1 is $\frac{1}{n}$-homogeneous. Let C denote a comb of this fan, as described in the proof of Theorem 5.1,

and let F_C denote the fan itself. We know already that F_C is an endpoint-homogeneous smooth fan, so we need only show it is $\frac{1}{n}$-homogeneous.

Note that X naturally divides I into $n = 2m+3$ sets:

$$\begin{aligned} T &= \{0\}, \\ E &= \{1\}, \\ A_1 &= \{a_1\},\ A_2 = \{a_2\}, \ldots,\ A_m = \{a_m\}, \\ D_1 &= (0, a_1),\ D_2 = (a_1, a_2), \ldots,\ D_m = (a_{m-1}, a_m),\ D_{m+1} = (a_m, 1). \end{aligned}$$



For every $B \in \mathcal{B}(F_C)$, there is a homeomorphism $\phi_B : I \to B$ mapping 1 to the tip of B such that $\phi_B[X] = B \cap \overline{E(F_C) \setminus B}$. Partition F_C into n classes of points by taking

$$P_Z = \bigcup_{B \in \mathcal{B}(F_C)} \phi_B[Z]$$

where Z is any of the classes listed above, T, E, $A_1, \ldots, A_m$, $D_1, \ldots, D_{m+1}$. It is clear that the P_Z partition F_C into n sets, and we claim that this partition witnesses the $\frac{1}{n}$-homogeneity of F_C.

First, it is not difficult to see that these classes can be defined topologically. For example, $T = \{t\}$ (the top of F_C) and $E = E(F_C)$. Given $i \leq m$, A_i consists of all those points $x \in F_C$ that are limits of endpoints, and such that the arc $[t, x]$ contains $i+1$ limits of endpoints (counting t and x). Given $i \leq m+1$, D_i consists of all those points $x \in F_C$ that are not limits of endpoints, and such that the arc $[t, x]$ contains i limits of endpoints.

Because the P_Z admit this kind of topological definition, every homeomorphism $F_C \to F_C$ preserves each of the P_Z. It remains to show that if $x, y \in P_Z$ for some particular Z, then there is a homeomorphism $h : F_C \to F_C$ with $h(x) = y$. This is trivial for $Z = T$, and it is true for $Z = E$ because F_C is endpoint-homogeneous. We must check the A_i and D_i.

Fix $i \leq m$, and let $x, y \in A_i$. Because F_C is endpoint-homogeneous, there is a homeomorphism $h : F_C \to F_C$ mapping the blade containing x to the blade containing y. But P_{A_i} contains exactly one point of every blade, and h preserves P_{A_i}, so this implies $h(x) = y$.

It remains to consider the D_i. Fix $i \leq m+1$. Instead of considering two arbitrary members of D_i, we begin by consider two members of P_{D_i} on B_0, the leftmost blade of C. Fix $z_1 = (0, y_1)$ and $z_2 = (0, y_2)$ in $P_{D_i} \cap B_0$. Because $[z_1, z_2] \subseteq P_{D_i}$ and $P_{D_i} \cap \overline{\mathcal{E}(C)} = \emptyset$, there is an open $U \supseteq [z_1, z_2]$ such that $U \cap \mathcal{E}(C) = \emptyset$. Shrinking U if necessary (and relabelling z_1 and z_2 so that z_1 is between t and z_2), we may (and do) assume that

$$U = V \times (y_1 - \varepsilon, y_2 + \varepsilon)$$

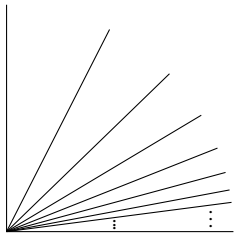
for some clopen $V \subseteq K$ containing 0. There is a homeomorphism $h_0 : I \to I$ such that $h_0(y_1) = y_2$ and $h_0(y) = y$ for all $y \notin (y_1 - \varepsilon, y_2 + \varepsilon)$. But then

$$h(z) = \begin{cases} (x, h_0(y)) & \text{if } z = (x,y) \in U, \\ z & \text{if not} \end{cases}$$

is a homeomorphism $F_C \to F_C$ mapping z_1 to z_2.

Thus for any two points of $P_{D_i} \cap B_0$, there is a homeomorphism $F_C \to F_C$ mapping one to the other. But also, because F_C is endpoint-homogeneous and P_{D_i} is homeomorphism-invariant, for every $x \in P_{D_i}$ there is a homeomorphism $F_C \to F_C$ mapping it to a point of B_0. By inverting and composing homeomorphisms, it follows that if $x, y \in P_{D_i}$ then there is a homeomorphism $F_C \to F_C$ mapping x to y. □

Observe that not all $\frac{1}{n}$-homogeneous fans need be smooth, and not all $\frac{1}{n}$-homogeneous smooth fans need be endpoint-homogeneous. The picture following Definition 2.4 shows two non-smooth fans; the one on the left is $\frac{1}{7}$-homogeneous and the one on the right is $\frac{1}{5}$-homogeneous. Drawn below is a $\frac{1}{5}$-homogeneous smooth fan that is not endpoint-homogeneous.



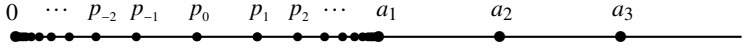
Theorem 6.2. *There is an endpoint-homogeneous, $\frac{1}{n}$-homogeneous smooth fan for all $n \geq 5$.*

Proof. After the previous theorem, it remains to show there is an endpoint-homogeneous, $\frac{1}{n}$-homogeneous smooth fan for all even $n \geq 6$.

Let $\{p_i : i \in \mathbb{Z}\}$ be a subset of I such that $p_i < p_j$ whenever $i < j$, and $\lim_{i\to-\infty} p_i = 0$, and $\lim_{i\to\infty} p_i < 1$. Let $a_1 = \lim_{i\to\infty} p_i$, and if $n \geq 8$ fix some $a_2, \ldots, a_m$ such that $a_1 < a_2 < \cdots < a_m < 1$, where $m = \frac{n-4}{2}$. Let

$$X = \{p_i : i \in \mathbb{Z}\} \cup \{0, a_1\} \cup \{a_2, \ldots, a_m\}.$$

This set X is illustrated below for $n = 10$.



We claim that the X-EPG fan constructed in the proof of Theorem 5.1 is $\frac{1}{n}$-homogeneous. Let C denote a comb of this fan, as described in the proof of Theorem 5.1, and let F_C denote the fan itself. We know already that F_C is an endpoint-homogeneous smooth fan, so we need only show it is $\frac{1}{n}$-homogeneous.

Like in the previous proof, X naturally divides I into n sets:

$$\begin{aligned} T &= \{0\}, \\ E &= \{1\}, \\ L &= \{p_i \colon i \in \mathbb{Z}\}, \\ G &= \bigcup \{(p_i, p_{i+1}) \colon i \in \mathbb{Z}\} \\ A_1 &= \{a_1\},\ A_2 = \{a_2\},\ \ldots,\ A_{m-1} = \{a_{m-1}\},\ A_m = \{a_m\}, \\ D_1 &= (a_1, a_2),\ D_2 = (a_2, a_3),\ \ldots,\ D_{m-1} = (a_{m-1}, a_m),\ D_m = (a_m, 1). \end{aligned}$$

For every $B \in \mathcal{B}(F_C)$ there is a homeomorphism $\phi_B : I \to B$ mapping 1 to the tip of B such that $\phi_B[X] = B \cap \overline{E(F_C) \setminus B}$. Partition F_C by taking

$$P_Z = \bigcup_{B \in \mathcal{B}(F_C)} \phi_B[Z],$$

where Z is any of the classes listed above, $T, E, P, I, A_1, \ldots, A_m, D_1, \ldots, D_m$. It is clear that these sets P_Z form a partition F_C into n sets, and we claim that this partition witnesses the $\frac{1}{n}$-homogeneity of F_C.

As in the previous proof, it is not difficult to see that each of the partition pieces P_Z admits a purely topological definition. Explicitly: $T = \{t\}$ is just the top of F_C; $E = E(F_C)$; L is the set of all those x that are a limit of endpoints, and such that $[t, x] \cap \overline{\mathcal{E}(C) \setminus [t, x]}$ is a decreasing convergent sequence; G is defined similarly, except that x is not a limit of endpoints; A_i is all those x that are a limit of endpoints, and such that, if e denotes the endpoint on the blade of x, then $[x, e]$ contains $m - i$ limits of endpoints; D_i is defined similarly, except that x is not a limit of endpoints.

Because each of the P_Z admits a purely topological definition, every homeomorphism $F_C \to F_C$ preserves each of the P_Z. It remains to show that if $x, y \in P_Z$ for some particular Z, then there is a homeomorphism $h : F_C \to F_C$ with $h(x) = y$.

This is trivial for $Z = T$, and it is true for $Z = E$ because F_C is endpoint-homogeneous. The case $Z = A_i$ (for any $i \leq m$) follows from the endpoint-homogeneity of F_C just as in the previous proof.

Like with the A_i, the argument for the D_i is essentially the same as in the proof of the previous theorem. Recall the basic structure of this 2-part argument. First, we show that for any two points of $P_{D_i} \cap B_0$, there is a homeomorphism $F_C \to F_C$ mapping one to the other. (This was obtained by leaving all X-coordinates alone, and applying a small nudge to the Y-coordinates of all points in some relatively clopen $U \subseteq K \times (0, 1]$.) Second, because F_C is endpoint-homogeneous and P_{D_i} is homeomorphism-invariant, observe that for every $x \in P_{D_i}$ there is a homeomorphism $F_C \to F_C$ mapping x to a point of B_0. Thus by inverting and composing homeomorphisms, it follows that if $x, y \in P_{D_i}$ then there is a homeomorphism $F_C \to F_C$ mapping x to y.

The first part of this argument for the P_{D_i}, as spelled out in the previous proof, can be duplicated to tell us something about the class P_G as well:

(∗) If $i \in \mathbb{Z}$ and $z_1, z_2 \in \{0\} \times (p_i, p_{i+1})$, then there is a homeomorphism $F_C \to F_C$ mapping z_1 to z_2.

This fact will be used to examine P_G later, but next we look at P_L.

Like with the D_i, the argument is in two parts, and we begin by consider two members of P_L on B_0. Fix $(0, p_i)$ and $(0, p_j)$ in $P_L \cap B_0$, with $i < j$. Let $\phi : I \to I$ be the homeomorphism such that

- $\phi(0) = 0$ and $\phi(z) = z$ for all $z \geq a_1$.
- $\phi(p_k) = p_{k+j-i}$ for all $k \in \mathbb{Z}$, and
- ϕ maps (p_k, p_{k+1}) linearly onto $(p_{k+j-i}, p_{k+j-i+1})$ for all $k \in \mathbb{Z}$.

Note, in particular, that $\phi[X] = X$ and $\phi(p_i) = p_j$. We aim to find a homeomorphism $h : F_C \to F_C$ such that $h(0, y) = (0, \phi(y))$ for all $y > 0$ (in other words, h restricts to ϕ on $B_0 \setminus \{t\}$).

In the proof of Theorem 5.1, our construction begins with a countable dense $D \subseteq X$. Because each of the p_k is isolated in X, $\{p_k : k \in \mathbb{Z}\} \subseteq D$. The construction proceeds by fixing an infinite sequence $\langle y_n : n \in \mathbb{N}\rangle$ such that each member of D appears infinitely often in the sequence. In particular,

$$Y_k = \{n : y_n = p_k\}$$

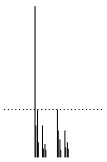
is infinite for every $k \in \mathbb{N}$. Let $\psi : \mathbb{N} \to \mathbb{N}$ be a permutation of $\mathbb{N}$ such that

- $\psi[Y_k] = Y_{k+j-i}$ for all $k \in \mathbb{Z}$, and
- ψ acts as the identity map on $\mathbb{N} \setminus \bigcup_{k\in\mathbb{Z}} Y_k$.

Observe that $\phi(y_n) = y_{\psi(n)}$ for all $n \in \mathbb{N}$.

Recalling more notation from the proof of Theorem 5.1, $C_{>0} = C \cap (0, 1]$, and for each $n \in \mathbb{N}$ there is a relatively clopen subset V_n^1 of $C_{>0}$ containing the blade at $x_n^1 = 2/3^n$. Furtheremore, the V_n^1 form a partition of $C_{>0} \setminus B_0$.

Recall that M denotes the largest member of X, and $M^{n+1}y_n$ is the Y-coordinate of the highest point on the leftmost blade of V_n^1 that is a limit of endpoints. We divide V_n^1 into two pieces overlapping at one point:



$$V_n^{1+} = \left\{(x, y) \in V_n^1 : y \geq M^{n+1}y_n\right\},$$

$$V_n^{1-} = \left\{(x, y) \in V_n^1 : y \leq M^{n+1}y_n\right\}.$$

In other words, V_n^{1+} is just the top part of the leftmost blade of V_n^1, and V_n^{1-} is everything else. In particular, note that V_n^{1+} is a vertical line segment.

In the proof of the Theorem 5.1, we show that for any $m, n \in \mathbb{N}$ there is a homeomorphism $V_n^1 \to V_m^1$, which in the proof is denoted $(h_m^1)^{-1} \circ h_n^1$. This homeomorphism scales and translates V_n^{1+} onto V_m^{1+} and (with a different scaling factor) V_n^{1-} onto V_m^{1-}. For the present proof, we need a homeomorphism $V_n^1 \to V_{\psi(n)}^1$, defined by modifying $(h_{\psi(n)}^1)^{-1} \circ h_n^1$ as follows.

The idea is that we would like to adjust the map $(h_{\psi(n)}^1)^{-1} \circ h_n^1 : V_n^1 \to V_{\psi(n)}^1$ on the vertical segment V_n^{1+} only, so that instead of a simple linear mapping

onto $V^{1+}_{\psi(n)}$, it mimics $\phi \restriction [0, y_n]$. More specifically, for each n let ℓ_n denote the linear mapping $V^{1+}_n \to [0, y_n]$ sending the topmost point of V^{1+}_n to y_n:

$$\ell_n(2/3^n, y) = \tfrac{y_n}{y_n - M^{n+1} y_n}(y - M^{n+1} y_n).$$

Observe that these mappings are invertible, and $(\ell_{\psi(n)})^{-1}$ is the linear mapping $[0, y_{\psi(n)}] \to V^{1+}_{\psi(n)}$ that sends $y_{\psi(n)}$ to the topmost point of $V^{1+}_{\psi(n)}$:

$$(\ell_{\psi(n)})^{-1}(y) = \Big(\tfrac{2}{3^{\psi(n)}}, \tfrac{y_{\psi(n)} - M^{\psi(n)+1} y_{\psi(n)}}{y_{\psi(n)}} y + M^{\psi(n)+1} y_{\psi(n)}\Big).$$

Define $h_n : V^1_n \to V^1_{\psi(n)}$ by setting

$$h_n(x) = \begin{cases} (h^1_{\psi(n)})^{-1} \circ h^1_n(z) & \text{if } z \in V^{1-}_n, \\ (\ell_{\psi(n)})^{-1} \circ \phi \circ \ell_n(z) & \text{if } z \in V^{1+}_n. \end{cases}$$

Note that ℓ_n is a linear mapping $V^{1+}_n \to [0, y_n]$, ϕ maps $[0, y_n]$ homeomorphically to $[0, y_{\psi(n)}]$, and $(\ell_{\psi(n)})^{-1}$ is a linear mapping $[0, y_{\psi(n)}] \to V^{1+}_{\psi(n)}$. Thus the composition $(\ell_{\psi(n)})^{-1} \circ \phi \circ \ell_n$ is a homeomorphism $V^{1+}_n \to V^{1+}_{\psi(n)}$. Furthermore, $(h^1_{\psi(n)})^{-1} \circ h^1_n(z)$ is a homeomorphism $V^{1-}_n \to V^{1-}_{\psi(n)}$, and it is straightforward to check that each of these homeomorphisms sends the common point of $V^{1-}_n \cap V^{1+}_n$ to the common point of $V^{1-}_{\psi(n)} \cap V^{1+}_{\psi(n)}$. It follows that h_n is a homeomorphism $V^1_n \to V^1_{\psi(n)}$.

Define a mapping $h : F_C \to F_C$ as follows:

$$h(z) = \begin{cases} t & \text{if } z = t, \\ (0, \phi(y)) & \text{if } z = (0, y) \in B_0 \text{ and } y > 0, \\ h_n(z) & \text{if } z \in V^1_n \text{ for some } n. \end{cases}$$

Observe that $F_C = \{t\} \cup C_{>0} = \{t\} \cup B_0 \cup \bigcup_{n \in \mathbb{N}} V^1_n$, so h is defined on all of F_C. Because ψ is a permutation of $\mathbb{N}$, and each $h_{n,\psi(n)}$ is a bijection $V^1_n \to V^1_{\psi(n)}$, and because $\phi : I \to I$ is a bijection with $\phi(0) = 0$, h is a bijection. We claim that h is also continuous.

Because the V^1_n are all clopen and the $h_{n,\psi(n)}$ are all homeomorphisms, it is clear that h is continuous at every $z \in \bigcup_{n \in \mathbb{N}} V^1_n$.

Let $z = (0, y) \in B_0 \setminus \{t\}$ (so $y \in (0, 1]$). We show next that h is continuous at z in $\{z\} \cup \bigcup_{n \in \mathbb{N}} V^1_n$.

To this end, let $\langle z_n : n \in \mathbb{N} \rangle$ be a sequence of points in $\bigcup_{n \in \mathbb{N}} V^1_n$ converging to z. For each n let $\bar{x}_n = \pi_1(z_n)$ and $\bar{y}_n = \pi_2(z_n)$, so that $z_n = (\bar{x}_n, \bar{y}_n)$.

For each n, let k_n denote the integer such that $z_n \in V^1_k$. Because each V^1_k is clopen in C and $\lim_{n \to \infty} z_n = (0, y) \notin \bigcup_{k \in \mathbb{N}} V^1_k$, no single V^1_k can contain infinitely many of the z_n. Consequently, $\lim_{n \to \infty} k_n = \infty$.

Next, observe that the maximum Y-coordinate of V^{1-}_k is $M^{k+1} y_n$. Because $M = a_m < 1$, this goes to 0 as k increases. Because $\lim_{n \to \infty} k_n = \infty$ and $\lim_{n \to \infty} \bar{y}_n = y > 0$, it follows that $z_n \notin V^{1-}_{k_n}$ for sufficiently large values of n. Consequently, $z_n \in V^{1+}_{k_n}$ for all sufficiently large n.

Thus, for sufficiently large n, we have

$$\bar{x}_n = \tfrac{2}{3^{k_n}} \qquad \text{and} \qquad \bar{y}_n \in [M^{k_n+1} y_n, y_n].$$

Plugging these values into the formulas above, we get $h(\bar{x}_n, \bar{y}_n) =$

$$\left(\tfrac{2}{3^{\psi(k_n)}}, \tfrac{y_{\psi(k_n)} - M^{\psi(k_n)+1} y_{\psi(k_n)}}{y_{\psi(k_n)}} \phi\left(\tfrac{y_{k_n}}{y_{k_n} - M^{k_n+1} y_{k_n}} (y - M^{k_n+1} y_{k_n}) \right) + M^{\psi(k_n)+1} y_{\psi(k_n)} \right)$$

for all sufficiently large n. Recall that $\lim_{n\to\infty} k_n = \infty$, and (because ψ is a bijection) this also implies $\lim_{n\to\infty} \psi(k_n) = \infty$. In particular, $\lim_{n\to\infty} \frac{2}{3^{k_n}} = 0$ and, because $M < 1$, $\lim_{n\to\infty} M^{k_n+1} = 0$ and $\lim_{n\to\infty} M^{\psi(k_n)+1} = 0$. Therefore, taking the limit of the above expression for $h(\bar{x}_n, \bar{y}_n)$ as $n \to \infty$ gives

$$\lim_{n\to\infty} h(\bar{x}_n, \bar{y}_n) = (0, \phi(y)) = \phi(z).$$

Thus h is continuous at z in the space $\{z\} \cup \bigcup_{n\in\mathbb{N}} V_n^1$.

But now observe that h is clearly continuous at z in $F_C \setminus \bigcup_{n\in\mathbb{N}} V_n^1$, because this space is just a copy of I on which h acts like ϕ.

Because h is continuous at z both in $\{z\} \cup \bigcup_{n\in\mathbb{N}} V_n^1$ and in $F_C \setminus \bigcup_{n\in\mathbb{N}} V_n^1$, it follows that h is continuous at z in F_C. Because z was an arbitrary point of $B_0 \setminus \{t\}$, and because we already observed h is continuous on $F_C \setminus B_0 = \bigcup_{n\in\mathbb{N}} V_n^1$, this shows that h is continuous on $F_C \setminus \{t\}$.

But if h is continuous on $F_C \setminus \{t\}$, while $h(t) = t$, then it is continuous on F_C. (Generally speaking, any continuous bijection $F_C \setminus \{t\} \to F_C \setminus \{t\}$ extends to a continuous bijection $F_C \to F_C$ by mapping t to t, because F_C is the one-point compactification of $F_C \setminus \{t\}$.)

Hence h is a continuous bijection $F_C \to F_C$. Because F_C is compact, h is a homeomorphism. Furthermore, $h(0, p_i) = (0, \phi(p_i)) = (0, p_j)$ and so, as desired, we have found a homeomorphism sending one arbitrary point of $B_0 \cap P_L$ to another.

Because F_C is endpoint-homogeneous and P_L is homeomorphism-invariant, for every $x \in P_L$ there is a homeomorphism $F_C \to F_C$ mapping it to a point of B_0. By inverting and composing homeomorphisms, it follows that if $x, y \in P_L$ then there is a homeomorphism $F_C \to F_C$ mapping x to y.

To complete the proof of the theorem, it now remains only to show that the same is true for the final piece of our partition, P_G. Thankfully, this follows relatively easily from the work we have already done for P_L, plus the statement $(*)$ from earlier in the proof.

Observe that with p_i, p_j as before, the homeomorphism h described above does not only map $(0, p_i)$ to $(0, p_j)$; it maps the entire vertical line segment $\{0\} \times (p_i, p_{i+1})$ onto $\{0\} \times (\phi(p_i), \phi(p_{i+1})) = \{0\} \times (p_j, p_{j+1})$.

Fix $x, y \in B_0 \cap P_G$. There are some $i, j \in \mathbb{Z}$ such that $x \in \{0\} \times (p_i, p_{i+1})$ and $y \in \{0\} \times (p_j, p_{j+1})$. As mentioned above, there is also a homeomorphism $F_C \to F_C$ that sends $\{0\} \times (p_i, p_{i+1})$ to $\{0\} \times (p_j, p_{j+1})$. By the statement $(*)$, there is a homeomorphism mapping any given point of $\{0\} \times (p_i, p_{i+1})$ to any other point of $\{0\} \times (p_i, p_{i+1})$. Composing homeomorphisms, it follows that there is a homeomorphism $F_C \to F_C$ mapping x to y.

Thus for any two points of $B_0 \cap P_G$, there is a homeomorphism $F_C \to F_C$ mapping one to the other. But also, because F_C is endpoint-homogeneous and P_G is homeomorphism-invariant, for every $x \in P_G$ there is a homeomorphism $F_C \to F_C$ mapping it to a point of B_0. By inverting and composing homeomorphisms, it follows that if $x, y \in P_G$ then there is a homeomorphism $F_C \to F_C$ mapping x to y. □

The previous two theorems combine to prove theorem (5) listed in the introduction.

7. Open problems, and some non-smooth fans

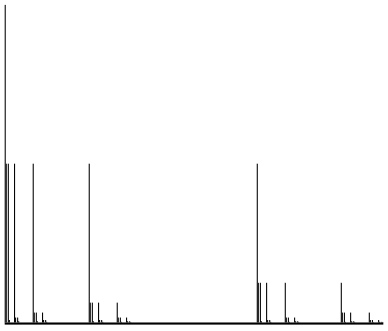
In this final section of the paper we state some open questions relating to endpoint-homogeneous and endpoint-generated fans. We also describe some relevant examples of non-smooth fans, most importantly one proving statement (4) from the introduction. Some of these examples are meant more as illustrations than formal theorems. Accordingly, we do not include full proofs of some of our claims about these examples.

Problem 7.1. *Suppose F and G are smooth, endpoint-homogeneous fans with $EPG(F) = EPG(G)$. Are they homeomorphic?*

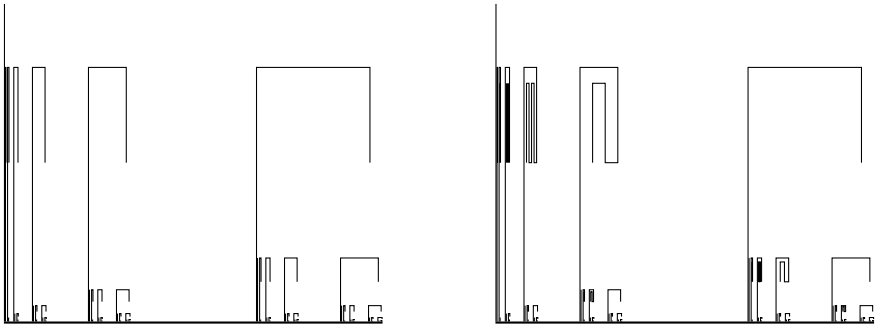
Conjecture: Yes, except for the degenerate case of $\emptyset$-EPG fans (the n-ods).

Some partial results are known already: see Section 4. It is worth pointing out that the above conjecture becomes false if the smoothness assumption is removed. The following example shows that if we allow non-smooth fans, then there are distinct homeomorphism types of $\{0, 1/2\}$-EPG fans. In fact, while there seems to be only one homeomorphism type of smooth $\{0, 1/2\}$-EPG fan, the pictures below reveal that there are many non-smooth types.

Example 7.2. Consider the following picture, which shows our construction of the comb for a $\{0, 1/2\}$-EPG endpoint-homogeneous smooth fan:



Here are two ways of modifying this picture to obtain planar dendroids that, while no longer combs over K, still have the property that collapsing the bottom line to a point produces a (non-smooth) fan:



These fans are still endpoint-homogeneous, with essentially the same proof as for the original version. (All the pieces of this picture are non-uniformly scaled versions of the whole, after making a topological adjustment to the leftmost blade.) This shows that there are at least two homeomorphism types of non-smooth, endpoint-homogeneous, $\{1/2\}$-EPG fans. ⊣

It should be clear that the two pictures shown here can be modified in many other ways, producing a zoo of non-smooth, endpoint-homogeneous, $\{0, 1/2\}$-EPG fans. And there is nothing special about the set $\{0, 1/2\}$: the same thing works for any of the sets considered in the proof of Theorem 5.1. In light of this, we do not expect a nice classification of endpoint-homogeneous non-smooth fans, along the lines of what the conjecture following Problem 7.1 proposes for smooth fans.

The results in Sections 4 and 5 characterize precisely those X such that there is a smooth X-EPG fan. What about non-smooth fans?

Problem 7.3. *For which $X \subseteq I$ is there an X-EPG fan?*

The following example shows that it is possible to have a non-smooth $[1/2, 1]$-EPG fan, despite the fact that there are no smooth $[1/2, 1]$-EPG fans. This example gives a proof for main result (4) listed in the introduction.

Example 7.4. The Cantor space K has a basis of clopen sets, each homeomorphic to K, and also contains many nowhere dense sets homeomorphic to K (as can be seen from Brouwer's characterization of K). Using these two facts, a straightforward recursive argument can be used to find a sequence $\langle K_n : n \in \mathbb{N} \rangle$ of nowhere dense subsets of K, each homeomorphic to K, such that for any sequence of points $\langle d_n : n \in \mathbb{N} \rangle$ such that $d_n \in K_n$ for all n, the set $\{d_n : n \in \mathbb{N}\}$ is dense in K.

Let C_K denote the Cantor comb, viewed as a subset of the X-Y plane in 3-dimensional space: i.e., $C_K = (K \times I \times \{0\}) \cup (I \times \{0\} \times \{0\})$.

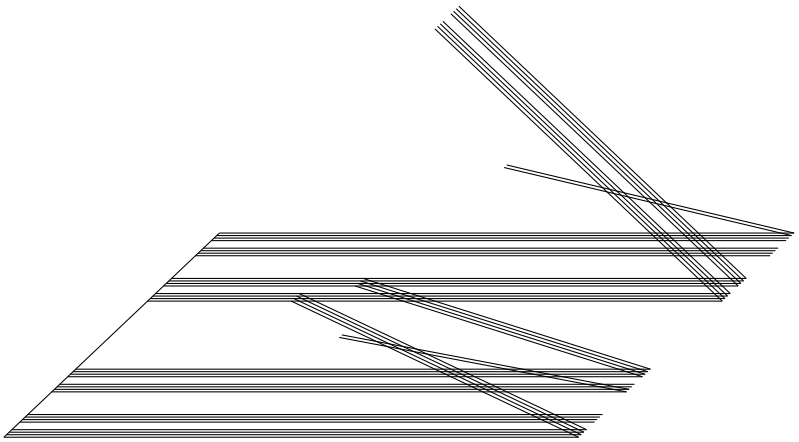
Let C_L be a comb whose corresponding fan is the Lelek fan (that is, $E(C_L)$ is dense in C_L). For each $n \in \mathbb{N}$, define $C_L^n \subseteq \mathbb{R}^3$ as follows. First, fix a homeomorphism $\phi_n : K \to K_n$. Next, define a function $C_L \to \mathbb{R}^3$ by

$$(x, y) \mapsto \left(\phi_n(x),\, 1 - \tfrac{y}{2},\, \tfrac{y}{2n}\right).$$

Finally, let C_L^n be the image of C_L under this mapping. Roughly speaking, C_L^n is a copy of the Lelek comb, with the bottom line removed, but (1)

situated over K_n instead of over K, and (2) beginning at the top of the Cantor comb C_K, instead of on the X-axis, and (3) tilted down towards the Cantor comb, so that C_L^n approaches C_K as n increases. Let

$$C = C_K \cup \bigcup_{n\in\mathbb{N}} C_L^n.$$



Here we have illustrated $C_K \cup \bigcup_{n=1}^{5} C_L^n$ (but with the copies of the Lelek fan drawn more like the Cantor fan, so that they show up better; and we have made no attempt to show that the K_n are nowhere dense).

It is not difficult to check that C is a dendroid. Furthermore (like with combs) collapsing $I \times \{0\} \times \{0\}$ to a point yields a (non-smooth) fan. We claim it is a $[1/2, 1]$-EPG fan.

Now let B be a blade of the fan. These come in two types: some lie flat in the X-Y plane (for example, the co-meagerly many blades above points in $K \setminus \bigcup_{n=1}^{\infty} K_n$ in C_K), and some have a piece angled out of the X-Y plane. Let us called these the type I and type II blades, respectively.

Because of how we chose the K_n, and because the endpoints of the Lelek comb are dense in the Lelek comb, the closure of the endpoints of the type II blades is all of $K \times [1/2, 1] \times \{0\}$, i.e., the top half of the comb C_K.

If B is a type I blade, then the top half of B is contained in $\overline{E(C) \setminus B}$, by the previous paragraph. But no end points of C have X-coordinate below $1/2$, so $B \cap \overline{E(C) \setminus B}$ is precisely the top half of B.

If B is a type II blade, then $B = B' \cup B''$, where B' lies flat in the X-Y plane and B'' is angled up from the X-Y plane. As in the previous paragraph, $B' \cap \overline{E(C) \setminus B}$ is precisely the top half of B'. But observe that B'' is a blade in a copy of the Lelek fan attached to C_K, which means that all of B'' is contained in $\overline{E(C) \setminus B}$. Hence $B \cap \overline{E(C) \setminus B}$ is the top half of B' plus all of B'', which is a closed final segment of B. ⊣

Problem 7.5. *Is the fan from Example 7.4 planar?*

Let us observe that this example of a non-smooth $[1/2, 1]$-EPG fan is not endpoint-homogeneous. (This is because only some of the blades of the fan witness its non-smoothness. More precisely, say $B \in \mathcal{B}(F)$ is *smooth*

if for every sequence $\langle x_n : n \in \mathbb{N} \rangle$ in $F \setminus B$ converging to a point $x \in B$, we have $\lim_{n\to\infty}[t, x_n] = [t, x]$. Some of the blades in our [1/2]-EPG fan are smooth (the type II blades), while some are not (the type I blades), and no homeomorphism can map a smooth blade onto a non-smooth blade.)

Problem 7.6. *For which $X \subseteq I$ is there an endpoint-homogeneous X-EPG (but not necessarily smooth) fan?*

While the previous example shows there is an endpoint-generated fan that is not endpoint-homogeneous, we do not have a smooth example.

Problem 7.7. *Is every X-EPG smooth fan endpoint-homogeneous?*

If the answer to this question is yes, then our conjecture above (if true) would provide a classification of all endpoint-generated smooth fans.

Problem 7.8. *Up to homeomorphism, are there only two $\frac{1}{6}$-homogeneous smooth fans?*

Note that the proof of Theorem 6.2, replacing X with $X \cup \{1\}$ does not hurt the argument, so there are at least two $\frac{1}{6}$-homogeneous smooth fans. As mentioned already, Acosta, Hoehn, and Juárez prove in [2] that there is no $\frac{1}{4}$-homogeneous smooth fan. They do not rule out the possibility of a $\frac{1}{4}$-homogeneous non-smooth fan, however; they even prove some results about what such a fan would look like (if it exists).

Problem 7.9. *Is there a $\frac{1}{4}$-homogeneous (non-smooth) fan?*

Given $X \subseteq I$, let us say $x, x' \in X$ are *equivalent in X* if there is an order-preserving homeomorphism $\phi : I \to I$ mapping X onto X such that $\phi(x) = x'$.

Problem 7.10. *Suppose $X \subseteq I$ has n equivalence classes of points with respect to the above relation. If there is an X-EPG fan, must it be $\frac{1}{n}$-homogeneous? More specifically, is this true for the smooth fans constructed in Section 5?*

We end with a question less directly related to the material of the paper. Endpoint-homogeneous fans have rich automorphism groups in one sense. But there are other senses in which a space can have a rich automorphism group, for example, the automorphism group's universal minimal flow can be metrizable. The Lelek fan has this property [5], and the fans constructed in Section 5 are, in a sense, generalizations of the Lelek fan. This suggests the following question.

Problem 7.11. *Except for the degenerate cases of the simple n-ods and the star, is the universal minimal flow of the automorphism group of an endpoint-homogeneous fan metrizable? More specifically, is this true for the smooth fans constructed in Section 5?*

References

[1] J. M. Aarts and L. G. Oversteegen, "The homeomorphism group of the hairy arc," *Compositio Math.* **96** (1995), pp. 283–292.

[2] G. Acosta, L. Hoehn, and Y. Pacheco-Juárez, "Homogeneity degree of fans," *Topology and its Applications* **231** (2017), pp. 320–328.

[3] G. Acosta and Y. Pacheco-Juárez, "$\frac{1}{3}$-homogeneous dendrites," *Topology and its Applications* **219** (2017), pp. 55–77.

[4] D. Bartošová and A. Kwiatkowska, "Lelek fan from a projective Fraïssé limit," *Fundamenta Mathematicae* **231** (2015), pp. 57–79.

[5] D. Bartošová and A. Kwiatkowska, "The universal minimal flow of the group of homeomorphisms of the Lelek fan," *Transactions of the American Mathematical Society* **371** (2019), pp. 6995–7027.

[6] W. Brian and R. Gril Rogina, "The Classification of Two Lines Mahavier Products," *Journal of Dynamical and Control Systems* **31** (2025)

[7] W. Bula, L. Oversteegen, "A characterization of smooth Cantor bouquets," *Proceedings of the American Mathematical Society* **108** (1990), pp. 529–534.

[8] J. J. Charatonik and W. J. Charatonik, "Images of the Cantor fan," *Topology and its Applications* **33** (1989), pp. 163–172.

[9] W. J. Charatonik, "On fans," *Dissertationes Math. Rozprawy Mat.* **54** (1967), no. 39.

[10] W. Charatonik, "The Lelek fan is unique," *Houston Journal of Mathematics* **15** (1989), pp. 27–34.

[11] C. Eberhart, "A note on smooth fans," *Colloq. Math.* **20** (1969), pp. 89–90.

[12] R. Gril Rogina, "Towards the Complete Classification of the Two Lines Inverse Limit," *Journal of Dynamical and Control Systems* **29** (2023), pp. 1525–1545.

[13] R. Hernández-Gutiérrez and L. Hoehn, "Smoth fans that are endpoint rigid," *Applied General Topology* **24** (2023), pp. 407–422.

[14] S. B. Nadler, Jr., *Continuum Theory, an Introduction*, CRC Press, Oregon, 1992.

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