Endpoint-homogeneous fans

joint work with Rene Gril Rogina

Will Brian

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University of North Carolina at Charlotte

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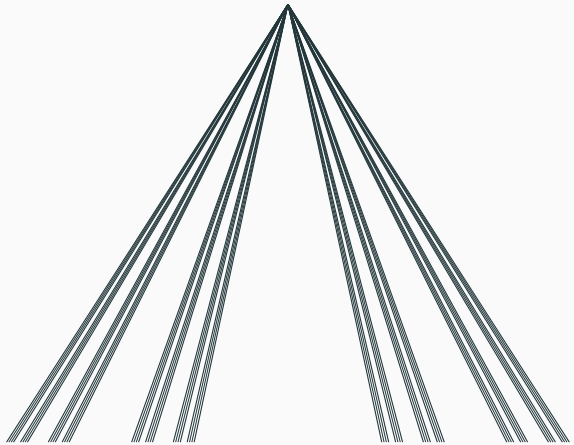

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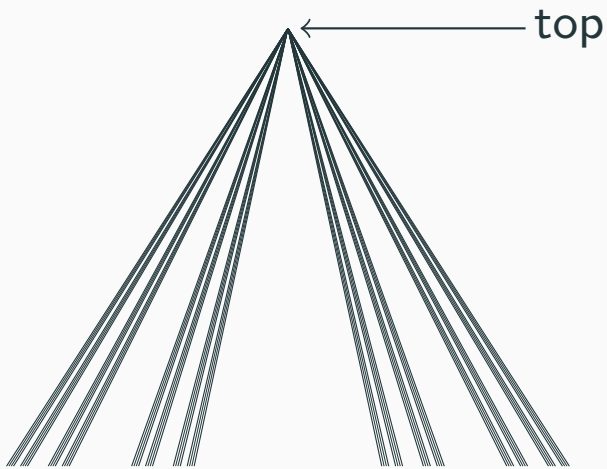

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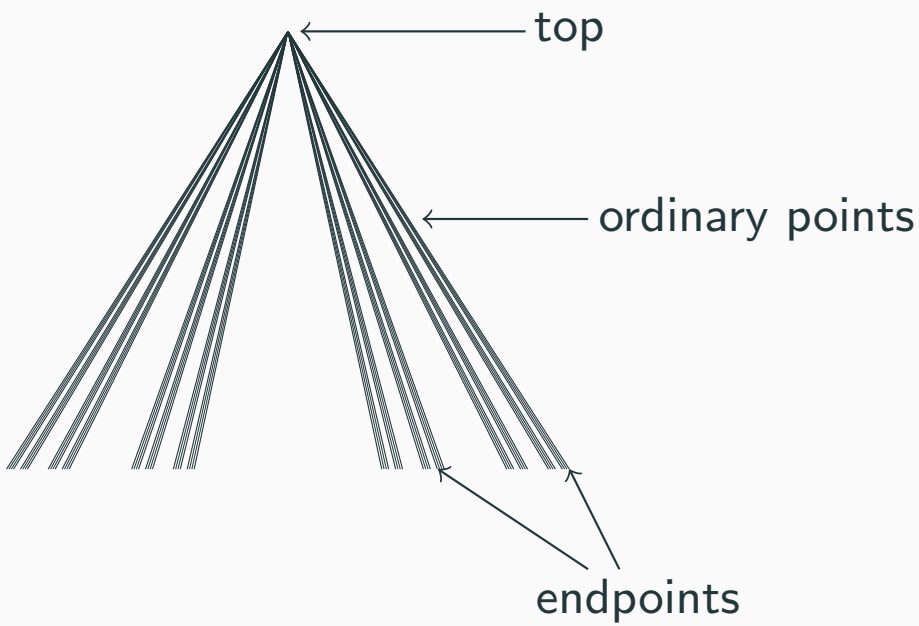

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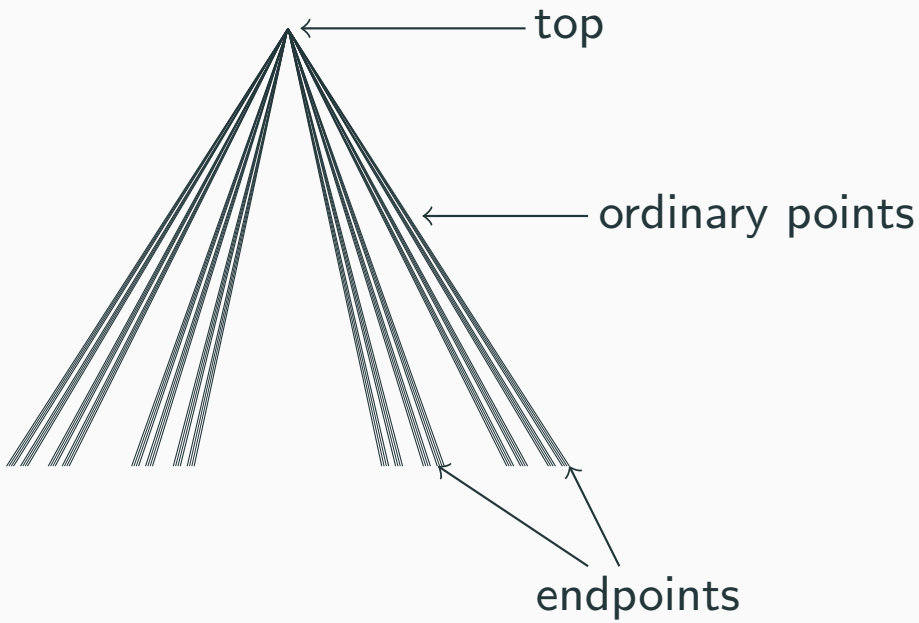

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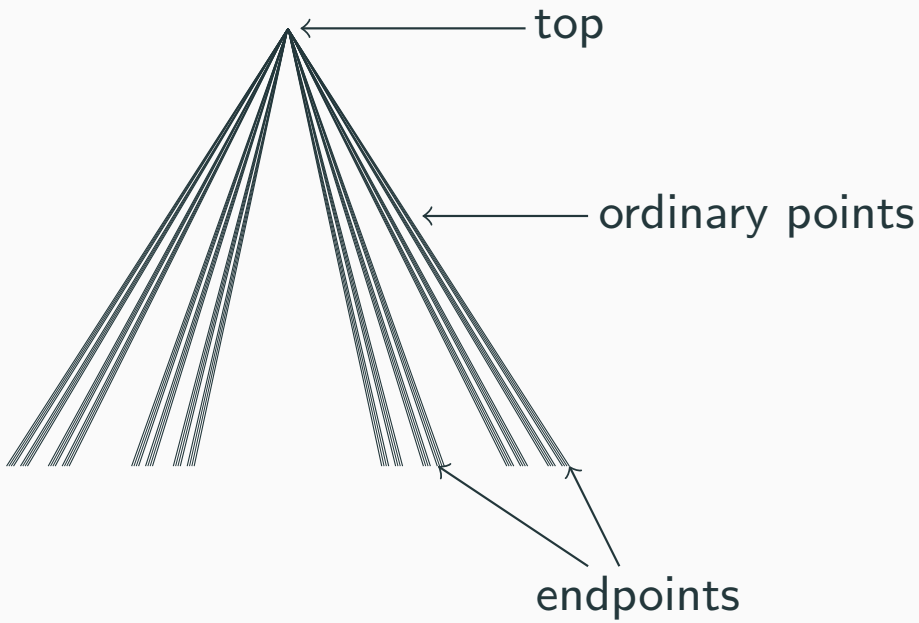

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So . . . what's the next best thing?

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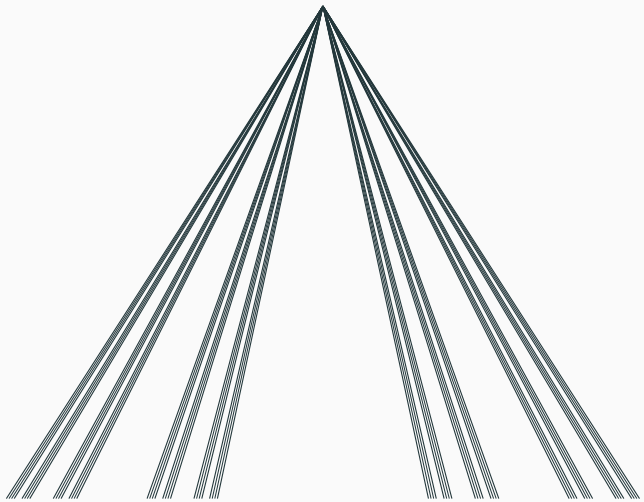

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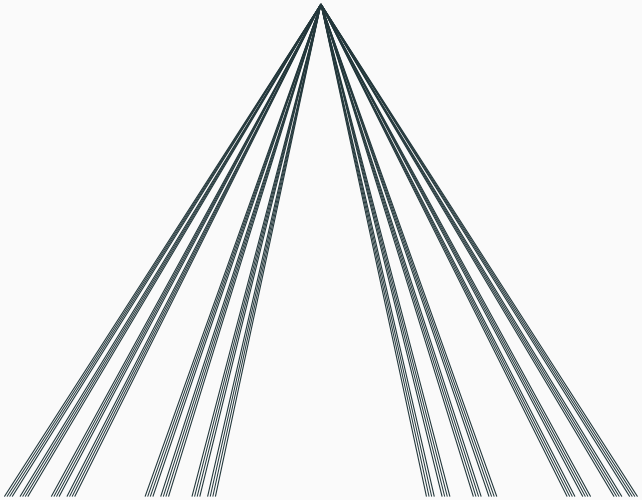

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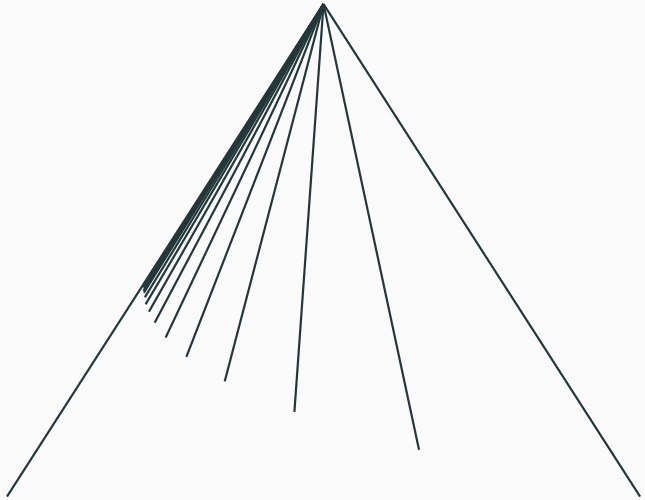

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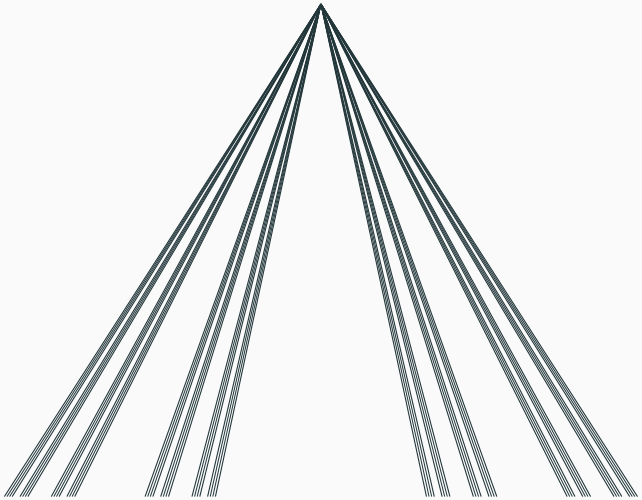

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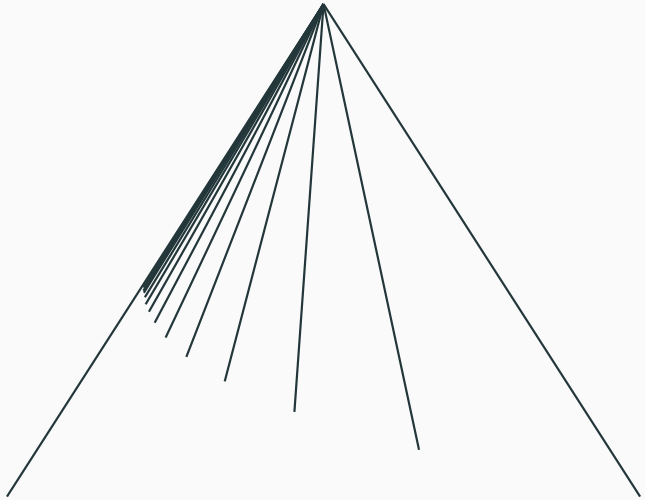

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Note that a fan is endpoint-homogeneous if and only if any two blades of the fan are topologically indistinguishable.

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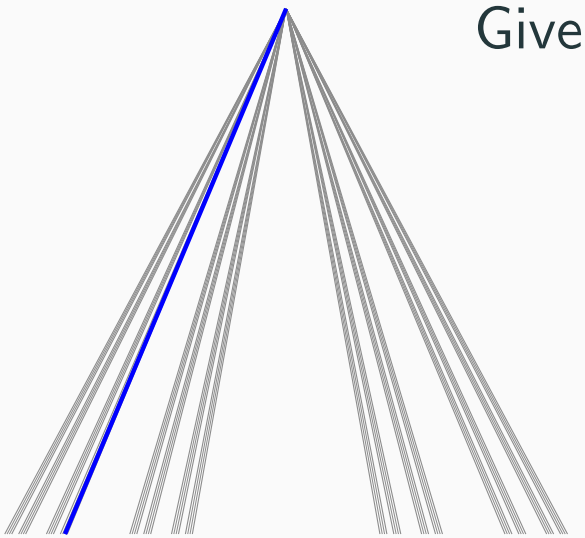

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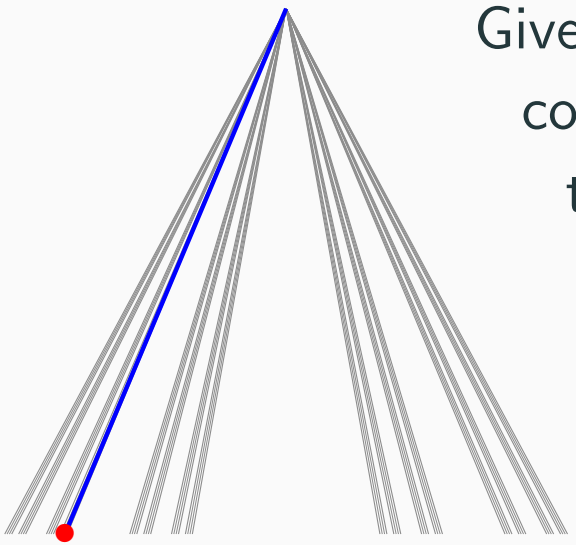

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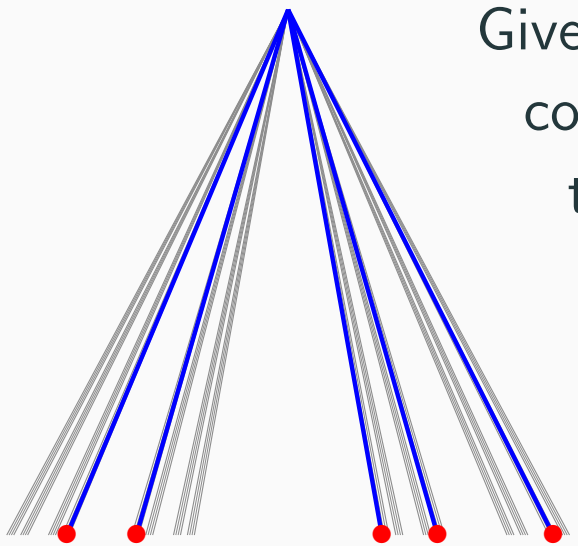

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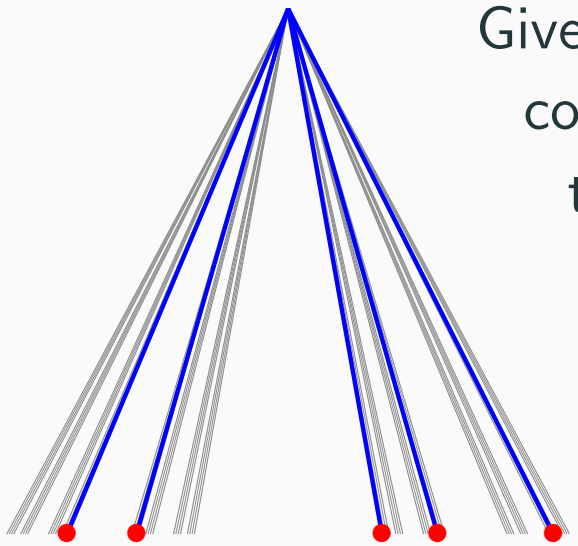

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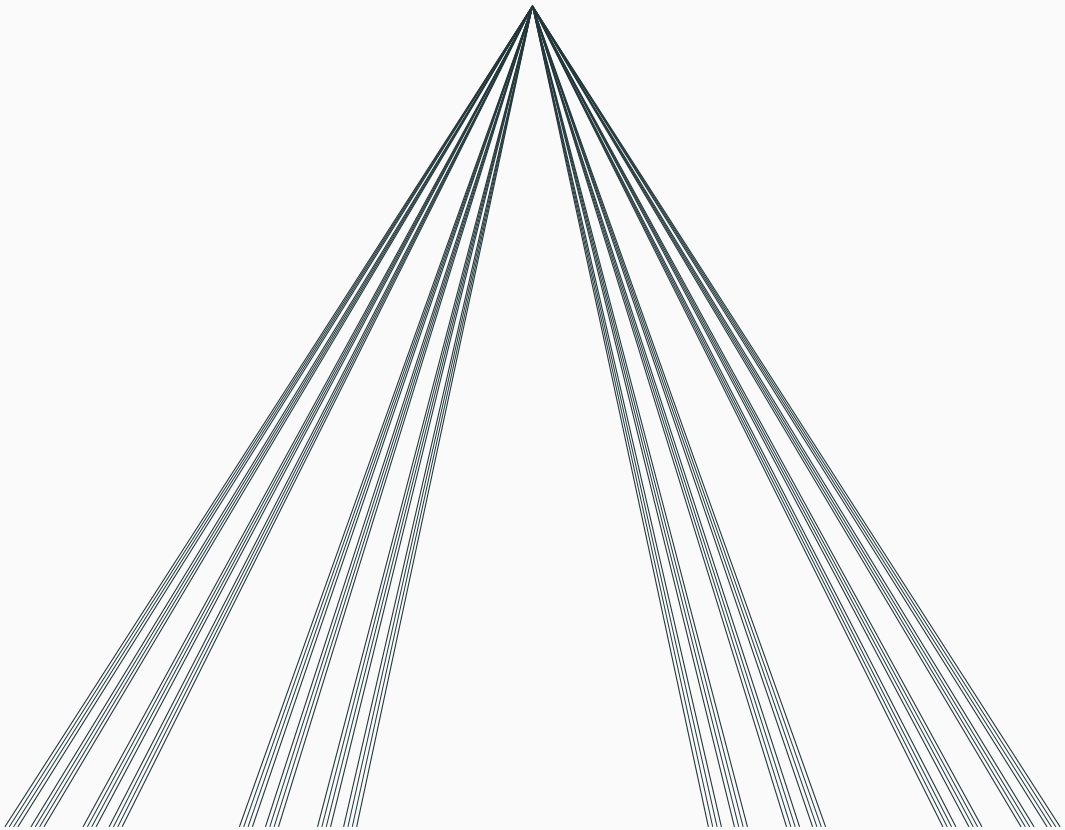

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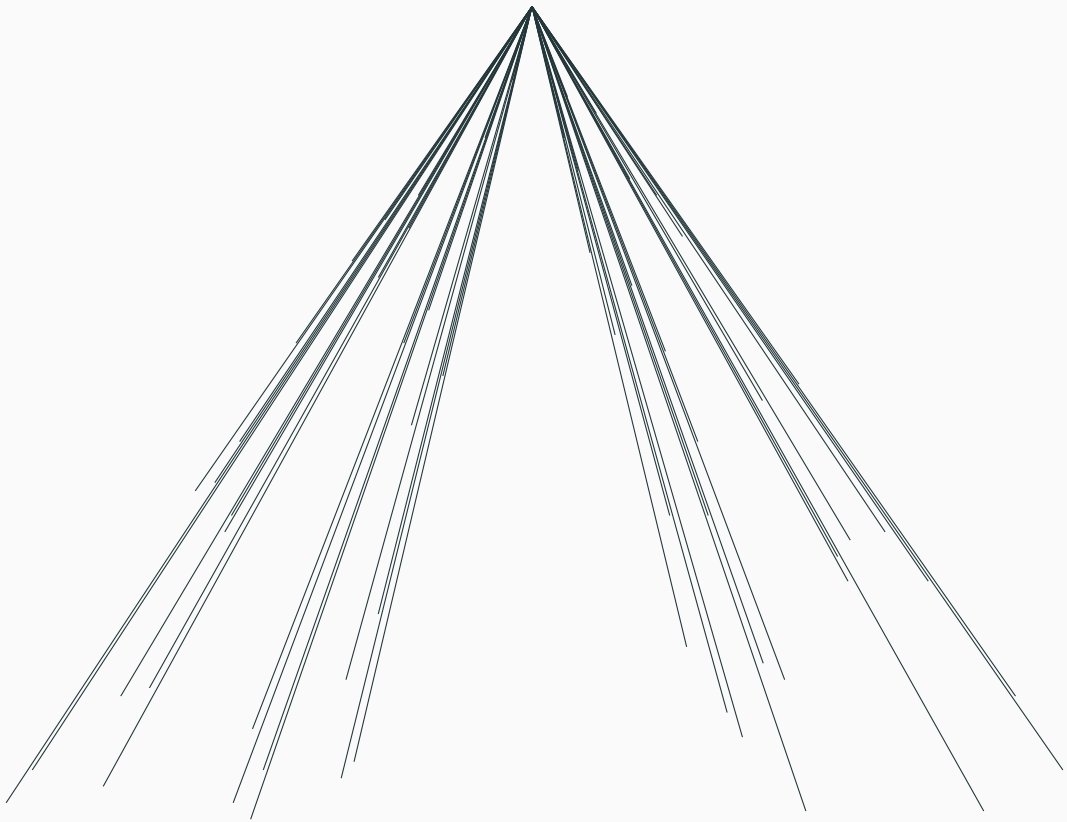

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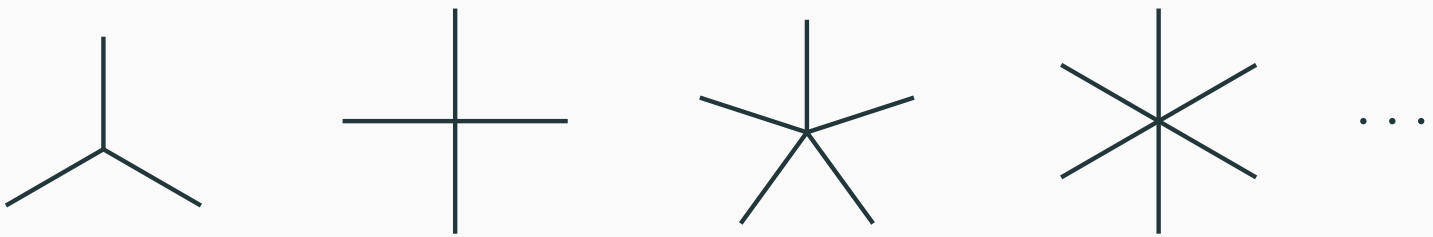

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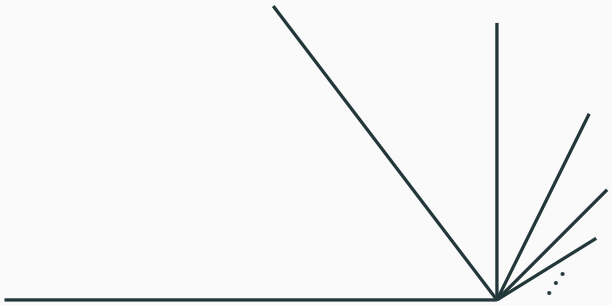
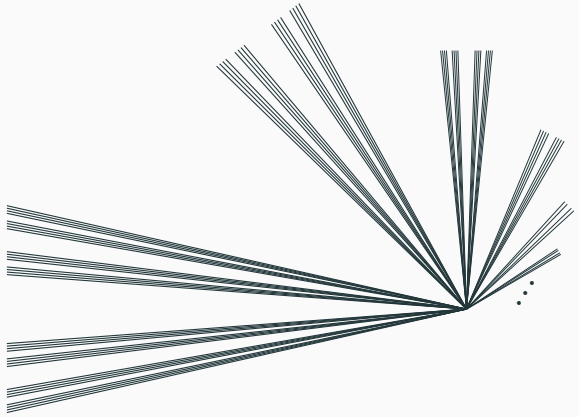
- Examples for $X = \{0\}$ and $\{0,1\}$ are also not difficult to find.





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The proof of this theorem has two main parts:

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2. Prove that no other sets are possible (at least not for smooth fans).

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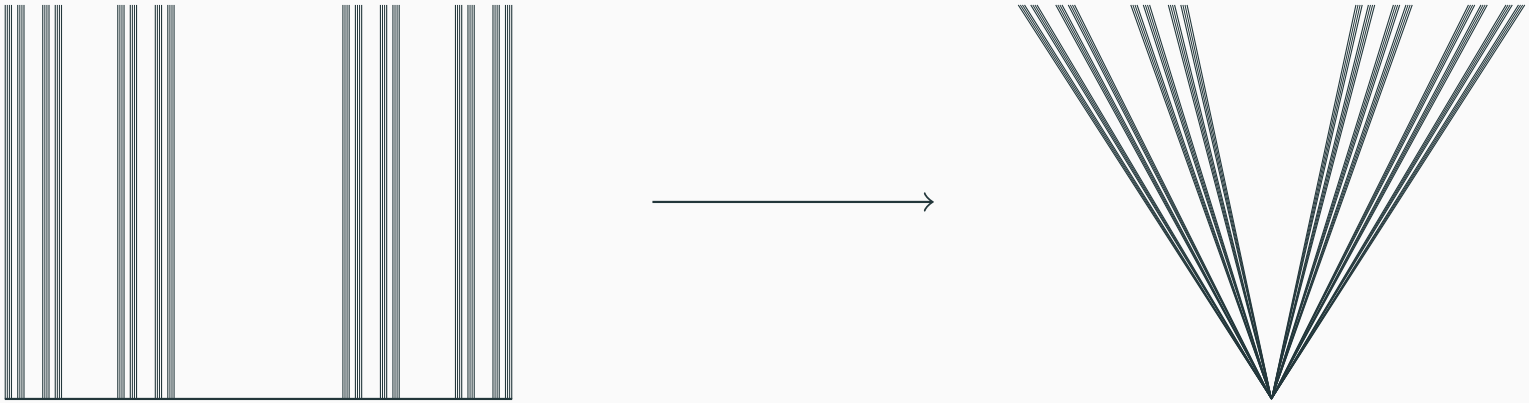

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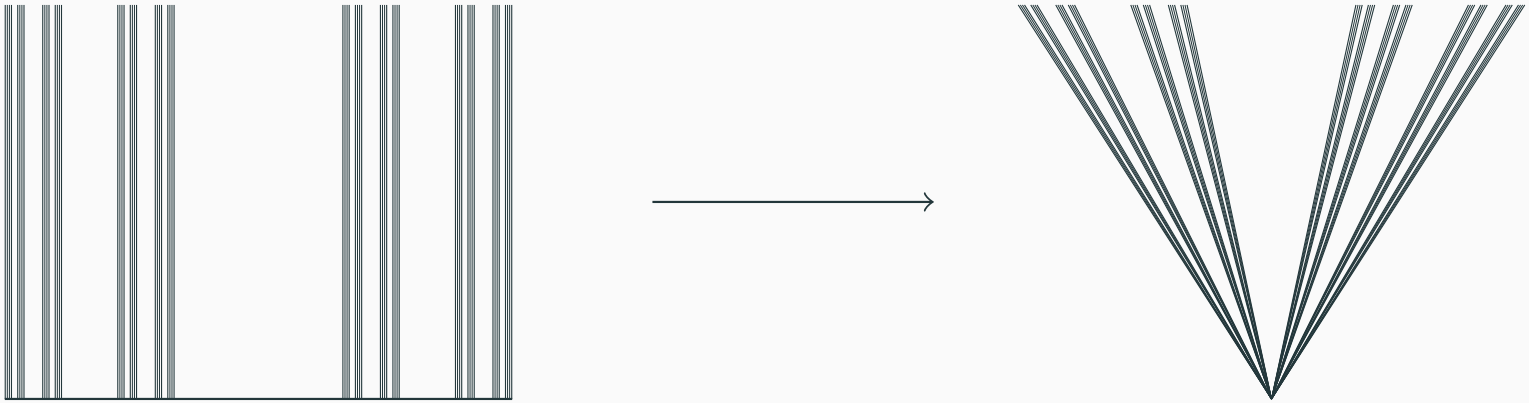

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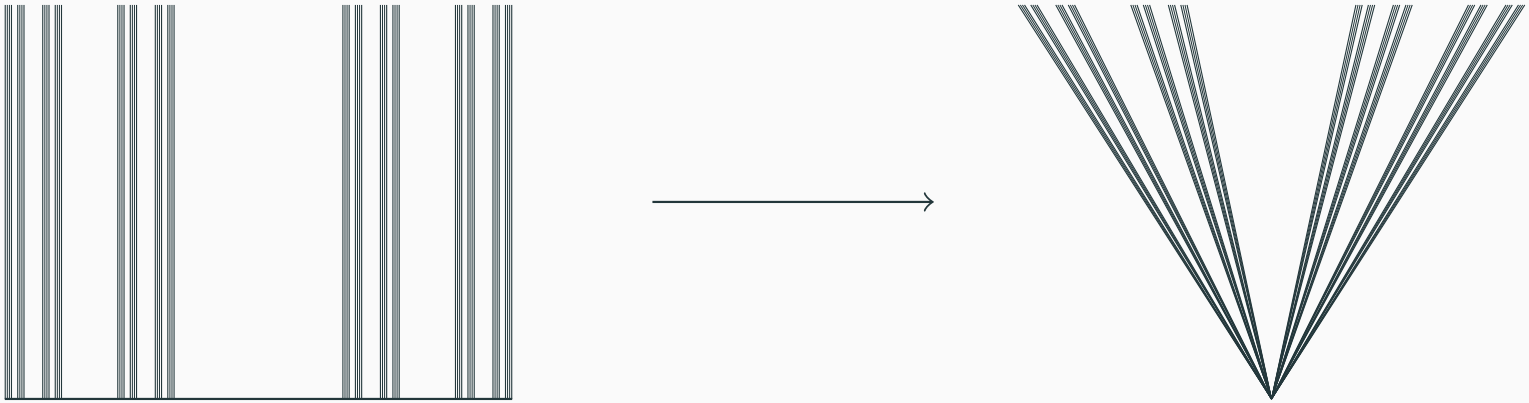

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Every smooth fan can be represented by a comb.

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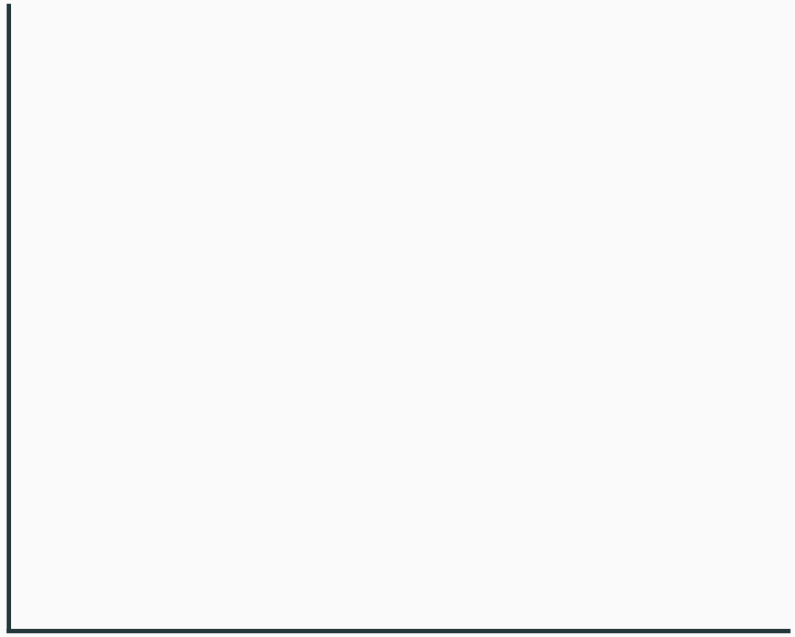

Begin with just one blade on the far left side.

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Add a sequence of new blades, with the endpoints converging to two points on the old blade as shown.

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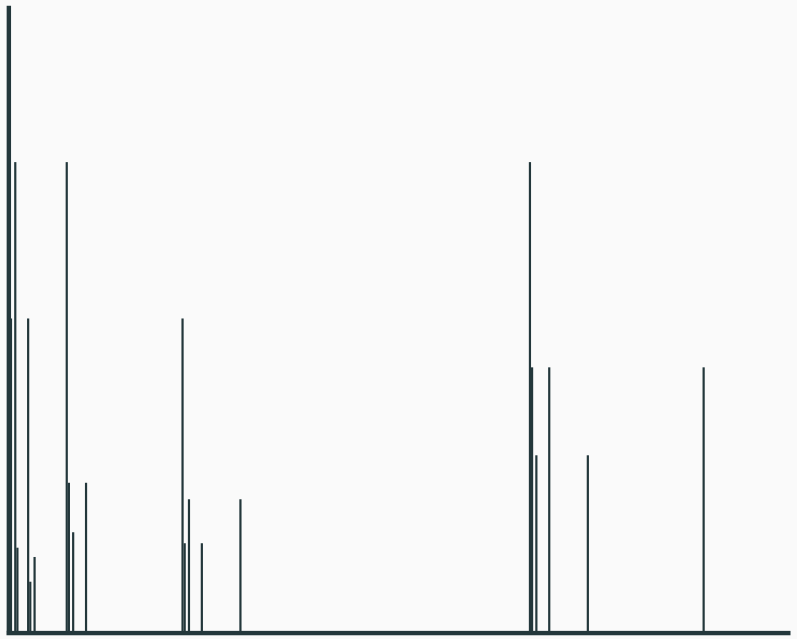

Do this again for each of the just-added blades, making sure to put the limit points lower and lower as we move left, to prevent the unwanted accumulation of extra limit points on the leftmost blade.

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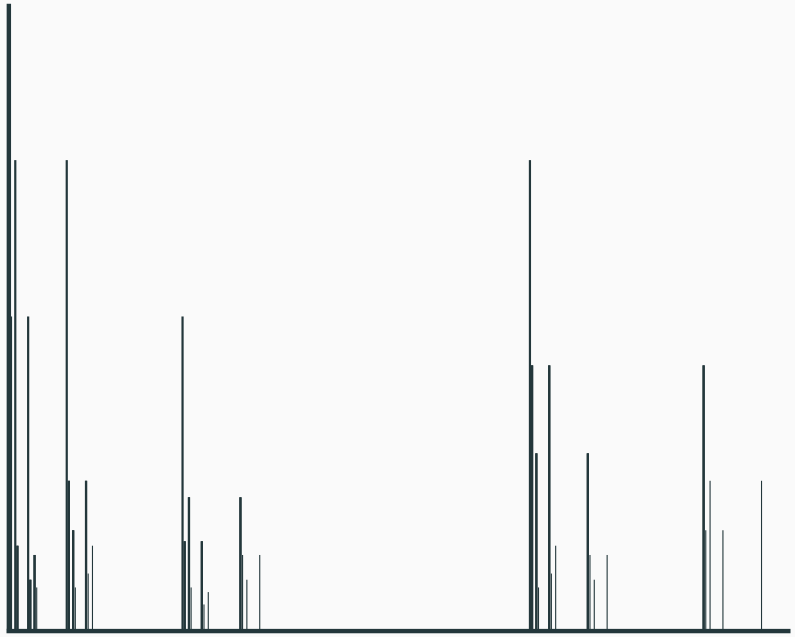

All the other steps of the construction look the same: add a new sequence of blades converging to every just-added blade.

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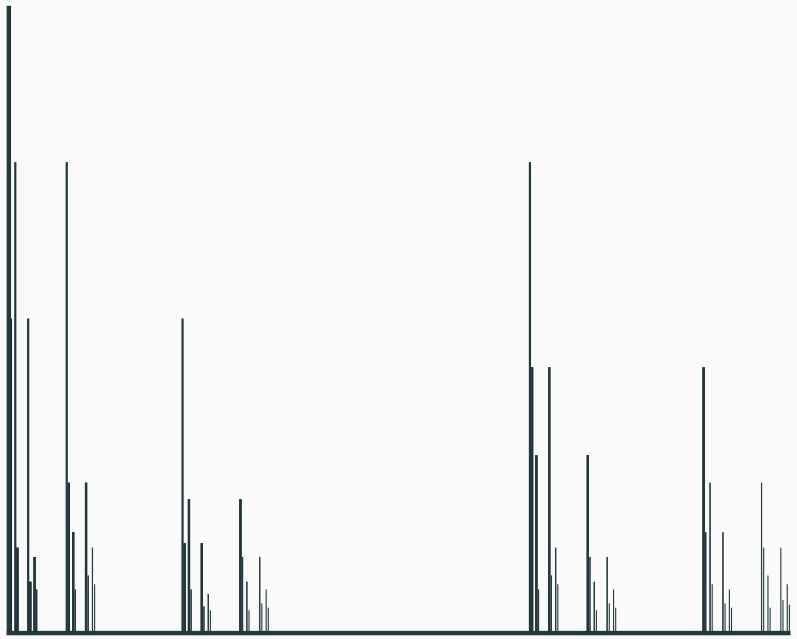

After infinitely many steps, we get the desired comb.

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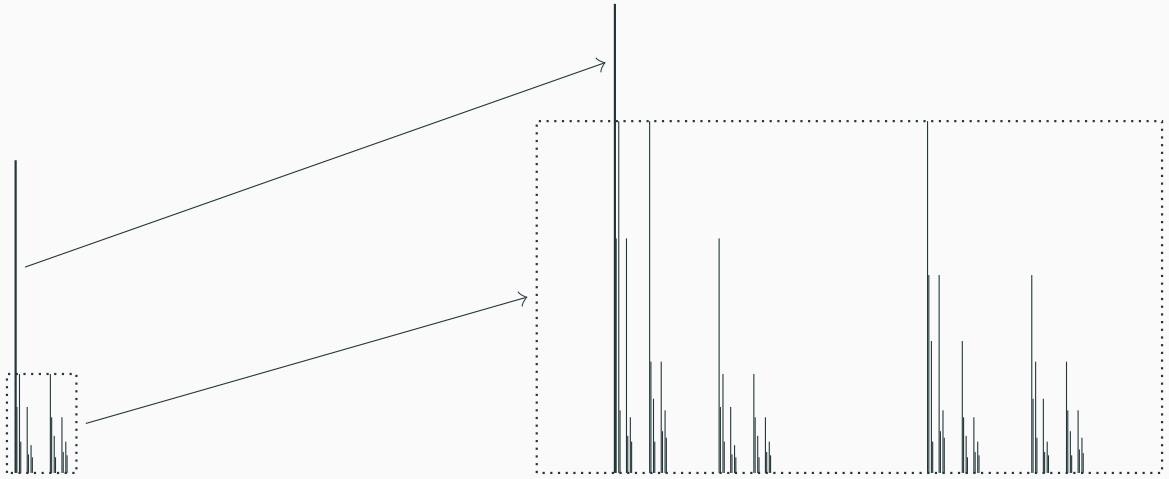

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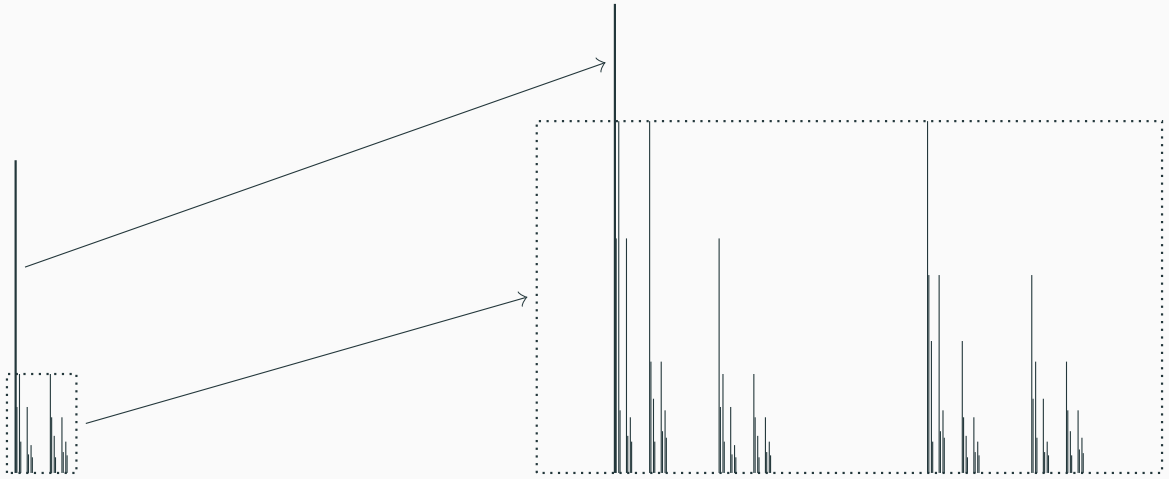

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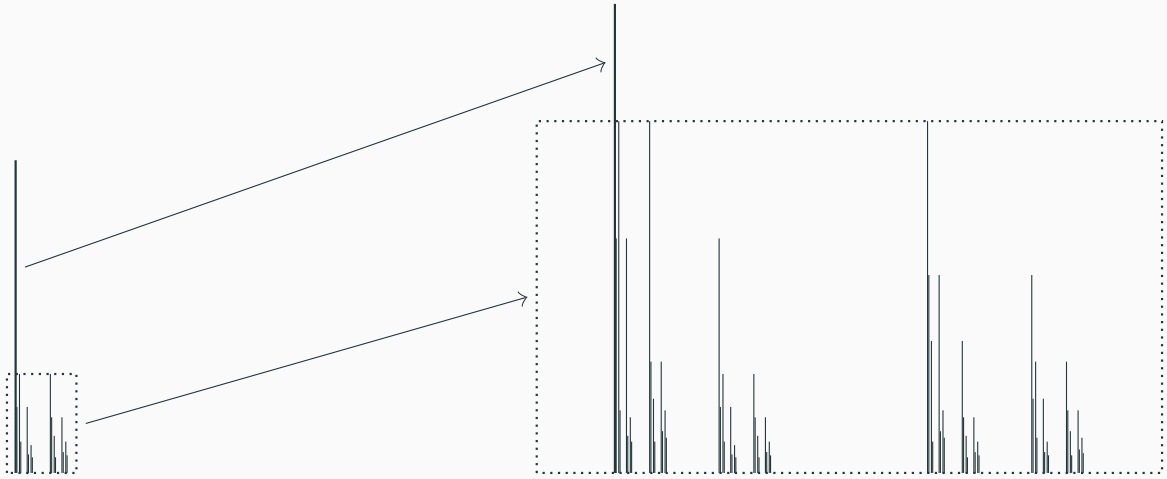

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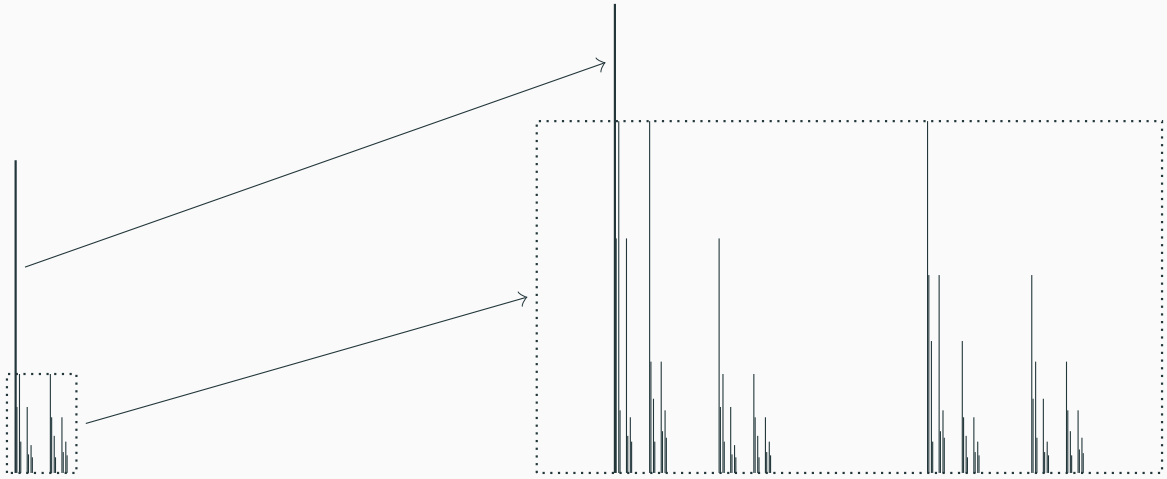

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To get $1 \in X$, multiply the whole comb by the Cantor space before collapsing the bottom to a point.

Part 2: ruling out other sets

Recall the main theorem of this talk:

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Part 2: ruling out other sets

Recall the main theorem of this talk:

Theorem (B. and Gril Rogina, 2024)
A set $X \subseteq [0,1]$ is equal to $\mathrm{EPG}(F)$ for an endpoint-homogeneous smooth fan F if and only if

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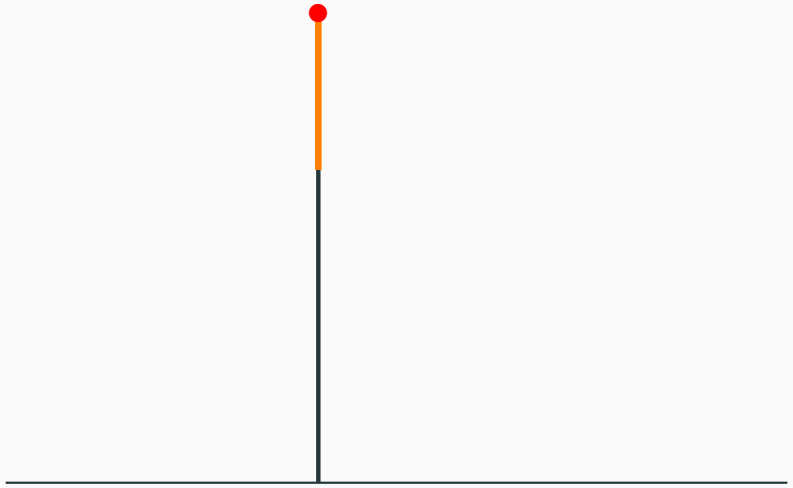

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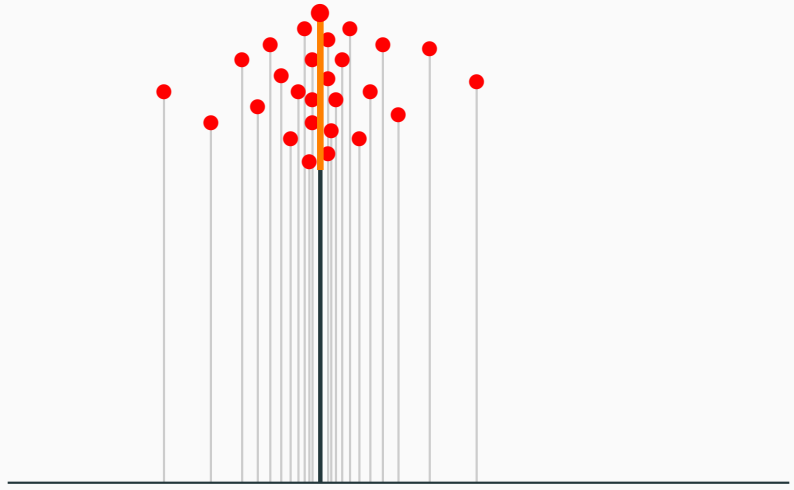

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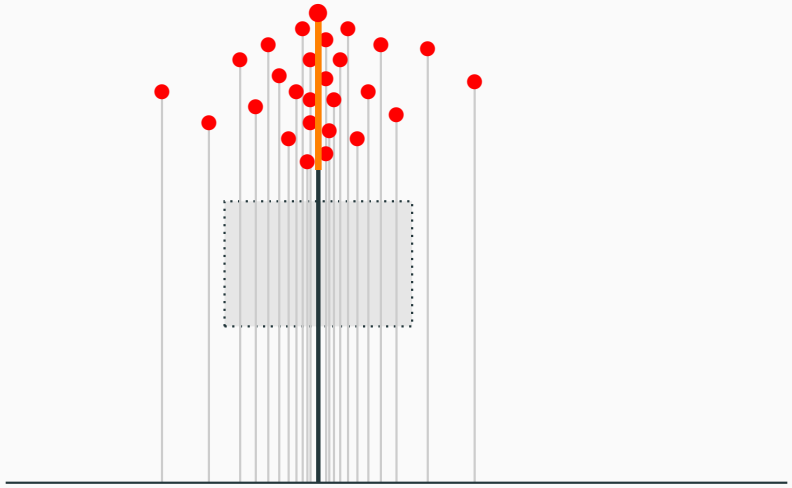

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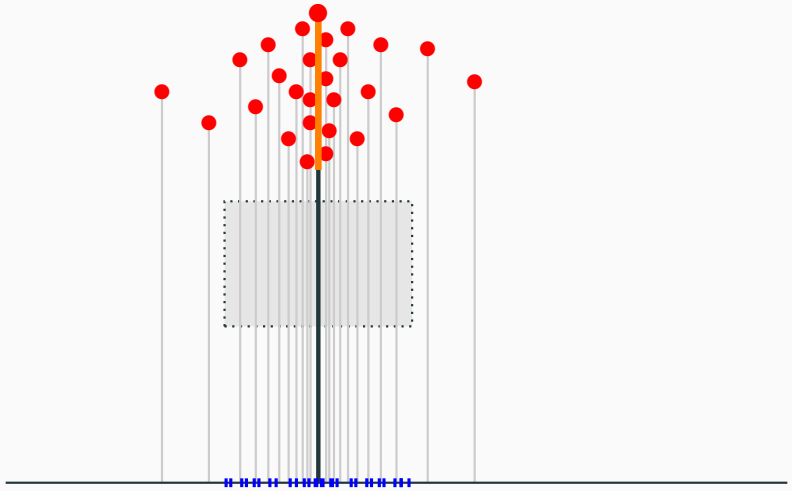

The endpoints of the other blades limit onto the top half of B. Fix a basic open neighborhood V of a point in the bottom half of B that avoids the endpoints of all the other blades. Let X be the set of x-coordinates of those blades with endpoints above V.

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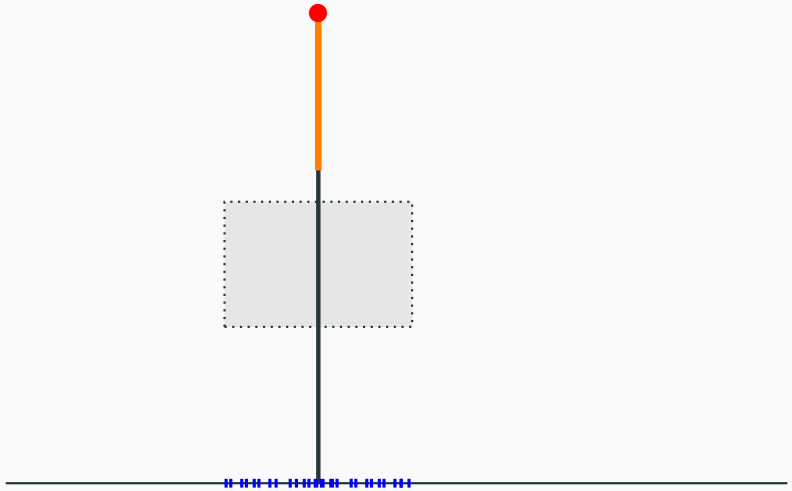

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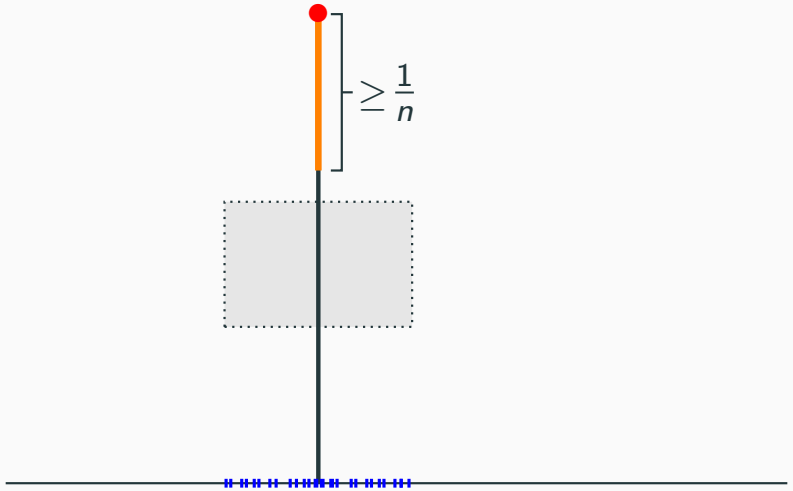

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$$E_{1/n} = \left\{ x : \text{the top and bottom of } B_x \cap \overline{\mathcal{E}(F)} \text{ are } \geq \frac{1}{n} \text{ apart} \right\}$$

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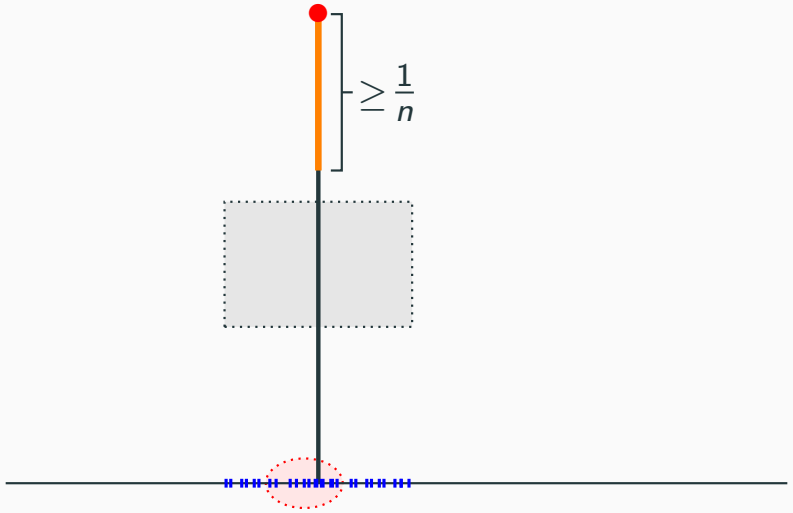

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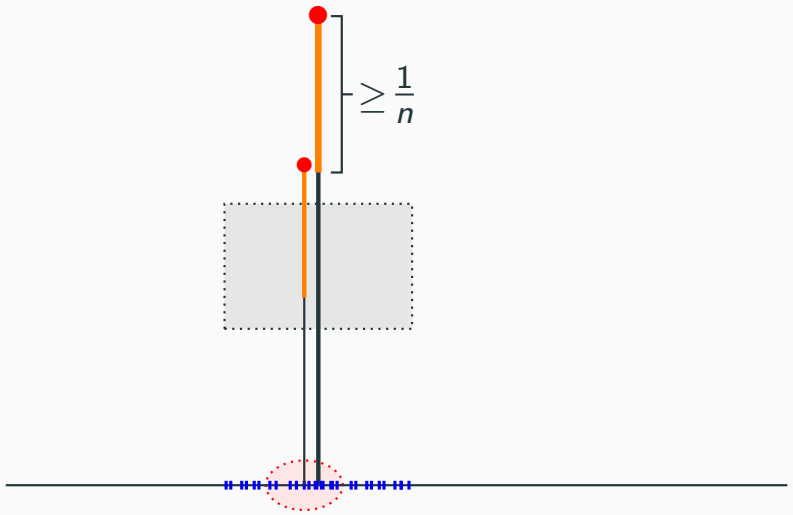

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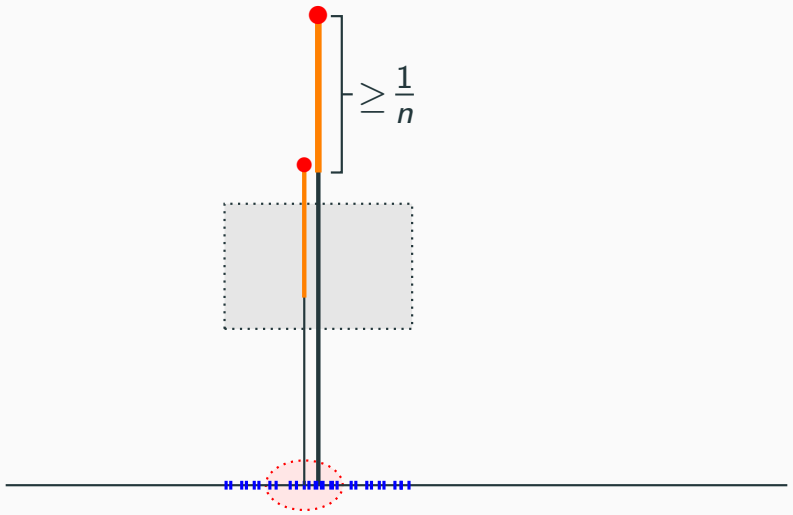

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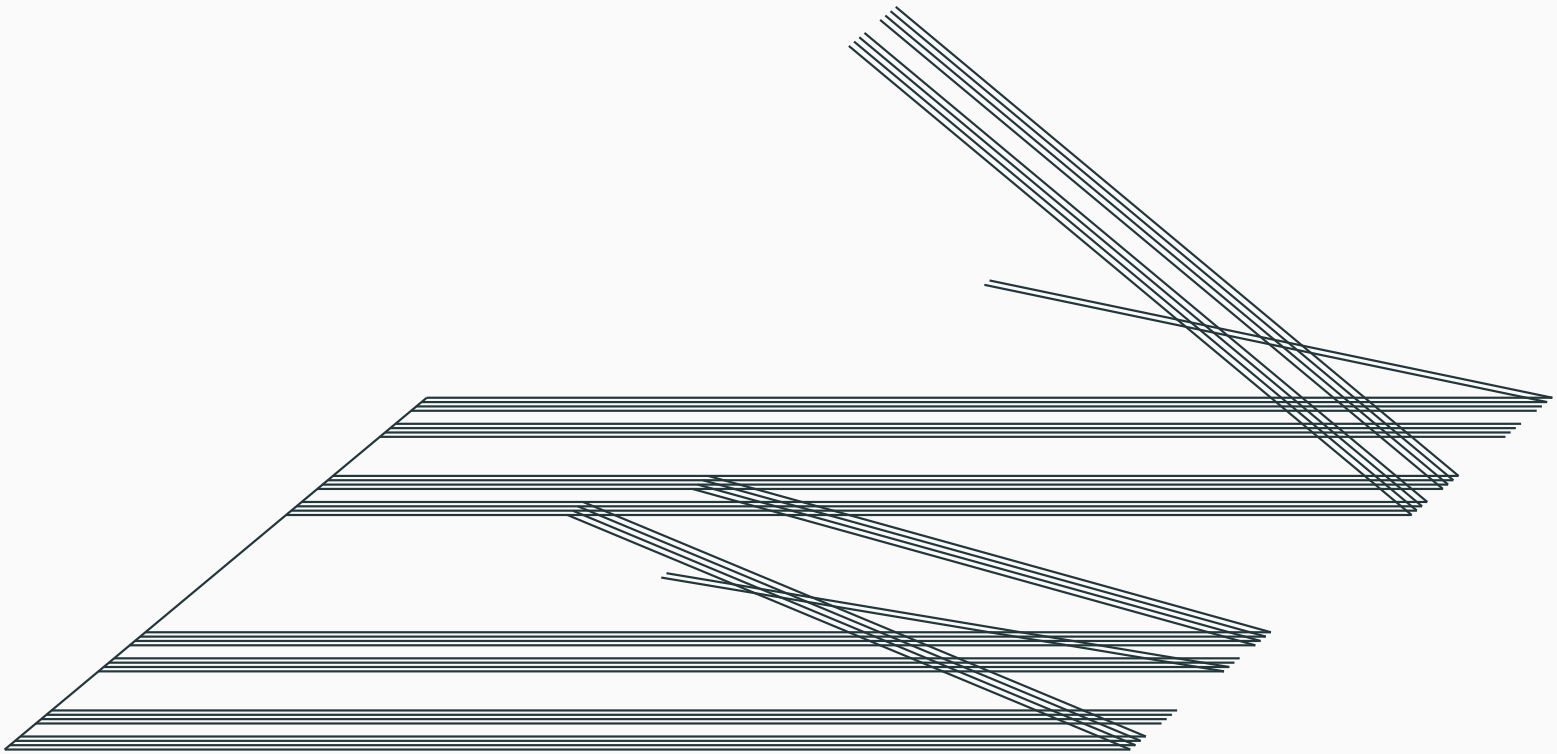

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For which $X \subseteq [0, 1]$ is there a (possibly non-smooth) endpoint-generated fan F with $\mathrm{EPG}(F) = X$? Can certain X be achieved for endpoint-generated fans but not for endpoint-homogeneous fans?

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Is there a non-smooth fan with exactly 4 *types of points?*

Thank you for listening!

Any questions?