Does $\mathcal{P}(\omega)/\mathrm{Fin}$ know its right hand from its left?

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The inverse of a trivial automorphism is trivial, and the composition of two trivial automorphisms is trivial. So the trivial automorphisms form a subgroup of $Aut(\mathcal{P}(\omega)/\mathrm{Fin})$.

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Theorem (W. Rudin, 1956)

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Over the years, this theorem has been built upon by Shelah, Steprāns, Veličković, Farah, Moore, and others. We now know that OCA implies all automorphisms of $\mathcal{P}(\omega)/\mathrm{Fin}$ are trivial.

When two automorphisms are the same

Two automorphisms α and β of ${}^{\mathcal{P}(\omega)}\!/_{\mathrm{Fin}}$ are *conjugate* if there is a third automorphism γ such that $\gamma \circ \alpha = \beta \circ \gamma$.

$$\begin{array}{ccc} {}^{\mathcal{P}(\omega)}\!/_{\mathrm{Fin}} & \xrightarrow{\alpha} & {}^{\mathcal{P}(\omega)}\!/_{\mathrm{Fin}} \\ \gamma \downarrow & & \downarrow \gamma \\ {}^{\mathcal{P}(\omega)}\!/_{\mathrm{Fin}} & \xrightarrow{\beta} & {}^{\mathcal{P}(\omega)}\!/_{\mathrm{Fin}} \end{array}$$

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Are σ and σ^{-1} conjugate?

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Are σ and σ^{-1} conjugate?

In other words, can $\mathcal{P}(\omega)/\mathrm{Fin}$ tell its right from its left?

Is the shift map conjugate to its inverse?

Theorem (van Douwen, 1983)

If γ *is a conjugacy mapping between* σ *and* σ^{-1} *(i.e.,* γ *is an automorphism such that* $\gamma \circ \sigma = \sigma^{-1} \circ \gamma$*), then* γ *is non-trivial.*

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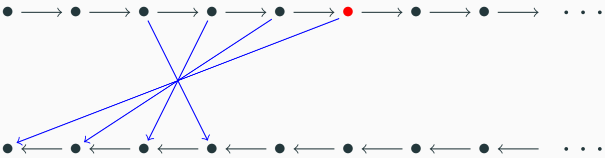

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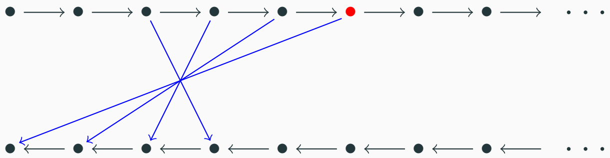

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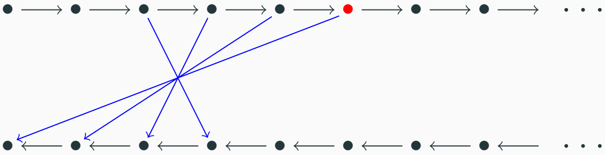

Every good point n is followed by $< f(n)$ more good points. This implies there are infinitely many bad points. Among these, we can find an infinite set B such that $f[B+1] \cap (f[B] - 1) = \emptyset$, which means in particular that $\alpha_f \circ \sigma([B]_{\mathrm{Fin}}) \neq \sigma^{-1} \circ \alpha_f([B]_{\mathrm{Fin}})$.

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In the remainder of this talk, we will describe a few results related to this theorem, and sketch some of the ideas that go into proving them.

van Douwen's index theorem

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$$\mathrm{Ind}(f) = |\mathrm{domain}(f) \setminus \mathrm{image}(f)| - |\mathrm{image}(f) \setminus \mathrm{domain}(f)|.$$

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In particular, it makes sense to write $\mathrm{Ind}(\alpha_f)$, not just $\mathrm{Ind}(f)$.

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Theorem (B. and Farah, 2024)

Let α and β be trivial automorphisms of $\mathcal{P}(\omega)/\mathrm{Fin}$. TFAE:

1. *α and β are conjugate in a forcing extension.*
2. *CH proves α and β are conjugate.*
3. *$\mathrm{Ind}(\alpha)$ and $\mathrm{Ind}(\beta)$ have the same parity, and the structures $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \alpha\rangle$ and $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \beta\rangle$ are elementarily equivalent.*

$\mathbb{H}$ and $\mathbb{M}$

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Let $\mathbb{H}^* = \beta\mathbb{H} \setminus \mathbb{H}$ and $\mathbb{M}^* = \beta\mathbb{M} \setminus \mathbb{M}$ denote the Čech-Stone remainders of these two spaces.

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Just as $\mathbb{H}$ can be obtained from $\mathbb{M}$ by gluing some points together, there is an equivalence relation $\sim$ on $\mathbb{M}^*$ such that $\mathbb{H}^* = \mathbb{M}^*/\sim$.

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$$\langle x_n \rangle_u \;=\; u\text{-}\lim_{n \in \omega}(n, x_n) \qquad \text{(the limit is taken in } \beta\mathbb{M})$$

is a point of $\mathbb{M}^*$ for every $u \in \omega^*$.

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$$\langle 1, 1, 1, \ldots \rangle_u \sim \langle 0, 0, 0, \ldots \rangle_{\sigma(u)}$$

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$\mathbb{H}^*$ is obtained from $\mathbb{M}^*$ by gluing these I_u together, the right endpoint of I_u being glued to the left endpoint of $I_{\sigma(u)}$. Each of these I_u is called a *standard subcontinuum* of $\mathbb{H}^*$.

An order-reversing autohomeomorphism

A self-homeomorphism $f : \mathbb{H}^* \to \mathbb{H}^*$ is called *order-reversing* if

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Theorem (Vignati, 2021)

OCA $+$ MA *implies all self-homeomorphisms of* $\mathbb{H}^*$ *are trivial, and in particular there is no order-reversing self-homeomorphism of* $\mathbb{H}^*$.

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(As before, we're using σ to denote the Stone dual of the shift map on $\mathcal{P}(\omega)/\mathrm{Fin}$, which maps an ultrafilter $u \in \omega^*$ to the ultrafilter generated by $\{A+1 : A \in u\}$.)

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Using the theorem on the previous slide (and using CH again), there is an order-preserving self-homeomorphism $F : \mathbb{M}^* \to \mathbb{M}^*$ such that $\pi \circ F = f$.

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We now have two self-homeomorphisms of $\mathbb{M}^*$, F and G. G is order-reversing and F is order-reversing, so their composition $H = F \circ G$ is an order-reversing self-homeomorphism of $\mathbb{M}^*$.

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Furthermore, because G sends each I_u to itself (only reversed), H maps each I_u to $I_{f(u)}$, just like F.

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Thus $H = F \circ G$ maps the set $\{\bar{1}_u, \bar{0}_{\sigma(u)}\}$ to the set $\{\bar{0}_{f(u)}, \bar{1}_{\sigma^{-1}(f(u))}\}$, which is also an equivalence class of $\sim$.

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Because H preserves the equivalence classes of $\sim$, the function $[x]_\sim \mapsto [H(x)]_\sim$ is a well-defined mapping on $\mathbb{H}^*$. This function is the sought-after order-reversing self-homeomorphism of $\mathbb{H}^*$.

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Let f *be a permutation of* ω *with infinitely many* $\mathbb{Z}$*-like orbits, and let* g *be an almost permutation with infinitely many* $\mathbb{Z}$*-like orbits and one* $\mathbb{N}$*-like orbit. Is* $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \alpha_f \rangle \equiv \langle \mathcal{P}(\omega)/\mathrm{Fin}, \alpha_g \rangle$*?*

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