An uncountable analogue of Fraïssé's theorem

Will Brian
(joint work with Jacob Page)

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University of North Carolina at Charlotte

Part $\aleph_0$

The random graph and the FEP

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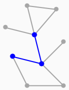

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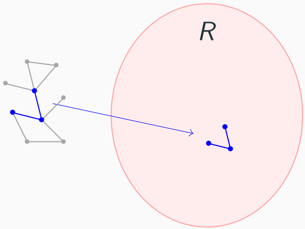

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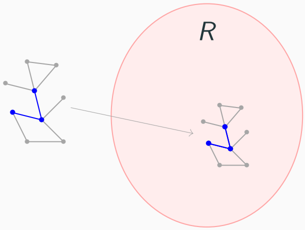

FEP: Suppose A and B are finite graphs with $A \subseteq B$, and $e : A \to R$ is an embedding. Then there is an embedding $\bar{e} : B \to R$ that extends e.

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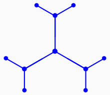

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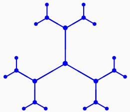

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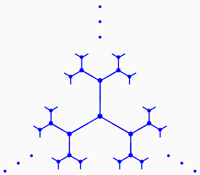

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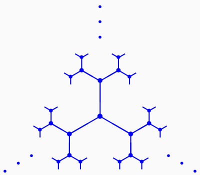

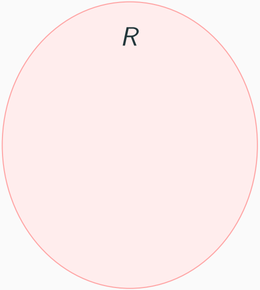

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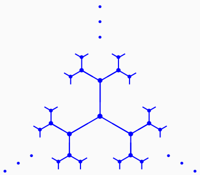

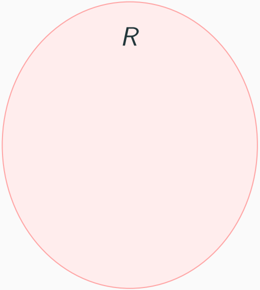

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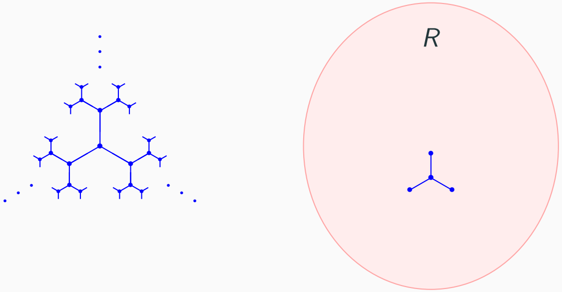

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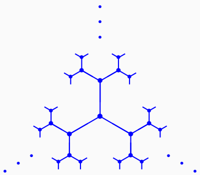

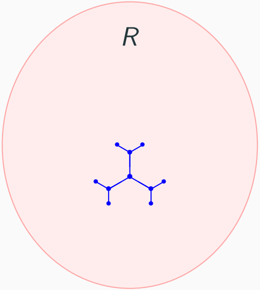

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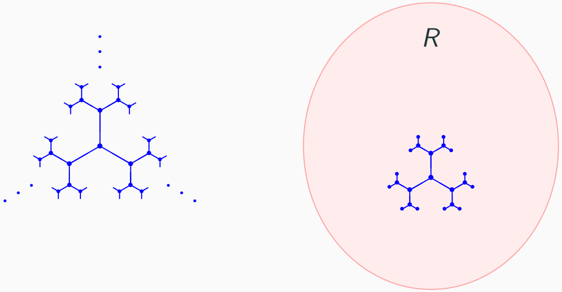

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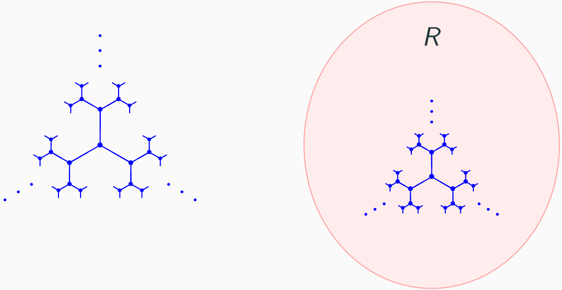

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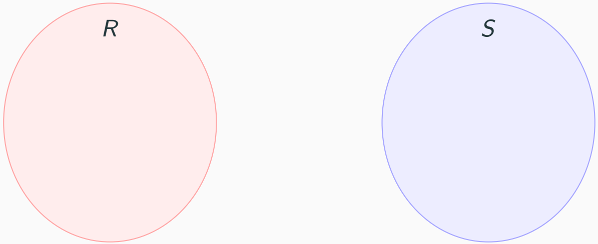

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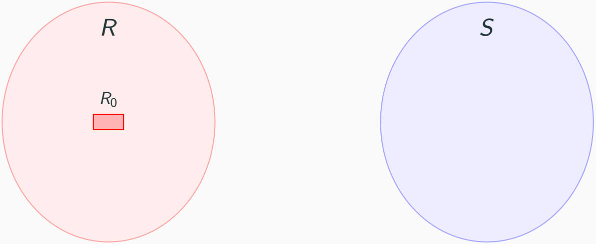

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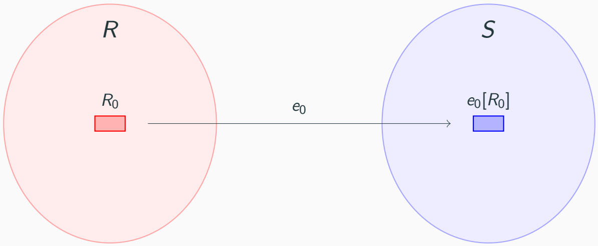

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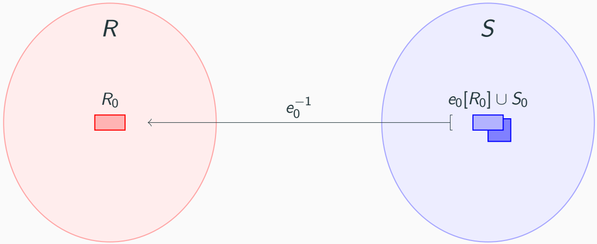

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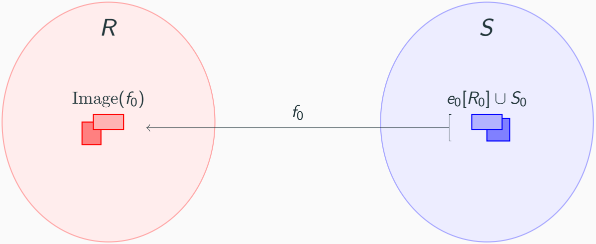

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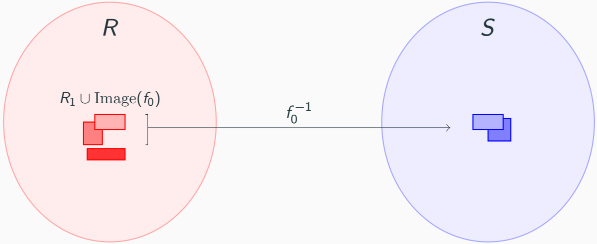

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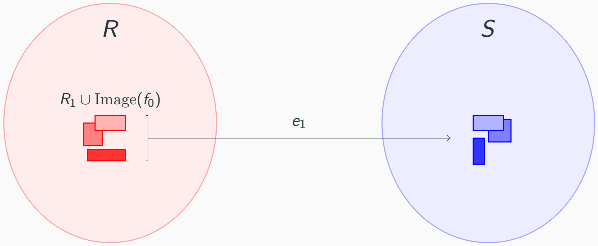

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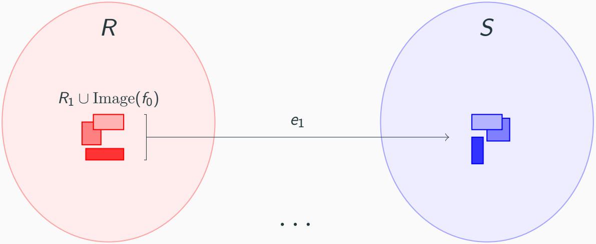

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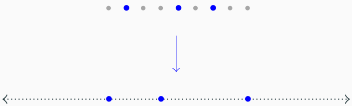

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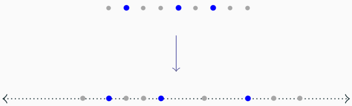

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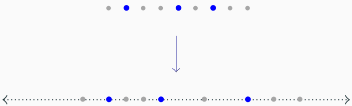

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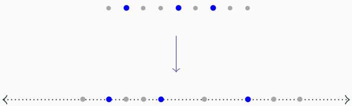

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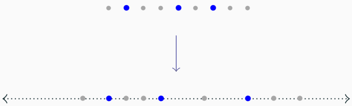

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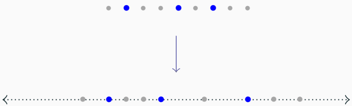

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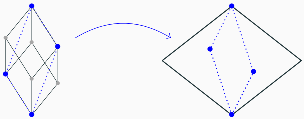

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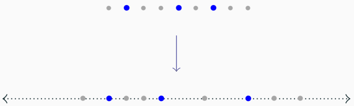

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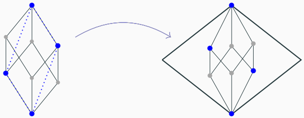

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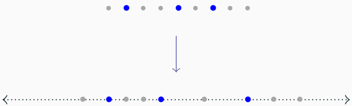

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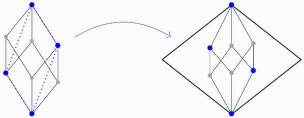

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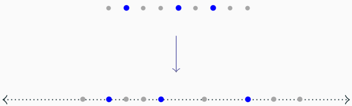
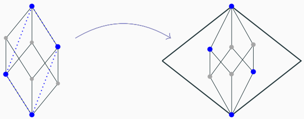


And $\langle \mathbb{Q}, \leq \rangle$ is $\aleph_0$-universal: it contains a copy of every countable linear order.

The countable atomless Boolean algebra $\langle \mathbb{C}, \leq, \mathbf{0}, \mathbf{1} \rangle$ is is the analogous object for BA's. It is the unique countable Boolean algebra with the FEP.



Furthermore, $\langle \mathbb{C}, \leq, \mathbf{0}, \mathbf{1} \rangle$ is $\aleph_0$-universal: it contains a copy of every countable BA.

Not only do these structures enjoy analogous properties, but the proofs that they have these properties are essentially the same.

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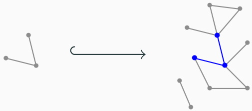

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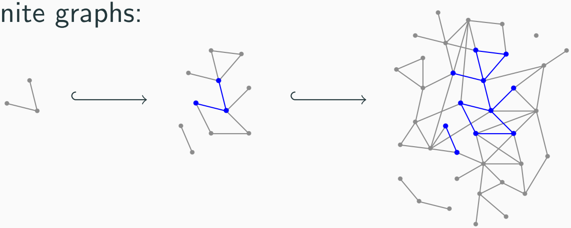

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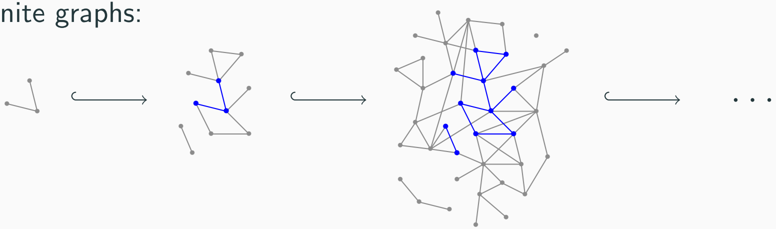

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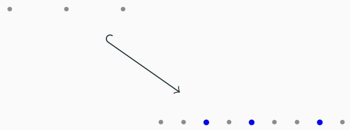

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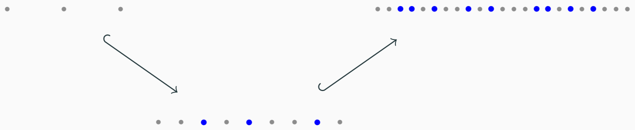

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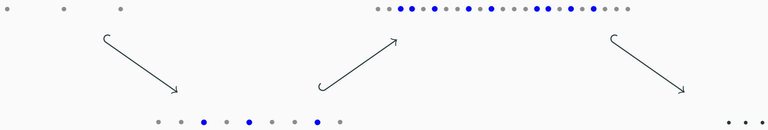

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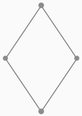

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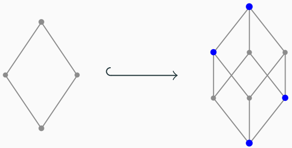

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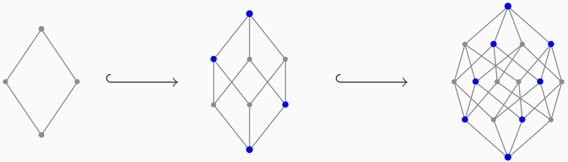

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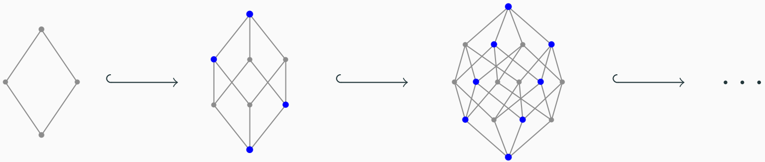

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Rule(👍): If you see an "essentially countable" mathematical object of class C with lots of self-morphisms, and if it is universal (in some sense) for other essentially countable objects of class C, then you should suspect that you are looking at a Fraïssé limit.

Two non-examples

A connected compact metric space is called a *metric continuum*.

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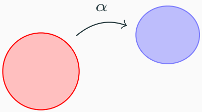

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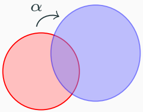

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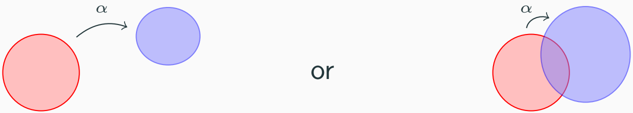

Theorem: (Geschke, 2015) There is no universal countable incompressible dynamical system.

When does a class of finite structures have a Fraïssé limit?

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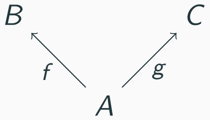

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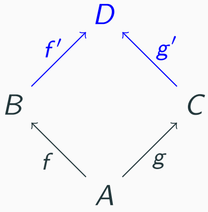

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When $\mathcal{K}$ satisfies these properties, there is exactly one such structure up to isomorphism, called the *Fraïssé limit* of $\mathcal{K}$.

Part $\aleph_1$

Bigger Fraïssé limits?

To summarize: if a class $\mathcal{K}$ of finite structures is "nice enough" then there is a very special $\mathcal{K}$-structure of size $\aleph_0$, the Fraïssé limit of $\mathcal{K}$. It is $\aleph_0$-universal and is characterized by the FEP.

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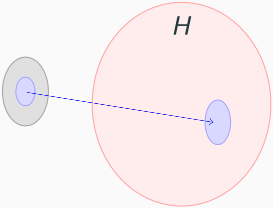

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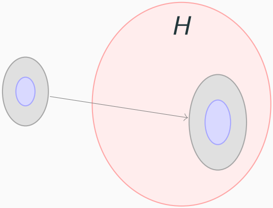

CEP: Suppose A and B are countable graphs with $A \subseteq B$, and $e : A \to H$ is an embedding. Then there is an embedding $\bar{e} : B \to H$ that extends e.

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To summarize: if a class $\mathcal{K}$ of finite structures is "nice enough" then there is a very special $\mathcal{K}$-structure of size $\aleph_0$, the Fraïssé limit of $\mathcal{K}$. It is $\aleph_0$-universal and is characterized by the FEP.

Assuming CH, something similar happens at $\aleph_1$.

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The proof is via a transfinite back-and-forth argument of length ω_1, using the CEP at successor steps.

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It seems that every one of our Fraïssé limits has a size-$\aleph_1$ analogue, provided that we assume CH.

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Theorem: Suppose $\mathcal{K}$ is a class of finite structures with the Hereditary Property, the Joint Embedding Property, and the Amalgamation Property (i.e., a class with a Fraïssé limit). Then there is a size-$\mathfrak{c}$ structure M with the CEP.

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Surprisingly, such classes can reacquire universal models at $\aleph_1$.

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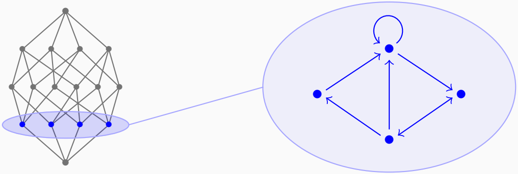

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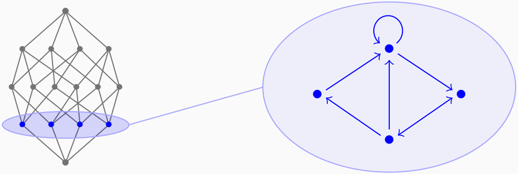

Furthermore, the incompressibility of σ means that the digraphs obtained this way are always strongly connected.

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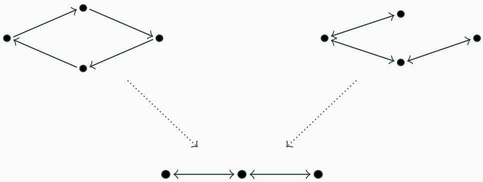

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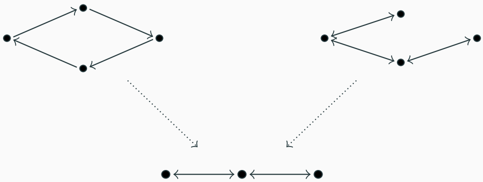

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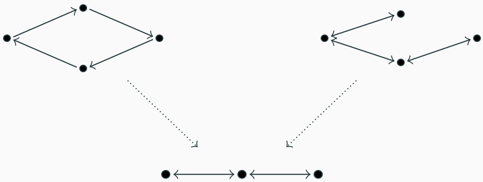

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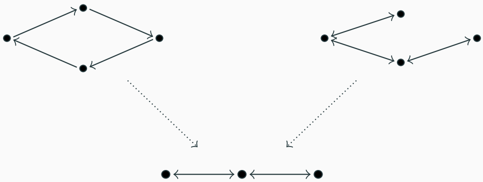

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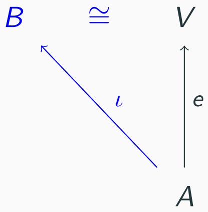

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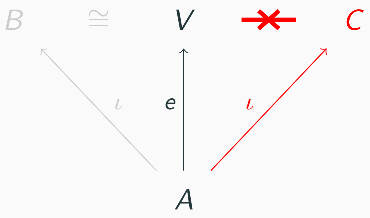

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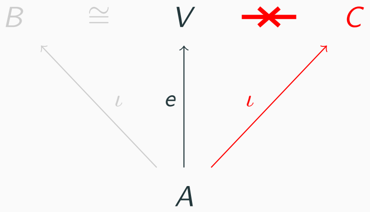

Theorem: (B., 2024) $\langle {}^{\mathcal{P}(\omega)}/_{\mathrm{Fin}}, \xrightarrow{\sigma} \rangle$ is a decisive $\mathcal{K}_\sigma$-structure.

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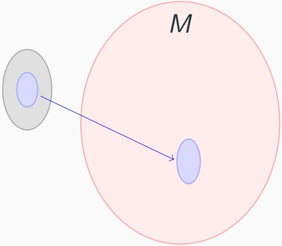

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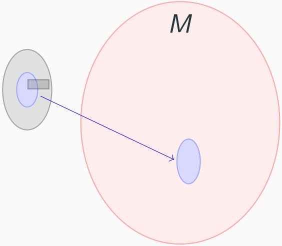

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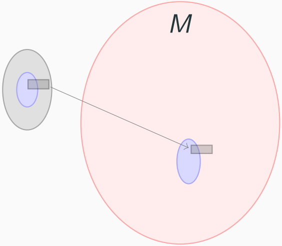

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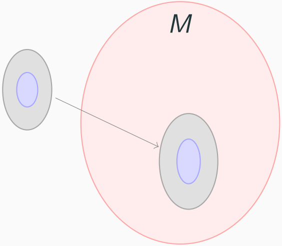

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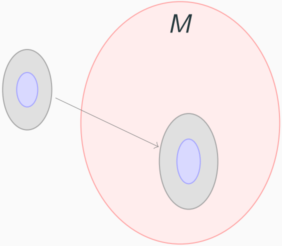

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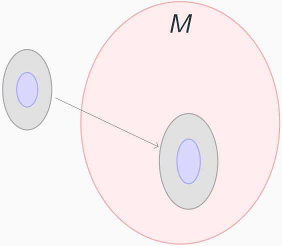

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Theorem: (B., 2024) $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \xrightarrow{\sigma} \rangle$ has the w-CEP.

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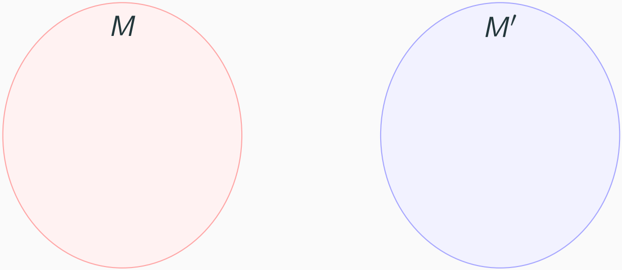

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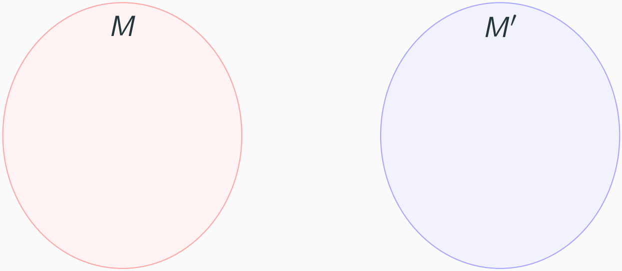

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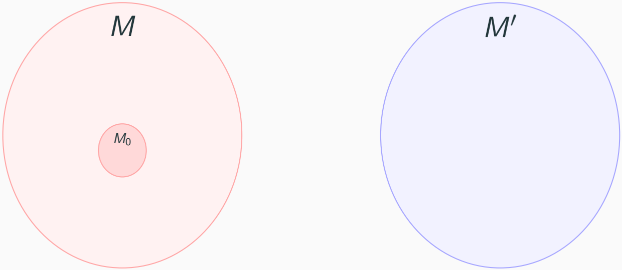

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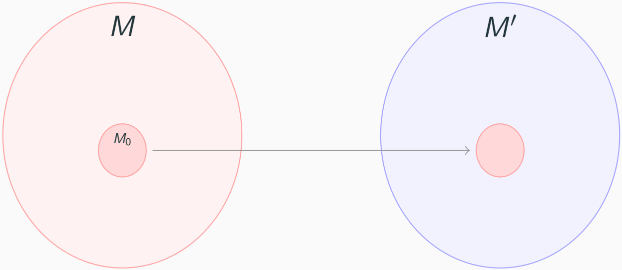

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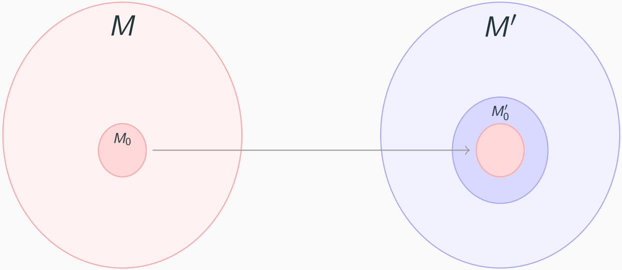

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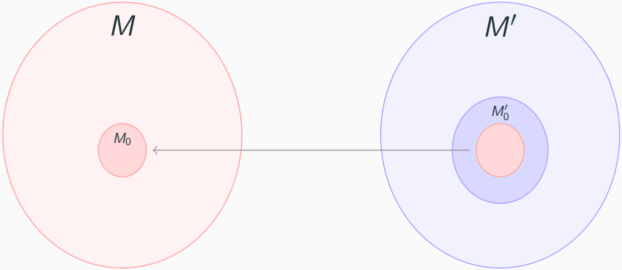

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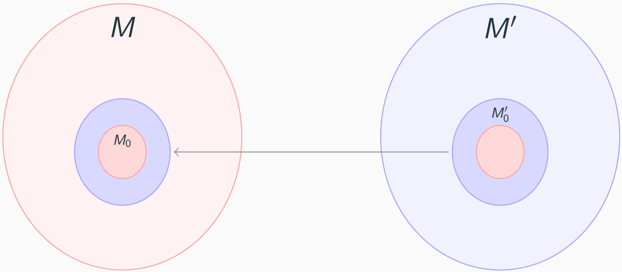

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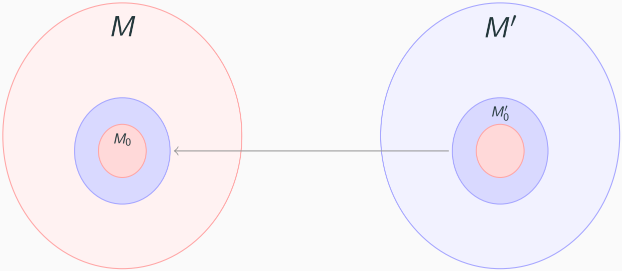

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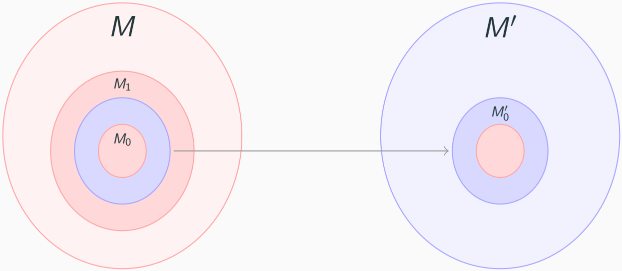

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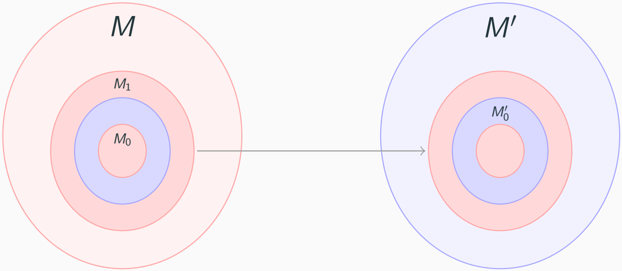

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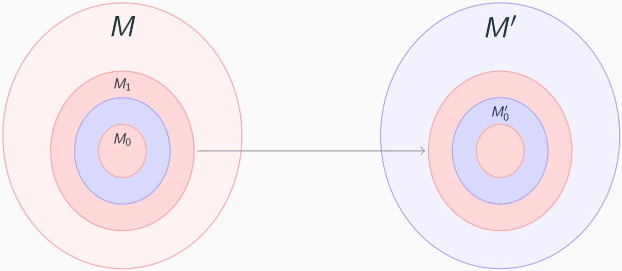

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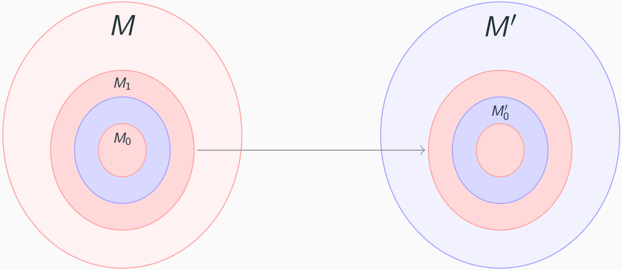

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Diagram of implications between various properties

For a given class $\mathcal{K}$ of finite structures, the following implications hold for all $\mathcal{K}$-structures: