Order-reversing maps on $\mathbb{N}^*$ and $\mathbb{H}^*$

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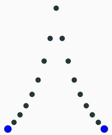

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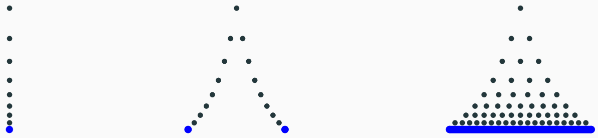

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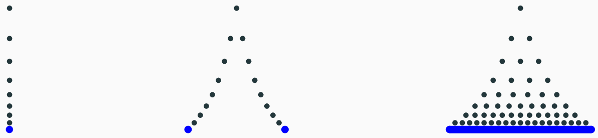

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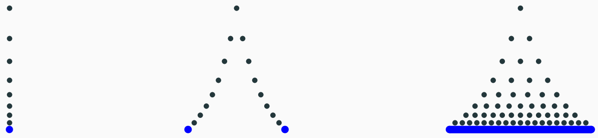

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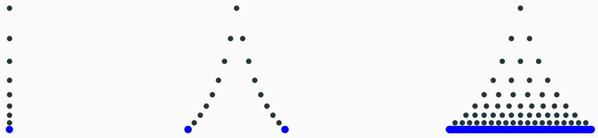

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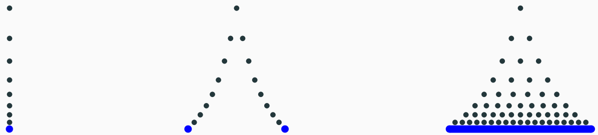

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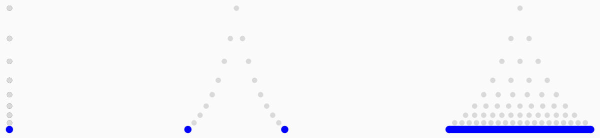

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The *remainder* of a compactification Y of X is $Y \setminus X$.

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This map σ transfers to $\mathbb{N}^*$ something of the order on $\mathbb{N}$. Using σ, slick proofs can be found for deep theorems of arithmetic combinatorics, like Hindman's Theorem or van der Waerden's Theorem.

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Thus van Douwen's question is whether there is any topological difference in $\mathbb{N}^*$ between a right shift and a left shift.

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Further work has given a robust understanding of when CH proves two trivial self-homeomorphisms are conjugate (B. & Farah, 2024).

A few words on the proof

Suppose $\mathcal{A}$ is a partition of $\mathbb{N}^*$ into finitely many clopen sets. The action of σ on $\mathcal{A}$ is represented by a digraph $\langle \mathcal{A}, \xrightarrow{\sigma} \rangle$, where

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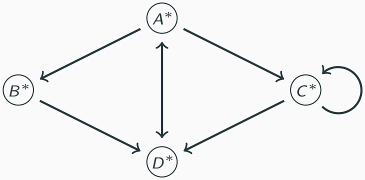

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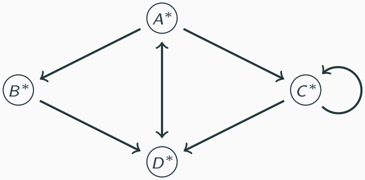

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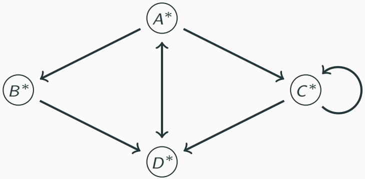

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Lemma

A finite digraph is strongly connected if and only if it is isomorphic to $\langle \mathcal{A}, \xrightarrow{\sigma} \rangle$ for some partition $\mathcal{A}$ of $\mathbb{N}^*$ into clopen sets.

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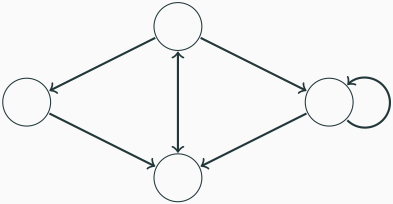

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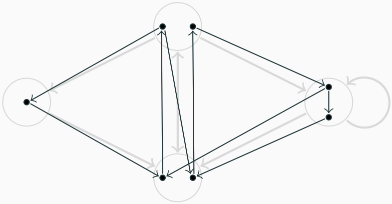

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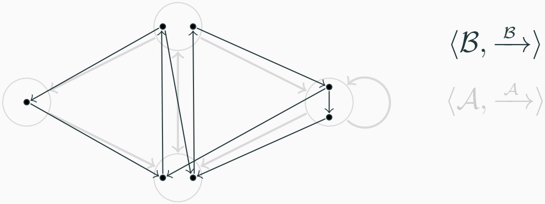

Lemma

Suppose $\mathcal{A}$ and $\mathcal{B}$ are partitions of $\mathbb{N}^*$ into clopen sets. If $\mathcal{B}$ is a refinement of $\mathcal{A}$, then the natural mapping $\mathcal{B} \to \mathcal{A}$ is an epimorphism from $\langle \mathcal{B}, \xrightarrow{\sigma} \rangle$ to $\langle \mathcal{A}, \xrightarrow{\sigma} \rangle$.

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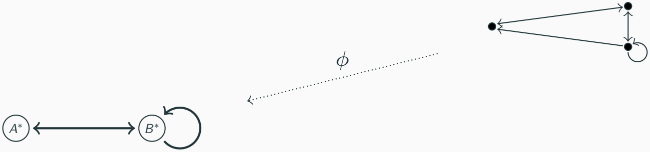

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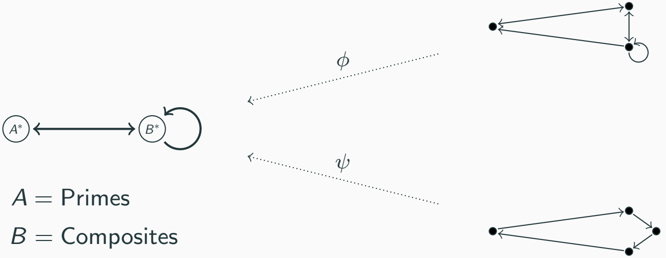

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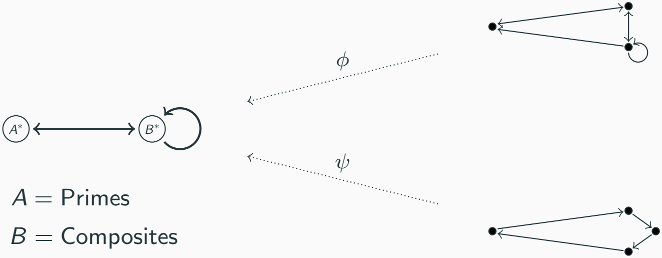

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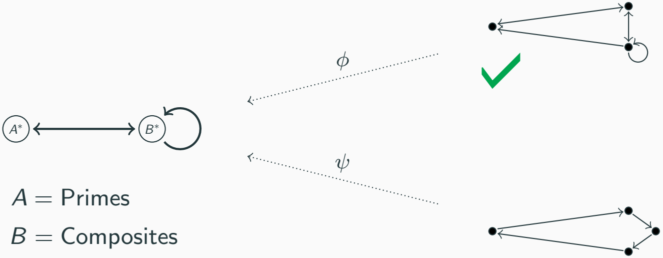

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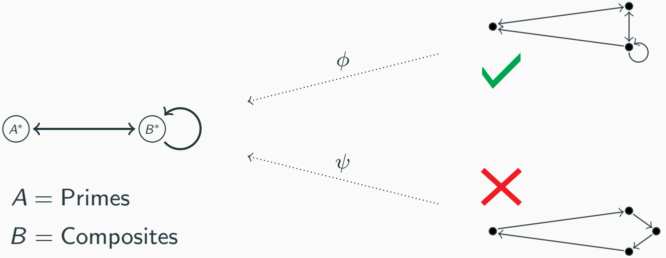

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$$\begin{array}{c} \langle \mathcal{V}, \longrightarrow \rangle \\ \Big\downarrow \phi \\ \langle \mathcal{A}, \xrightarrow{\sigma} \rangle \end{array}$$

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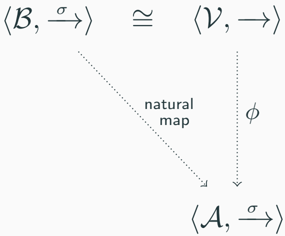

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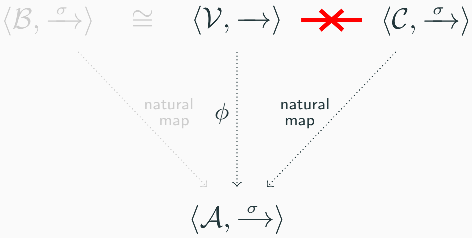
1. *ϕ is realizable as a refinement of $\mathcal{A}$, or*
2. *there is a refinement $\mathcal{C}$ of $\mathcal{A}$ such that the natural map $\mathcal{C} \to \mathcal{A}$ is incompatible with ϕ.*



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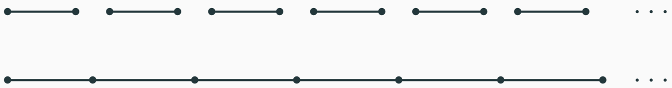

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Just as $\mathbb{H}$ can be obtained from $\mathbb{M}$ by gluing some points together, there is an equivalence relation $\sim$ on $\mathbb{M}^*$ such that $\mathbb{H}^* = \mathbb{M}^*/\sim$.

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Theorem (Vignati, 2021)
$\mathrm{OCA} + \mathrm{MA}$ *implies all self-homeomorphisms of* $\mathbb{H}^*$ *are trivial, and in particular there is no order-reversing self-homeomorphism of* $\mathbb{H}^*$.

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Thus $H = F \circ G$ maps the set $\{\bar{1}_u, \bar{0}_{\sigma(u)}\}$ to the set $\{\bar{0}_{f(u)}, \bar{1}_{\sigma^{-1}(f(u))}\}$, which is also an equivalence class of $\sim$.

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Because H preserves the equivalence classes of $\sim$, the function $[x]_\sim \mapsto [H(x)]_\sim$ is a well-defined mapping on $\mathbb{H}^*$. This function is the sought-after order-reversing self-homeomorphism of $\mathbb{H}^*$.

A few open questions

Open questions:

1. Is it consistent with ZFC that every self-homeomorphism of $\mathbb{N}^*$ is conjugate to its inverse? Does this follow from CH?
2. Assuming CH, how many conjugacy classes are there in the self-homeomorphism group of $\mathbb{N}^*$?
3. If there is a non-trivial self-homeomorphism of $\mathbb{N}^*$, must there be one that is conjugate to σ?
4. Is it consistent with ¬CH that $(\mathbb{N}^*, \sigma)$ and $(\mathbb{N}^*, \sigma^{-1})$ are conjugate?
5. Assuming CH, how many conjugacy classes are there in the self-homeomorphism group of $\mathbb{N}^*$?
6. Is there a self-homeomorphism of $\mathbb{H}^*$ that is neither order-preserving nor order-reversing, but a bit of both?

Thank you for listening!

Any questions?