Does $^{\mathcal{P}(\omega)}/_{\mathrm{Fin}}$ know its right hand from its left?

Part 1

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are naturally isomorphic to one another.

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Theorem (W. Rudin, 1956)

The Continuum Hypothesis implies there are $2^{\mathfrak{c}}$ *automorphisms of* $\mathcal{P}(\omega)/\mathrm{Fin}$. *In particular,* CH *implies there are many nontrivial automorphisms of* $\mathcal{P}(\omega)/\mathrm{Fin}$.

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- (Farah, Moore, and Vignati, 2024) OCA implies all automorphisms of $\mathcal{P}(\omega)/\mathrm{Fin}$ are trivial.

When are two automorphisms the same?

Two automorphisms α and β of ${}^{\mathcal{P}(\omega)}\!/_{\mathrm{Fin}}$ are *conjugate* if there is a third automorphism γ such that $\gamma \circ \alpha = \beta \circ \gamma$.

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In other words, can $\mathcal{P}(\omega)/\mathrm{Fin}$ tell its right from its left?

Is the shift map conjugate to its inverse?

Theorem (van Douwen, 1983)

If γ is a conjugacy mapping between σ and σ^{-1} (i.e., γ is an automorphism such that $\gamma \circ \sigma = \sigma^{-1} \circ \gamma$), then γ is nontrivial.

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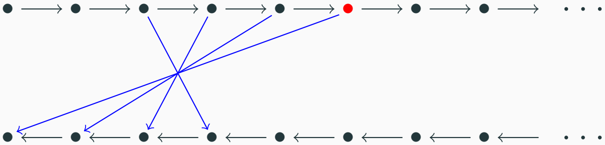

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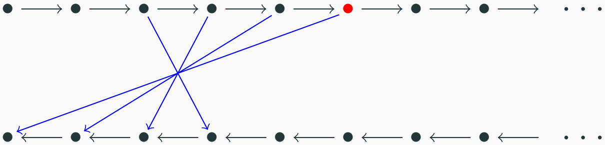

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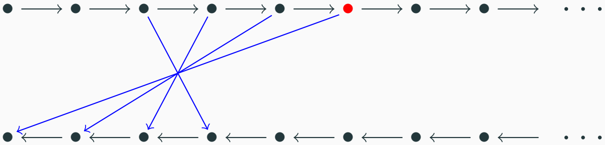

Every good point n is followed by $< f(n)$ more good points. This implies there are infinitely many bad points. Among these, we can find an infinite set B such that $f``(B+1) \cap (f``(B) - 1) = \emptyset$, which means in particular that $\alpha_f \circ \sigma([B]_{\mathrm{Fin}}) \neq \sigma^{-1} \circ \alpha_f([B]_{\mathrm{Fin}})$.

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Via Stone duality, these results can be phrased topologically:

CH implies the topological dynamical systems (ω^*, σ) and (ω^*, σ^{-1}) are conjugate, while $\mathrm{OCA} + \mathrm{MA}$ implies there is not even a factor mapping from either one onto the other.

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- For any finite partial isomorphism $\phi_0 : (F, \leq) \to (G, \leq)$, where F and G are finite subsets of X and Y respectively, and for any given $x \in X \setminus F$, there is an extension of ϕ_0 to $F \cup \{x\}$ (and similarly when the roles of X and Y are interchanged).

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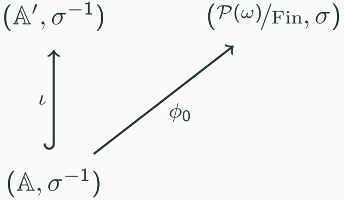

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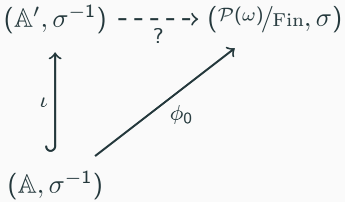

The second item asks for a kind of "lifting property" for $(\mathcal{P}(\omega)/\mathrm{Fin}, \sigma)$: given two embeddings as shown, is there an embedding of $(\mathbb{A}', \sigma^{-1})$ that completes the diagram?

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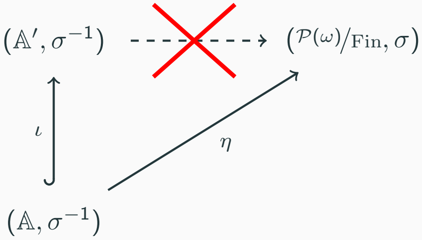
The second bullet point on the previous slide is not generally true. More precisely, there is a countable substructure $(\mathbb{A}, \sigma^{-1})$ of $(\mathcal{P}(\omega)/\mathrm{Fin}, \sigma^{-1})$, and an $x \in \mathcal{P}(\omega)/\mathrm{Fin} \setminus \mathbb{A}$, and an embedding η of $(\mathbb{A}, \sigma^{-1})$ into $(\mathcal{P}(\omega)/\mathrm{Fin}, \sigma)$ such that if $\mathbb{A}' \supseteq \mathbb{A} \cup \{x\}$ then there is no embedding $\bar{\eta}$ of $(\mathbb{A}', \sigma^{-1})$ into $(\mathcal{P}(\omega)/\mathrm{Fin}, \sigma)$ with $\bar{\eta} \circ \iota = \eta$.



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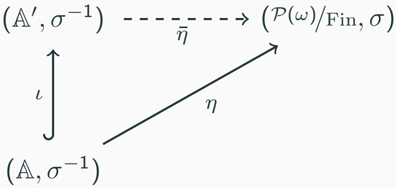
Lemma (the Lifting Lemma)

Let $(\mathbb{A}, \sigma^{-1})$ *and* $(\mathbb{A}', \sigma^{-1})$ *be countable substructures of* $(\mathcal{P}(\omega)/\mathrm{Fin}, \sigma^{-1})$ *with* $\mathbb{A} \subseteq \mathbb{A}'$*, and suppose* η *is an elementary embedding from* $(\mathbb{A}, \sigma^{-1})$ *into* $(\mathcal{P}(\omega)/\mathrm{Fin}, \sigma)$*. Then* η *extends to an embedding* $\bar{\eta}$ *of* $(\mathbb{A}', \sigma^{-1})$ *into* $(\mathcal{P}(\omega)/\mathrm{Fin}, \sigma)$*, with* $\bar{\eta} \circ \iota = \eta$*.*



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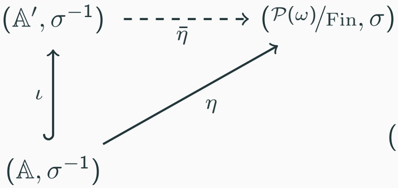

In particular, $\bar{\eta}$ exists if

$(\eta[\mathbb{A}], \sigma^{-1}) \prec (\mathcal{P}(\omega)/\mathrm{Fin}, \sigma^{-1})$.

A better back-and-forth argument

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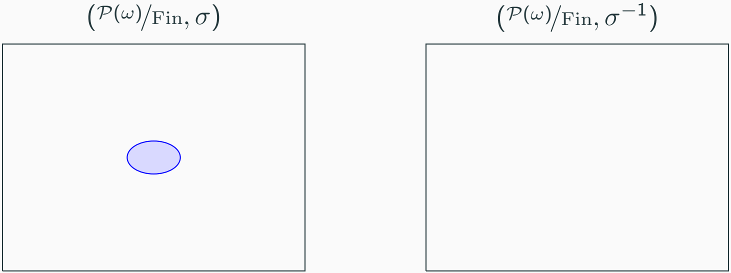

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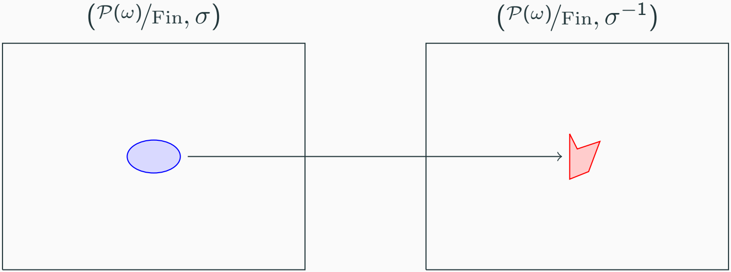

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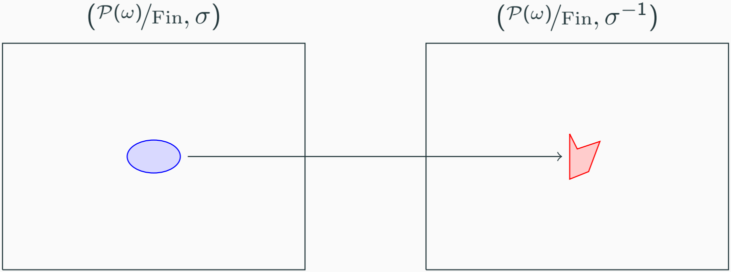

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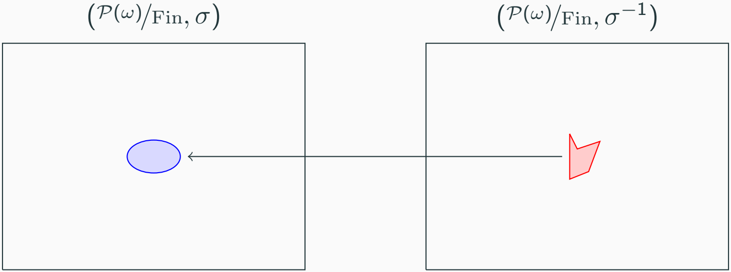

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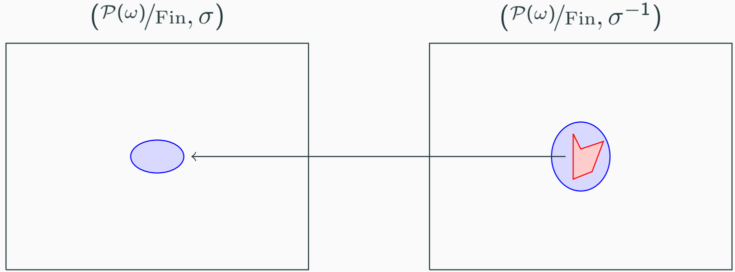

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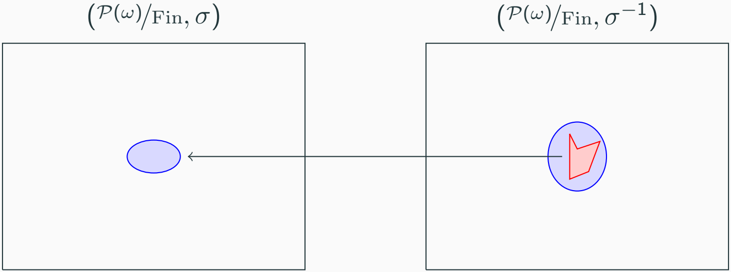

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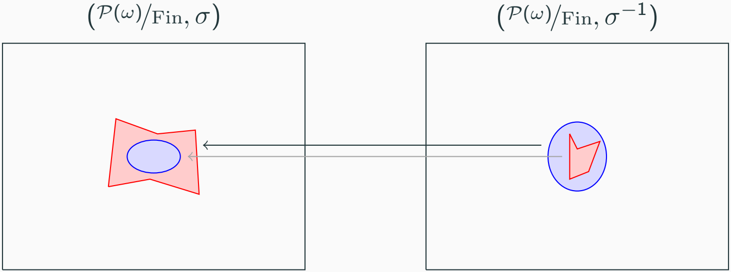

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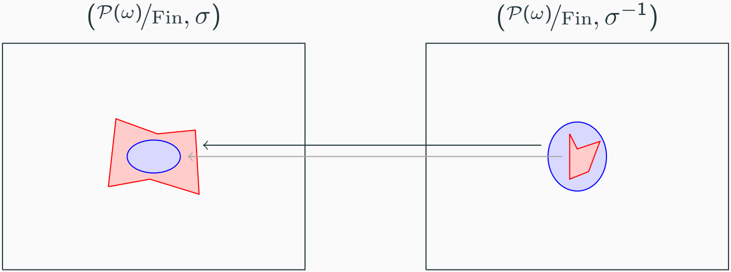

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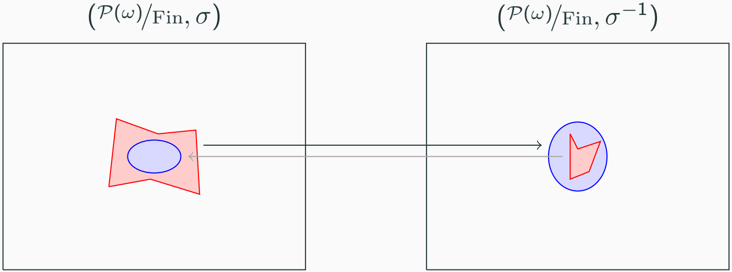

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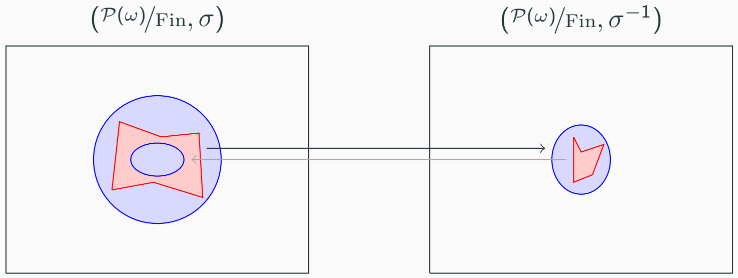

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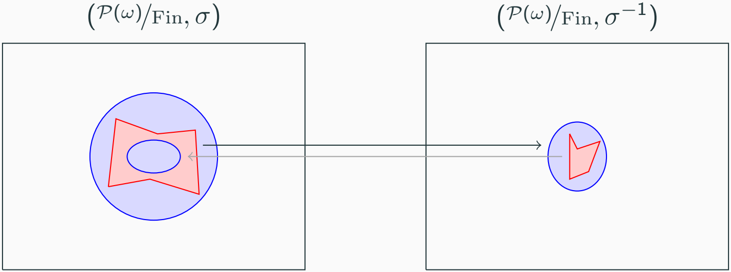

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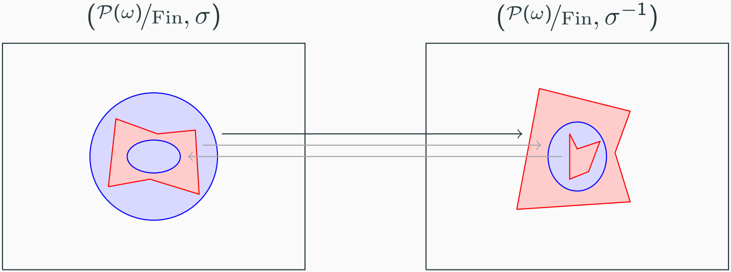

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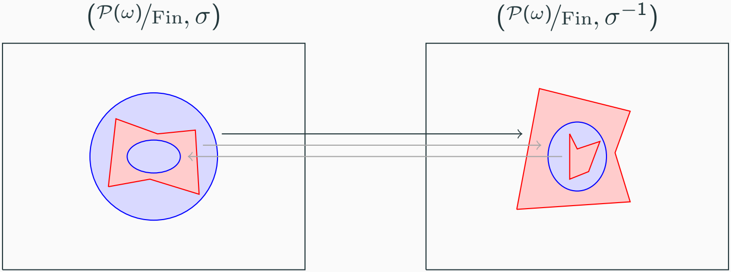

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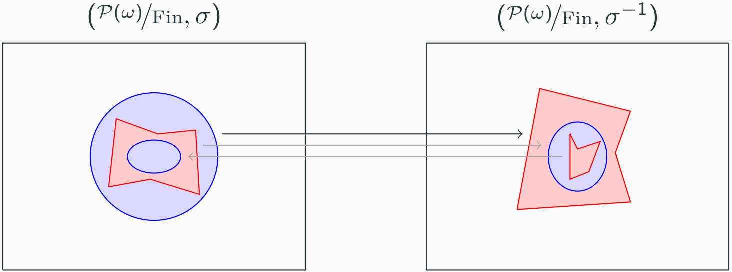

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12. At stage α, be sure that the elementary substructure used on each side contains the α^{th} member of ${}^{\mathcal{P}(\omega)}/_{\mathrm{Fin}}$ (according to the well order fixed at the beginning of the proof).

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In the end, this construction produces an isomorphism between $(\mathcal{P}(\omega)/\mathrm{Fin}, \sigma)$ and $(\mathcal{P}(\omega)/\mathrm{Fin}, \sigma^{-1})$.

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Note that this is a result of ZFC (no CH required).

Thank you for listening!

Any questions?