Does $\mathcal{P}(\omega)/\mathrm{Fin}$ know its right hand from its left?

Part 3

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Back-and-forth again with the Lifting Lemma

This talk focuses on three further consequences of the Lifting Lemma and the main theorem presented in the previous talks, as well as some related open questions.

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Recall that the Lifting Lemma can be used in a back-and-forth argument to prove, assuming CH, that σ and σ^{-1} are conjugate.
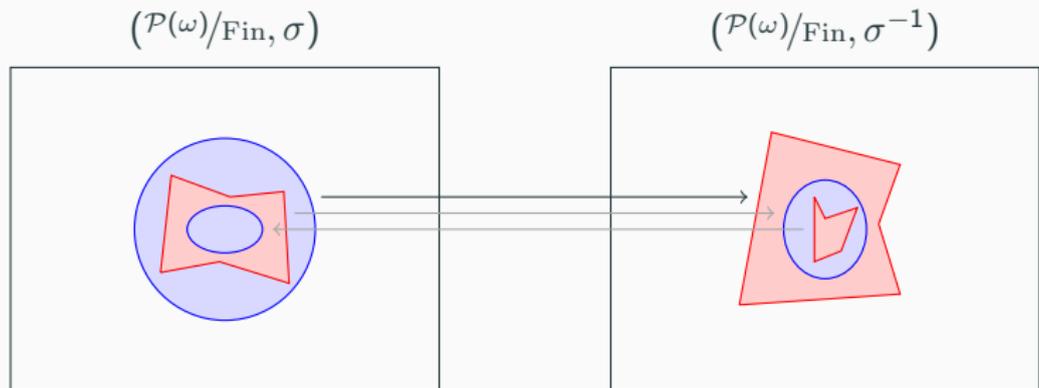
$\left(\mathcal{P}(\omega)/\mathrm{Fin}, \sigma\right)$ $\left(\mathcal{P}(\omega)/\mathrm{Fin}, \sigma^{-1}\right)$

In fact, our argument shows something stronger. Assuming CH:

Theorem

Suppose $\langle \mathbb{A}, \sigma^{-1} \rangle$ is a countable elementary substructure of $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \sigma^{-1} \rangle$, and η is an embedding of $\langle \mathbb{A}, \sigma^{-1} \rangle$ into $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \sigma \rangle$. Then there is an isomorphism ϕ from $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \sigma^{-1} \rangle$ to $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \sigma \rangle$ with $\phi \restriction \mathbb{A} = \eta$.

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Corollary

There is a nontrivial automorphism of $\mathcal{P}(\omega)/\mathrm{Fin}$ that commutes with σ, i.e., an automorphism ϕ such that $\phi \circ \sigma = \sigma \circ \phi$.

Square roots of trivial mappings

Let f denote some permutation of ω with exactly one $\mathbb{Z}$-like orbit.

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Lemma

If σ and σ^{-1} are conjugate, there is a (necessarily nontrivial) automorphism ϕ of $\mathcal{P}(\omega)/\mathrm{Fin}$ such that $\phi \circ \phi = \alpha_f$. Furthermore, some such nontrivial automorphism ϕ is conjugate to σ.

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Proof: Let f denote a $\mathbb{Z}$-like orbit on ω as before:

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Thus we get two consequences of CH regarding the shift map:

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van Douwen's index theorem

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Theorem (B. and Farah, 2024)
Let α and β be trivial automorphisms of $\mathcal{P}(\omega)/\mathrm{Fin}$. TFAE:

1. α *and* β *are conjugate in a forcing extension.*

2. *CH proves* α *and* β *are conjugate.*

3. $\mathrm{Ind}(\alpha)$ *and* $\mathrm{Ind}(\beta)$ *have the same parity, and the structures* $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \alpha \rangle$ *and* $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \beta \rangle$ *are elementarily equivalent.*

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Suppose α and β are trivial automorphisms of $\mathcal{P}(\omega)/\mathrm{Fin}$ and the structures $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \alpha \rangle$ and $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \beta \rangle$ are elementarily equivalent. Does this imply α and β have the same index parity?

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Suppose α and β are trivial automorphisms of $\mathcal{P}(\omega)/\mathrm{Fin}$ and the structures $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \alpha \rangle$ and $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \beta \rangle$ are elementarily equivalent. Does this imply α and β have the same index parity?

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Let f be a permutation of ω with infinitely many $\mathbb{Z}$-like orbits, and let g be an almost permutation with infinitely many $\mathbb{Z}$-like orbits and one $\mathbb{N}$-like orbit. Is $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \alpha_f \rangle \equiv \langle \mathcal{P}(\omega)/\mathrm{Fin}, \alpha_g \rangle$?

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Just as $\mathbb{H}$ can be obtained from $\mathbb{M}$ by gluing some points together, there is an equivalence relation $\sim$ on $\mathbb{M}^*$ such that $\mathbb{H}^* = \mathbb{M}^*/\sim$.

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$\mathbb{H}^*$ is obtained from $\mathbb{M}^*$ by gluing these I_u together, the right endpoint of I_u being glued to the left endpoint of $I_{\sigma(u)}$. Each of these I_u is called a *standard subcontinuum* of $\mathbb{H}^*$.

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Theorem (Vignati, 2021)

OCA + MA *implies all self-homeomorphisms of* $\mathbb{H}^*$ *are trivial, and in particular there is no order-reversing self-homeomorphism of* $\mathbb{H}^*$.

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Using the theorem on the previous slide (and using CH again), there is an order-preserving self-homeomorphism $F : \mathbb{M}^* \to \mathbb{M}^*$ such that $\pi \circ F = f$.

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We now have two self-homeomorphisms of $\mathbb{M}^*$, F and G. G is order-reversing and F is order-reversing, so their composition $H = F \circ G$ is an order-reversing self-homeomorphism of $\mathbb{M}^*$.

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Furthermore, because G sends each I_u to itself (only reversed), H maps each I_u to $I_{f(u)}$, just like F.

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Because H preserves the equivalence classes of $\sim$, the function $[x]_\sim \mapsto [H(x)]_\sim$ is a well-defined mapping on $\mathbb{H}^*$. This function is the sought-after order-reversing self-homeomorphism of $\mathbb{H}^*$.

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Open Question (Moore)

Can we characterize when CH *implies two structures of the form $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \alpha, \beta \rangle$ are isomorphic?*