The Laplace operator in the prolate spheroidal geometry: Neumann functions and image systems

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We apply the method of image charges from electrostatics to the study of the Neumann function for the Laplace operator, equivalent to the Green's function with a Neumann boundary condition imposed. Such an analysis has previously been given for the more general ellipsoidal case; however, the azimuthal symmetry of the prolate spheroid simplifies the analysis and the results may enjoy an undiminished application, as many natural phenomena conform to this geometry. Our results are twofold: we derive the Neumann function for a point source located in both the interior and exterior of the prolate spheroid, and then develop a system of fictitious charges that replicates the boundary conditions and thus becomes a useful problem-solving tool. In the present paper we consider two types of boundary conditions, each of which has a physically interesting interpretation in the prolate spheroidal geometry. It is hoped that this, together with the Neumann functions and image systems, will enable a range of physical applications.

1 Introduction and background

Given a source "charge" located within a region of interest and a set of conditions specifying the potential, or another useful physical quantity, on the boundary of the region, the method of image charges enables us to create a system of fictitious charges in the region's complement that collectively satisfy these boundary conditions and replace undesired elements from the original configuration. Crucially, the uniqueness theorem for Poisson's equation implies that this new system determines the same solution as the old one. This procedure can allow one to avoid solving a tough differential equation or to reduce the computational complexity involved in implementing a series solution to such an equation.

1.1 Green's Function

Consider a linear differential operator D over $\mathbb{R}^n$ and the following equation:

$$Du(x) = f(x)$$

The Green's function for D is the function G satisfying the following equation for some $x_0 \in \mathbb{R}^n$:

$$DG(x, x_0) = \delta(x - x_0)$$

It represents the response of the operator D to a point source. Intuitively, we can treat the forcing term above, $f(x)$, as an assemblage of uncountably many point sources $\delta(x - x_0)$. Hence it makes sense to integrate against G to determine u. First we use the definition of the delta function to note that [5]

$$f(x) = \int \delta(x - x_0) f(x_0) dx_0 = \int DG(x, x_0) f(x_0) dx_0$$

This yields

$$Du(x) = \int DG(x, x_0) f(x_0) dx_0$$

Then since D is linear and acts only on x, we have

$$Du(x) = D\left(\int G(x, x_0) f(x_0) dx_0 \right)$$

Which gives a solution to the differential equation above:

$$u(x) = \int G(x, x_0) f(x_0) dx_0$$

Hence determining the Green's function for a differential operator can facilitate a solution when other methods are impossible or otherwise too complicated. This shows that determining the Green's function has more than strictly physical applications. Thus the present paper intends to contribute theoretical tools for solving Poisson's equation in the prolate spheroidal geometry, as well as offering practical applications.

Xue and Deng [8] have derived the Green's function and accompanying image system for the case in which the region is a prolate spheroid and the boundary conditions are of the Dirichlet form. The goal of this paper is to do this when the condition on the boundary of the prolate spheroid is given by the partial derivative of the potential normal to the surface; this is also known as a Neumann boundary condition. Henceforth we will refer to the Green's function for this boundary value problem as the Neumann function. To our knowledge, this result remains unpublished. Importantly, deriving the Neumann functions will have theoretical significance while developing the image system will potentially yield useful practical applications.

Dassios and Sten [3] have derived the Neumann function for the general ellipsoidal case. However, this case lacks the symmetry of the prolate spheroid and consequently admits a solution that involves unwieldy ellipsoidal harmonics. Our aim is to simplify this analysis by exploiting the azimuthal symmetry of the prolate spheroid. Furthermore, many interesting physical phenomena do not require an ellipsoidal geometry to accurately model. For instance, using prolate spheroidal coordinates allows one to derive the wavefunction of electrons moving in the field of two confocal nuclei. In addition, the Neumann function in these coordinates might play a helpful role in calculating the reaction field within the solvation models of biomolecules such as proteins [7]. We now formally introduce the problem.

1.2 Poisson's equation in the prolate spheroidal geometry

A prolate spheroid is the region in $\mathbb{R}^3$ generated by the revolution of an ellipse about its major axis:

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{1}$$

Here, b is the length of the major axis, a is the length of the minor axis, and 2c is the interfocal distance with $c = \sqrt{b^2 - a^2}$. The prolate spheroidal coordinates are given by the 3-tuple (ξ, η, ϕ), where $\xi \in [1, \infty)$ is the radial variable, $\eta \in [-1, 1]$ is the angular variable, and $\phi \in [0, 2\pi]$ is the azimuthal variable.

The transformation to Cartesian coordinates is given by:

$$x = \sqrt{(\xi^2 - 1)(1 - \eta^2)} \cos(\phi) \tag{2}$$

$$y = \sqrt{(\xi^2 - 1)(1 - \eta^2)} \sin(\phi) \tag{3}$$

$$z = c\xi\eta \tag{4}$$

These transformations will be of great utility in developing the Neumann function image systems. Of equal importance is the fact that a prolate spheroid is determined by the constant $\xi_b = b/c$. This gives the following equivalent characterization of the prolate spheroid to be used later:

$$\frac{x^2+y^2}{c^2(\xi_b^2-1)} + \frac{z^2}{c^2\xi_b^2} = 1 \tag{5}$$

Now, let $\mathbf{r}_s$ denote the position vector of a source charge located within the interior of the prolate spheroid and $\mathbf{r}$ denote the point of measurement. Then the interior Neumann boundary value problem we aim to solve is the following:

$$\triangle N^i(\mathbf{r}, \mathbf{r}_s) = \delta(\mathbf{r} - \mathbf{r}_s) \qquad \xi_s < \xi < \xi_b \tag{6}$$

$$\frac{\partial}{\partial n} N^i(\mathbf{r}, \mathbf{r}_s) = \frac{1}{4\pi}\omega_{\xi_b}(\eta) \qquad \xi = \xi_b \tag{7}$$

The choice of boundary condition is physically intuitive as the prolate spheroid is not completely symmetric, and as such the flux through the surface should depend on the angular variable η. We first derive the Neumann function with this boundary condition, and then consider another non-constant boundary condition.

2.1 Interior Neumann problem: Boundary condition 1

It is known that the solution to the given Poisson's equation in free-space, i.e., without boundary constraints, is

$$N(\mathbf{r}, \mathbf{r}_s) = -\frac{1}{4\pi \mid \mathbf{r} - \mathbf{r}_s \mid} \tag{8}$$

Our goal is to derive a harmonic function, which we call the reflected solution $R^i(\mathbf{r}, \mathbf{r}_s)$, that satisfies the prescribed boundary conditions. Hence the complete solution will look like

$$N^i(\mathbf{r}, \mathbf{r}_s) = N(\mathbf{r}, \mathbf{r}_s) + R^i(\mathbf{r}, \mathbf{r}_s) \tag{9}$$

We first use separation of variables to attempt a solution of the above boundary value problem as a superposition of eigenfunctions. Setting $N^i(\mathbf{r}, \mathbf{r}_s) = A(\xi)B(\eta)C(\phi)$ in prolate spheroidal coordinates yields the following ordinary differential equations:

$$\frac{d}{d\xi}[(1-\xi^2)\frac{dA}{d\xi}] + n(n+1)A - \frac{m^2}{1-\xi^2}A = 0$$

$$\frac{d}{d\eta}[(1-\eta^2)\frac{dB}{d\eta}] + n(n+1)B - \frac{m^2}{1-\eta^2}B = 0$$

$$\frac{d^2C}{d\phi^2} = -m^2C$$

The first two are the well-known general Legendre differential equations which have as eigenfunctions the special functions $P_n^m(\xi)$, $P_n^m(\eta)$, called Legendre polynomials of the first kind which are singular at infinity, and $Q_n^m(\xi)$, $Q_n^m(\eta)$, called the Legendre polynomials of the second kind which are singular at ξ, $\eta = -1$, 1. The third equation has the typical eigenfunctions $\sin(m\phi)$ and $\cos(m\phi)$. However, exploiting the azimuthal symmetry of the prolate spheroid, we can choose to use only the even eigenfunction, $\cos(m\phi)$. Using these three eigenfunctions we can rewrite the free-space solution, a case of the well-known multipole expansion [8]:

$$-\frac{1}{4\pi \mid \mathbf{r}-\mathbf{r}_s \mid} = -\frac{1}{4\pi c}\sum_{n=0}^{\infty}\sum_{m=0}^{n} H_{mn} P_n^m(\xi_<) Q_n^m(\xi_>) P_n^m(\eta_s) P_n^m(\eta) cos(m(\phi-\phi_s)) \quad (10)$$

Where $\xi_> = \max\{\xi, \xi_s\}$, $\xi_< = \min\{\xi, \xi_s\}$, and where

$H_{mn} = (2n+1)(2-\delta_{m0})(-1)^m \left[\frac{(n-m)!}{(n+m)!}\right]^2$

Here δ is the Kronecker delta function. By demanding that the source charge be located on the xz-plane, (10) can be reduced to

$$-\frac{1}{4\pi \mid \mathbf{r}-\mathbf{r}_s \mid} = -\frac{1}{4\pi c}\sum_{n=0}^{\infty}\sum_{m=0}^{n} H_{mn} P_n^m(\xi_<) Q_n^m(\xi_>) P_n^m(\eta_s) P_n^m(\eta) cos(m\phi) \quad (11)$$

Similarly, we introduce the reflected part of the solution as a superposition of the eigenfunctions:

$$R^i(\mathbf{r}, \mathbf{r}_s) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \alpha_{mn} P_n^m(\xi) P_n^m(\eta)\cos(m\phi) \quad (12)$$

Here α_{mn} is a yet-to-be-determined expansion coefficient, which we will identify by applying the boundary condition. Hence the interior Neumann function becomes

$$N^i(\mathbf{r}, \mathbf{r}_s) = -\frac{1}{4\pi c}\sum_{n=0}^{\infty}\sum_{m=0}^{n} H_{mn} P_n^m(\xi_<) Q_n^m(\xi_>) P_n^m(\eta_s) P_n^m(\eta) cos(m\phi) + \quad (13)$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} \alpha_{mn} P_n^m(\xi) P_n^m(\eta)\cos(m\phi) \quad (14)$$

Our next step is critical: we judiciously factor out terms from the expansion coefficient α_{mn} to facilitate our subsequent analysis: we can rewrite it such that α_{mn} = -

$\frac{1}{4\pi c} H_{mn} P_n^m(\eta_s) P_n^m(\xi_s) A_{mn}$, where A_{mn} is now the undetermined expansion coefficient. This makes sense because we would expect the expansion coefficient to depend on the source charge. Substituting and collecting like terms, we obtain

$$N^i(\mathbf{r}, \mathbf{r}_s) = -\frac{1}{4\pi c} \sum_{n=0}^{\infty} \sum_{m=0}^{n} H_{mn} P_n^m(\xi_s) P_n^m(\eta_s) \big[Q_n^m(\xi) - A_{mn} P_n^m(\xi) \big] C_n^m(\eta, \phi) \tag{15}$$

The next step is to exploit the orthogonality relations of the prolate spheroidal harmonics in order to determine A_{mn}. We first give a quick detour to explain how this might happen. First note that the differential surface element of the prolate spheroid is given by

$$dS_\xi = h_\eta h_\phi d_\eta d_\phi = \frac{1}{\omega_\xi(\eta)} d_\eta d_\phi \tag{16}$$

where h_η, h_ϕ, as well as h_ξ are metric coefficients for the prolate spheroidal coordinates and are given by

$$h_\xi = c\sqrt{(\xi^2 - \eta^2)/(\xi^2 - 1)} \tag{17}$$

$$h_\eta = c\sqrt{(\xi^2 - \eta^2)/(1 - \eta^2)} \tag{18}$$

$$h_\phi = c\sqrt{(\xi^2 - 1)(1 - \eta^2)} \tag{19}$$

$$\tag{20}$$

And $\omega_\xi(\eta)$ is a geometric weighting function given by

$$\omega_\xi(\eta) = \frac{1}{c^2\sqrt{(\xi^2 - \eta^2)(\xi^2 - 1)}} \tag{21}$$

Now define the surface prolate spheroidal harmonics as the following combinations of eigenfunctions:

$C_n^m(\eta, \phi) = P_n^m(\eta) cos(m\phi)$, $S_n^m(\eta, \phi) = P_n^m(\eta) sin(m\phi)$

These harmonics are not orthogonal over the surface of the prolate spheroid in their present condition, but will be when multiplied by the geometric weighting function $\omega_\xi(\eta)$:

$$\iint_{S_\xi} S_n^m(\eta,\phi) S_N^M(\eta,\phi)\omega_\xi(\eta) dS_\xi = \int_{-1}^{1}\int_0^{2\pi} P_n^m(\eta)P_N^M(\eta) sin(m\phi) sin(M\phi) d\phi d\eta = \gamma_{mn}\delta_{nN}\delta_{mM}$$

$$\iint_{S_\xi} S_n^m(\eta,\phi) C_N^M(\eta,\phi)\omega_\xi(\eta) dS_\xi = \int_{-1}^{1}\int_0^{2\pi} P_n^m(\eta)P_N^M(\eta) sin(m\phi) cos(M\phi) d\phi d\eta = 0$$

$$\iint_{S_\xi} C_n^m(\eta,\phi) C_N^M(\eta,\phi)\omega_\xi(\eta) dS_\xi = \int_{-1}^{1}\int_0^{2\pi} P_n^m(\eta)P_N^M(\eta) cos(m\phi) cos(M\phi) d\phi d\eta = \gamma_{mn}\delta_{nN}\delta_{mM}$$

Here γ_{mn} is a normalization constant given by

$$\gamma_{mn} = \frac{2(n+m)!}{(2n+1)(n-m)!}(1+\delta_{0m})\pi \tag{22}$$

Later we will make use of the fact that $\gamma_{00} = 4\pi$. Finally, we note that

$$\frac{1}{4\pi}\iint_S \omega_{\xi_b}(\eta)\, ds = 1 \tag{23}$$

Using these facts we continue our analysis.

The normal derivative of $N^i(\mathbf{r},\mathbf{r}_s)$ can be expressed in the following way:

$$\frac{\partial}{\partial n}N^i(\mathbf{r},\mathbf{r}_s) = \frac{1}{h_{\xi_b}}\frac{\partial}{\partial \xi}N^i(\mathbf{r},\mathbf{r}_s) = \frac{a^2}{c}\omega_{\xi_b}(\eta)\frac{\partial}{\partial \xi}N^i(\mathbf{r},\mathbf{r}_s) \tag{24}$$

Hence the boundary condition implies that the following must hold:

$$-\frac{a^2}{c^2}\omega_{\xi_b}(\eta)\sum_{n=0}^{\infty}\sum_{m=0}^{n} H_{mn}P_n^m(\xi_s)P_n^m(\eta_s)\left[Q_n'^m(\xi_b) - A_{mn}P_n'^m(\xi_b)\right]C_n^m(\eta,\phi) = \omega_{\xi_b}(\eta) \tag{25}$$

Setting $n = 0$ and integrating over the prolate spheroid S_{ξ_b} we obtain:

$$-\frac{a^2}{c^2}Q_0^{'}(\xi_b) = 1 \Rightarrow Q_0^{'}(\xi_b) = -\frac{c^2}{a^2} \tag{26}$$

Which confirms that A_{00} is arbitrary, as we know that

$$Q_0^{'}(\xi_b) = \frac{d}{d\xi}\frac{1}{2}ln\Big(\frac{1+\xi}{1-\xi}\Big)\big|_{\xi=\xi_b} = \frac{1}{1-\xi_b^2} = \frac{1}{1-\frac{b^2}{c^2}} = \frac{c^2}{c^2-b^2} = -\frac{c^2}{a^2} \tag{27}$$

Now for each $i \geq 1, j = 0, ..., i$ we multiply by $C_i^j(\eta, \phi)$ and integrate over S_{ξ_b} to find A_{mn}:

$$-\frac{a^2}{c^2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\Big[\int_{-1}^{1}\int_{0}^{2\pi} P_n^m(\eta)P_i^j(\eta)cos(m\phi)cos(j\phi)d\phi d\eta\Big]H_{mn}P_n^m(\xi_s)P_n^m(\eta_s)\big[Q_n^{'m}(\xi_b) - A_{mn}P_n^{'m}(\xi_b)\big]$$
$$= \int_{-1}^{1}\int_{0}^{2\pi} P_i^j(\eta)cos(j\phi)d\phi d\eta$$

The lefthand integral is nonzero only when $n = i$ and $m = j$. Hence the equation reduces to

$$-\frac{a^2}{c^2}\gamma_{ji}H_{ji}P_i^j(\xi_s)P_i^j(\eta_s)\big[Q_i^j(\xi_b) - A_{ji}P_i^j(\xi_b)\big] = \int_{-1}^{1}\int_{0}^{2\pi} P_i^j(\eta)cos(j\phi)d\phi d\eta = 0 \tag{28}$$

Hence we have $A_{ji} = \frac{Q_i^{'j}(\xi_b)}{P_i^{'j}(\xi_b)}$

Which means

$$A_{mn} = \frac{Q_m^{'n}(\xi_b)}{P_m^{'n}(\xi_b)}, \qquad n \geq 1 \tag{29}$$

Substituting and letting $A_{00} = 0$ we obtain

$$N^i(\mathbf{r},\mathbf{r}_s) = -\frac{1}{4\pi c}\sum_{n=1}^{\infty}\sum_{m=0}^{n} H_{mn}P_n^m(\xi_s)P_n^m(\eta_s)\big[Q_n^m(\xi) - \frac{Q'^n_m(\xi_b)}{P'^n_m(\xi_b)}P_n^m(\xi)\big]C_n^m(\eta,\phi) \tag{30}$$

Which is equivalent to

$$N^i(\mathbf{r},\mathbf{r}_s) = -\frac{1}{4\pi \mid \mathbf{r}-\mathbf{r}_s \mid} + \frac{1}{4\pi c}\sum_{n=1}^{\infty}\sum_{m=0}^{n} H_{mn}P_n^m(\xi_s)P_n^m(\eta_s)\frac{Q'^n_m(\xi_b)}{P'^n_m(\xi_b)}P_n^m(\xi)C_n^m(\eta,\phi) \tag{31}$$

2.2 Interior Neumann problem: Boundary condition 2

We now consider a more general non-constant boundary condition:

$$\triangle N^i(\mathbf{r},\mathbf{r}_s) = \delta(\mathbf{r}-\mathbf{r}_s) \qquad \xi_s < \xi < \xi_b \tag{32}$$

$$\frac{\partial}{\partial n}N^i(\mathbf{r},\mathbf{r}_s) = f(\eta) \qquad \xi = \xi_b \tag{33}$$

We will show that this boundary value problem has almost the same solution as the one previous, when $f(\eta)$ takes on a particular form. Before choosing any particular Neumann boundary condition for this problem, it is crucial to note that the divergence theorem implies that [4]

$$\iint_S \frac{\partial N^i}{\partial n} = 1 \tag{34}$$

It is easy to see that this criterion is satisfied by our first boundary condition. We will show the second condition to be satisfied if we make the following choice of f:

$$f(\eta) = \frac{\alpha\eta + \beta}{\sqrt{\xi_b^2 - \eta^2}} \tag{35}$$

First we note that, by some simple algebra,

$$\frac{1}{\omega_{\xi_b}(\eta)} = ac\sqrt{\xi_b^2 - \eta^2}. \tag{36}$$

Now, integrating over the surface S_{ξ_b} we have

$$\iint_{S_{\xi_b}} \frac{\alpha\eta + \beta}{\sqrt{\xi_b^2 - \eta^2}} dS_{\xi_b} = 2\pi ac \int_{-1}^{1} (\alpha\eta + \beta) d\eta = 4\pi ac\beta \tag{37}$$

which evaluates to 1 if and only if

$$\beta = \frac{1}{4\pi ac}. \tag{38}$$

Since there is no current demand on the value of β, we may choose it to be that above. Our goal is now to derive the Neumann function given this requirement. In the following we shall simply use the symbol β for convenience.

As before, we impose the boundary condition, multiply by $C_i^j(\eta, \phi)$, integrate over the prolate spheroid S_{ξ_b}, and use the orthogonality relations:

$$-\frac{a^2}{4\pi c^2}\gamma_{ji} H_{ji} P_i^j(\xi_s) P_i^j(\eta_s) \left[Q_i'^j(\xi_b) - A_{ji} P_i'^j(\xi_b)\right] = \iint_{S_{\xi_b}} C_i^j(\eta, \phi) f(\eta) dS_{\xi_b} \tag{39}$$

Then analyzing the righthand term of (39) we have

$$\iint_S C_i^j(\eta, \phi) f(\eta) dS = \left(\int_{-1}^{1} P_i^j(\eta) \frac{\alpha\eta + \beta}{\sqrt{\xi_b^2 - \eta^2}} \frac{1}{\omega_{\xi_b}(\eta)} d\eta\right) \left(\int_0^{2\pi} cos(m\phi) d\phi\right) =$$

$$\begin{cases} 2\pi ac \int_{-1}^{1} P_i(\eta)(\alpha\eta+\beta)d\eta, & j=0, i<2 \\ 0, & \text{otherwise} \end{cases}$$

This follows from the theory of Legendre functions, which implies that this integral vanishes if the degree of $P_i(\eta)$ is greater than that of the polynomial with which it is multiplied. Since the degree of $\alpha\eta+\beta$ is 1, the integral is non-vanishing only for $i=0$ and $i=1$.
For $i=0$, we have

$$2\pi ac \int_{-1}^{1} P_0(\eta)(\alpha\eta+\beta)d\eta = 2\pi ac \int_{-1}^{1} (1)(\alpha\eta+\beta)d\eta = 4\pi ac\beta \tag{40}$$

Since $i=0$ and $j=0$, we obtain the A_{00} case from the first non-constant boundary condition:

$$-\frac{a^2}{4\pi c^2}\gamma_{00}(\eta)H_{00}P_0(\xi_s)P_0(\eta_s)Q_0^{'}(\xi_b) = 4\pi ac\beta \tag{41}$$

$$\Rightarrow \frac{-a^2}{c^2}(1)(1)(1)\frac{-c^2}{a^2} = 4\pi ac\beta \tag{42}$$

$$\Rightarrow \beta = \frac{1}{4\pi ac} \tag{43}$$

Fortunately, this is the exact value of β required above! Additionally, as long as β is chosen as this value, we have the identity (26) from the first problem which allows A_{00} to be arbitrary.

Now, for $i=1$, we have

$$-\frac{a^2}{4\pi c^2}H_{01}P_1(\xi_s)P_1(\eta_s)\left[Q_1^{'}(\xi_b)-A_{01}P_1^{'}(\xi_b)\right]\int_{-1}^{1}(P_1(\eta))^2 d\eta \int_0^{2\pi} d\phi = 2\pi ac\int_{-1}^{1} P_1(\eta)(\alpha\eta+\beta)d\eta \tag{44}$$

which, since $H_{01} = 3$ and $P_1(x) = x$, reduces to

$$-\frac{a^2}{c^2}\xi_s\eta_s\left[Q_1^{'}(\xi_b) - A_{01}\right] = 2\pi ac\int_{-1}^{1}(\alpha\eta^2 + \beta\eta)d\eta = \frac{4\pi ac\alpha}{3} \tag{45}$$

Which gives us

$$A_{01} = \frac{4\pi c^3\alpha}{3a\xi_s\eta_s} + Q_1^{'}(\xi_b), \qquad \eta_s \neq 0 \tag{46}$$

Note that when $\alpha = 0$, this is simply the value for A_{01} in the previous boundary value problem.

Finally, for $i \geq 2$, we have

$$-\frac{a^2}{4\pi c^2}\gamma_{0i}(\eta)H_{0i}P_i(\xi_s)P_i(\eta_s)\left[Q_i^{'}(\xi_b) - A_{0i}P_i^{'}(\xi_b)\right] = \iint_S C_i(\eta,\phi)f(\eta)dS = 0 \tag{47}$$

Hence

$$A_{0i} = \frac{Q_i^{'}(\xi_b)}{P_i^{'}(\xi_b)} \tag{48}$$

And since the righthand integral of (39) vanishes for $j > 0$, we have

$$A_{mn} = \frac{Q_n^{'m}(\xi_b)}{P_n^{'m}(\xi_b)}, \qquad n \geq 2, m = 0, 1, ... \quad \text{or} \quad n = 1, m = 1 \tag{49}$$

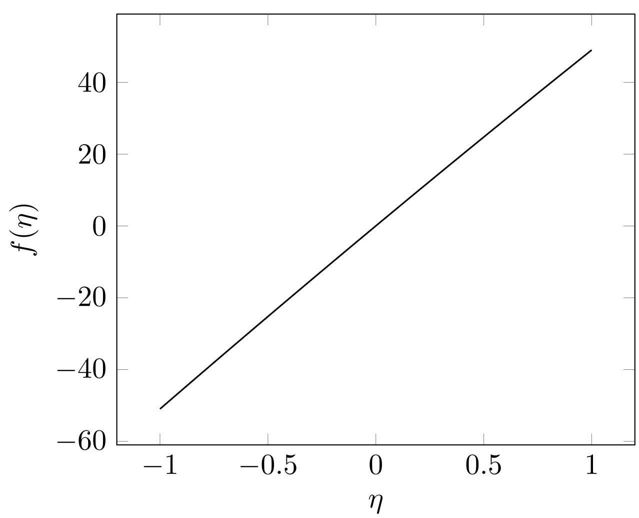
Hence the Neumann function for this boundary condition is almost identical to that for the first: only the A_{01} terms differ. Moreover, when $\alpha = 0$ this boundary value problem is identical to the first one; hence the second boundary condition is truly a more general version. Now, since our analysis imposes no demand on the value of α other than that it be nonzero, it is seen that our derived Neumann function solves uncountably many boundary value problems! Hence by varying α, a wider range of real physical problems might be simulated by the above boundary value problem.

2.3 Physical interpretation of boundary data

When we fix any two of a, b, or c, the graph of the Neumann boundary function prompts a physically interesting interpretation. For instance, if we let $b = 2$ and $c = 1$, then $a = \sqrt{3}$. Since α is allowed to be arbitrary, we pick a moderately large value: $\alpha = 100$. The boundary function then becomes

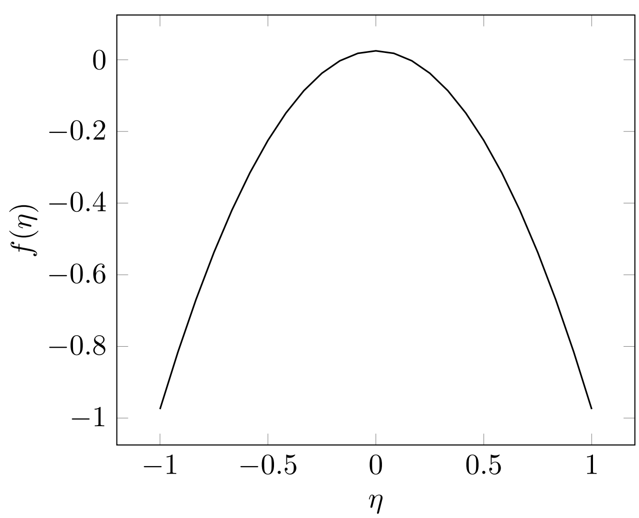
$$f(\eta) = \frac{100x + \frac{1}{4\pi\sqrt{3}}}{\sqrt{4 - x^2}} \tag{50}$$

and has the following graph:



We see that the function is approximately linear on its domain. Indeed, increasing α makes this approximate linearity more pronounced and vice versa. Moreover, the zero of the graph is very close to the origin. Since $\eta \in [-1, 1]$, this suggests the physical interpretation of a temperature flux over the surface of the prolate spheroid that is negative on the half below the xy-plane ($\eta \leq 0$) and positive on the half above ($\eta > 0$). These values for η are of course approximate, but also very accurate since the boundary function is nearly odd. This interpretation may correspond to the problem of finding the potential on the surface due to an interior point charge where some external source, such as another charged molecule, is brought near to the surface of the prolate spheroid parallel to its major axis.

Now, when $\alpha = 0$, the graph of f becomes



This suggests the interpretation of the prolate spheroid increasingly radiating heat from its surface the farther away from the center ($\eta = 0$) and toward the poles ($\eta = -1, 1$) one travels.

We now turn to the development of an image system for the interior Neumann function satisfying this latter boundary condition. The analysis is far simpler and it is hoped that the image system corresponding to the more general boundary condition may be easily derived from it.

3 Image system for the interior Neumann function

The construction of the image system for the interior Neumann function will proceed in a manner similar to that in Dassios [3] and Xue [8]: we first conjecture a candidate system and then perform an analysis to determine whether it in fact satisfies the boundary data. It is clear that a successful image system will not be unique, especially in the case of the prolate spheroid. Indeed, due to the azimuthal symmetry, the point charge component of the system may be placed in either the xz- or yz-plane. Hence the constitution of a particular image system will depend on the types of stipulations and analyses in which the researcher wishes to engage. In the following, we conjecture three components of the image system: a point image charge located on the xz-plane, a continuous line image extending from the point image onward to infinity, and a surface image in the shape of a prolate spheroid confocal with the one under analysis. Each of these components is to be located in the exterior of the given prolate spheroid ξ_b.

We begin with the point image charge. It will have strength Q and be located in the exterior at some point $\mathbf{r}_k = (x_k, 0, z_k) = (\xi_k, \eta_k, 0)$. By Coulomb's law, the contribution from the point image is

$$-\frac{Q}{4\pi \mid \mathbf{r} - \mathbf{r}_s \mid} \tag{51}$$

Now, the line image will extend from the point $\mathbf{r}_k$ to infinity along the curve $C : (\eta, \phi) =$

$(\eta_k, 0)$ with a density

$$\rho(\mathbf{t}') = \rho(\xi, \eta_k, 0) = \frac{\alpha q(\xi)}{h_\xi(\xi, \eta_k)} \tag{52}$$

where α is some real constant (different from α as used above) and $q(\xi)$ is a continuous function on $[\xi_k, \infty)$. Finally, the surface image will be the confocal prolate spheroid ξ_k with a density

$$d(\mathbf{s}) = d(\xi_k, \eta, \phi) = \omega_{\xi_k}(\eta) \sum_{n=2}^{\infty} \sum_{m=0}^{n} d_{mn} C_n^m(\eta, \phi) \tag{53}$$

Here we have demanded that the monopole ($n = 0$) and dipole ($n = 1$) terms vanish. Indeed, it is known from electrostatics that a vanishing monopole term impies a total surface strength of zero and vanishing dipole terms imply a symmetric charge distribution over the surface with the centroid at the origin. We now denote the total image system potential by

$$I^i(\mathbf{r}, \mathbf{r}_s) = -\frac{Q}{4\pi \mid \mathbf{r} - \mathbf{r}_k \mid} - \frac{1}{4\pi} \int_C \frac{\rho(\mathbf{t}')}{\mid \mathbf{r} - \mathbf{t}' \mid} d(\mathbf{t}') - \frac{1}{4\pi} \iint_{S_{\xi_k}} \frac{d(\mathbf{s}')}{\mid \mathbf{r} - \mathbf{s}' \mid} dS_{\xi_k}(\eta', \phi') \tag{54}$$

which becomes

$$I^i(\mathbf{r}, \mathbf{r}_s) = -\frac{Q}{4\pi \mid \mathbf{r} - \mathbf{r}_k \mid} - \frac{\alpha}{4\pi} \int_{\xi_k}^{\infty} \frac{q(\xi')}{\mid \mathbf{r} - \mathbf{t}' \mid} d(\xi') - \frac{1}{4\pi} \iint_{S_{\xi_k}} \frac{d(\mathbf{s}')}{\mid \mathbf{r} - \mathbf{s}' \mid} dS_{\xi_k}(\eta', \phi') \tag{55}$$

Using the multipole expansion from earlier we have

$$\frac{\alpha}{4\pi} \int_{\xi_k}^{\infty} \frac{q(\xi')}{\mid \mathbf{r} - \mathbf{t}' \mid} d(\xi') = \frac{1}{4\pi c} \sum_{n=0}^{\infty} \sum_{m=0}^{n} H_{mn} P_n^m(\xi) P_n^m(\eta_k) C_n^m(\eta, \phi) \left(\alpha \int_{\xi_k}^{\infty} q(\xi') Q_n^m(\xi') d(\xi') \right) \tag{56}$$

$$\frac{1}{4\pi}\iint_{S_{\xi_k}} \frac{d(\mathbf{s}')}{\mid \mathbf{r}-\mathbf{s}' \mid} dS_{\xi_k}(\eta',\phi') = \frac{1}{4\pi c}\sum_{n=2}^{\infty}\sum_{m=0}^{n} H_{mn}\gamma_{mn}d_{mn}Q_n^m(\eta_k)P_n^m(\xi)C_n^m(\eta,\phi) \tag{57}$$

$$\frac{Q}{4\pi \mid \mathbf{r}-\mathbf{r}_k \mid} = \frac{Q}{4\pi c}\sum_{n=0}^{\infty}\sum_{m=0}^{n} H_{mn}P_n^m(\xi)Q_n^m(\xi_k)P_n^m(\eta_k)P_n^m(\eta)cos(m\phi) \tag{58}$$

Now, it is crucial that we require the strength or potential of the reflected part of the Neumann function to equal the total potential of the image system. Setting $I^i(\mathbf{r},\mathbf{r}_s) = R^i(\mathbf{r},\mathbf{r}_s)$ we have

$$\begin{aligned}\frac{1}{4\pi c}\sum_{n=0}^{\infty}\sum_{m=0}^{n} H_{mn}P_n^m(\xi)P_n^m(\eta_k)C_n^m(\eta,\phi)\Big(QQ_n^m(\xi_k)+\alpha\int_{\xi_k}^{\infty} q(\xi')Q_n^m(\xi')d(\xi')\Big) \\ +\frac{1}{4\pi c}\sum_{n=2}^{\infty}\sum_{m=0}^{n} H_{mn}\gamma_{mn}d_{mn}Q_n^m(\eta_k)P_n^m(\xi)C_n^m(\eta,\phi) \\ = -\frac{1}{4\pi c}\sum_{n=1}^{\infty}\sum_{m=0}^{n} H_{mn}P_n^m(\xi_s)P_n^m(\eta_s)\frac{Q_m'^n(\xi_b)}{P_m'^n(\xi_b)}P_n^m(\xi)C_n^m(\eta,\phi)\end{aligned}$$

The next step is to examine the $n = 0, 1, 2$ terms in sequence in order to determine the strength Q of the point image, the location $\mathbf{r}_k$ of the point image, and the expansion constant d_{mn} from the surface image density. Taking $n = 0$, we have

$$QQ_0(\xi_k) + \alpha\int_{\xi_k}^{\infty} q(\xi')Q_0(\xi')d\xi' = 0 \tag{59}$$

Which yields

$$Q = -\frac{\alpha}{Q_0(\xi_k)}\int_{\xi_k}^{\infty} q(\xi')Q_0(\xi')d\xi' \tag{60}$$

We can in principle choose any continuous function $q(\xi)$ that guarantees the existence of the improper integral. However, it is natural to desire our case to approach the spherical limit. Hence we make a choice similar to Dassios [3]:

$$q(\xi) = 1 - \frac{1}{\xi Q_0(\xi)} \tag{61}$$

Next we examine the $n = 1$ case. When $n = 1$, $m = 0, 1$ and so we have:

$$\alpha \int_{\xi_k}^{\infty} q(\xi') \left[Q_1^m(\xi') - \frac{Q_1^m(\xi_k)}{Q_0(\xi_k)} Q_0(\xi') \right] d\xi' P_1^m(\eta_k) = -\frac{Q_1'^m(\xi_b)}{P_1'^m(\xi_b)} P_1^m(\xi_s) P_1^m(\eta_s) \tag{62}$$

Defining g as the improper integral:

$$g_1^m(\xi_k) = \int_{\xi_k}^{\infty} q(\xi') \left[Q_1^m(\xi') - \frac{Q_1^m(\xi_k)}{Q_0(\xi_k)} Q_0(\xi') \right] d\xi', \qquad m = 0, 1 \tag{63}$$

and using the fact that, for $\phi = 0$,

$$x = c\sqrt{\xi^2 - 1}\sqrt{1 - \eta^2} = -cP_1^1(\xi)P_1^1(\eta) \tag{64}$$

$$z = c\xi\eta = cP_1(\xi)P_1(\eta) \tag{65}$$

we have

$$x_k = -\frac{P_1^1(\xi_k)}{\alpha g_1(\xi_k)} \frac{Q_1'^1(\xi_b)}{P_1'^1(\xi_b)} \quad x_s \tag{66}$$

$$z_k = -\frac{P_1(\xi_k)}{\alpha g_1(\xi_k)} \frac{Q_1'(\xi_b)}{P_1'(\xi_b)} \quad z_s \tag{67}$$

Now, using the aforementioned fact that

$$\frac{x^2 + y^2}{c^2(\xi_b^2 - 1)} + \frac{z^2}{c^2 \xi_b^2} = 1 \tag{68}$$

we obtain

$$\frac{x_s^2}{c^2(\xi_k^2-1)}\frac{P_1^1(\xi_k)^2}{\alpha^2 g_1^1(\xi_k)^2}\frac{Q_1^{'1}(\xi_b)^2}{P_1^{'1}(\xi_b)^2}+\frac{z_s^2}{c^2\xi_k^2}\frac{P_1(\xi_k)^2}{\alpha^2 g_1(\xi_k)^2}\frac{Q_1^{'}(\xi_b)^2}{P_1^{'}(\xi_b)^2}=1 \tag{69}$$

which is equivalent to

$$\frac{x_s^2}{a^2}\frac{P_1^1(\xi_k)^2}{\alpha^2 g_1^1(\xi_k)^2}\frac{Q_1^{'1}(\xi_b)^2}{P_1^{'1}(\xi_b)^2}+\frac{z_s^2}{b^2}\frac{P_1(\xi_k)^2}{\alpha^2 g_1(\xi_k)^2}\frac{Q_1^{'}(\xi_b)^2}{P_1^{'}(\xi_b)^2}=1 \tag{70}$$

Given the current situation, specifically with α of arbitrary value, this nonlinear equation is not guaranteed to have a solution. However, we can guarantee a solution by demanding that α satisfy

$$\alpha^2=\beta^2\left[\frac{x_s^2}{a^2}\frac{P_1^1(\xi_b)^2}{g_1^1(\xi_b)^2}\frac{Q_1^{'1}(\xi_b)^2}{P_1^{'1}(\xi_b)^2}+\frac{z_s^2}{b^2}\frac{P_1(\xi_b)^2}{g_1(\xi_b)^2}\frac{Q_1^{'}(\xi_b)^2}{P_1^{'}(\xi_b)^2}\right] \tag{71}$$

for some $\beta\in\mathbb{R}$ such that $\beta^2>1$. (Again, note that α as used here is distinct from that used in deriving the Neumann function.) Then define a function $f:[\xi_b,+\infty)\to\mathbb{R}$ such that

$$f(\xi)=\frac{1}{\alpha^2}\left[\frac{x_s^2}{a^2}\frac{P_1^1(\xi)^2}{g_1^1(\xi)^2}\frac{Q_1^{'1}(\xi_b)^2}{P_1^{'1}(\xi_b)^2}+\frac{z_s^2}{b^2}\frac{P_1(\xi)^2}{g_1(\xi)^2}\frac{Q_1^{'}(\xi_b)^2}{P_1^{'}(\xi_b)^2}\right]-1 \tag{72}$$

It is then seen that

$$\begin{aligned}\frac{1}{\beta^2}-1=\frac{1}{\alpha^2}\left[\frac{x_s^2}{a^2}\frac{P_1^1(\xi_b)^2}{g_1^1(\xi_b)^2}\frac{Q_1^{'1}(\xi_b)^2}{P_1^{'1}(\xi_b)^2}+\frac{z_s^2}{b^2}\frac{P_1(\xi_b)^2}{g_1(\xi_b)^2}\frac{Q_1^{'}(\xi_b)^2}{P_1^{'}(\xi_b)^2}\right]-1\\=f(\xi_b)\end{aligned}$$

Since $\beta^2>1$, $\frac{1}{\beta^2}<1$, and so we have

$$f(\xi_b)=\frac{1}{\beta^2}-1<1-1=0$$

It is easily shown that f is continuous. Hence, by the intermediate value theorem, (70) has a solution in $(\xi_b, +\infty)$. This means that ξ_k exists and so the location of the point image can be determined. Finally, we show that the expansion coefficient d_{mn} can be determined. Examining the $n = 2$ case and doing some algebra, we obtain

$$d_{mn} = -\frac{1}{\gamma_{mn} Q_n^m(\xi_k)} \left[\frac{Q_n'^m(\xi_b)}{P_n'^m(\xi_b)} P_n^m(\xi_s) P_n^m(\eta_s) + \left[Q Q_n^m(\xi_k) + \alpha \int_{\xi_k}^{\infty} q(\xi') Q_n^m(\xi') d\xi' \right] P_n^m(\eta_k) \right] \tag{73}$$

Note that since we know x_k and z_k, η_k can be determined from the inverse of the coordinate transformation given in the introduction.

As stated before, it is hoped that the image system corresponding to the more general boundary condition might be derived from the one developed here. One method to achieve is to choose a particular value of α. Indeed, since α is arbitrary, we can demand that $A_{01} = 0$ by setting

$$\alpha = -\frac{3a\xi_s \eta_s Q_1'(\xi_b)}{4\pi c^3} \tag{74}$$

which is allowed as each of ξ_b, ξ_s, and η_s are fixed. This would enable us to derive the second image system by simply modifying the influence of A_{01}. However, this value of α is very likely small or negative and so, as noted above, may prevent the physically appealing interpretation motivated earlier. In order to guarantee this interpretation, we will have to account in our analysis for the presence of the different, nonzero A_{01} term. This should require a relatively uncomplicated modification to the system given above, but for reasons of brevity will not be pursued in the present paper.

4.1 Exterior Neumann problem: Boundary condition 1

The first exterior problem we solve, with the source charge located outside of the prolate spheroid, is given by

$$\triangle N^e(\mathbf{r},\mathbf{r}_s) = \delta(\mathbf{r}-\mathbf{r}_s), \qquad \xi_s > \xi > \xi_b \tag{75}$$

$$\frac{\partial}{\partial n} N^e(\mathbf{r},\mathbf{r}_s) = -\frac{1}{4\pi}\omega_{\xi_b}(\eta), \qquad \xi = \xi_b \tag{76}$$

$$N^e(\mathbf{r},\mathbf{r}_s) = O\left(\frac{1}{\mid \mathbf{r} \mid^2}\right), \qquad \mathbf{r}\to\infty \tag{77}$$

where (77) follows directly from (75) and (76). The solution here will be more brief than for the interior problem, as applying the same method yields a similar result. We again exploit the azimuthal symmetry of the prolate spheroid, which yields the following solution in terms of the even exterior prolate spheroidal harmonics:

$$N^e(\mathbf{r},\mathbf{r}_s) = -\frac{1}{4\pi \mid \mathbf{r}-\mathbf{r}_s \mid} + \sum_{n=0}^{\infty}\sum_{m=0}^{n} B_{mn} H_{mn} Q_n^m(\xi_s) P_n^m(\xi_s) Q_n^m(\xi) P_n^m(\eta)\cos(m\phi) \tag{78}$$

where B_{mn} is the expansion coefficient to be determined by a method similar to the interior Neumann function case. Applying the multipole expansion formula as done above, the solution can be written as

$$N^e(\mathbf{r},\mathbf{r}_s) = -\frac{1}{4\pi c}\sum_{n=0}^{\infty}\sum_{m=0}^{n} H_{mn} Q_n^m(\xi_s) P_n^m(\eta_s)\big[P_n^m(\xi) - B_{mn} Q_n^m(\xi)\big] C_n^m(\eta,\phi) \tag{79}$$

Applying the boundary condition as before yields

$$-\frac{a^2}{c^2}\omega_{\xi_b}(\eta)\sum_{n=0}^{\infty}\sum_{m=0}^{n} H_{mn} Q_n^m(\xi_s) P_n^m(\eta_s)\big[P_n'^m(\xi_b) - B_{mn} Q_n'^m(\xi_b)\big] C_n^m(\eta,\phi) = \omega_{\xi_b}(\eta) \tag{80}$$

Now we integrate over S:

$$-\frac{a^2}{c^2}(\eta)\sum_{n=0}^{\infty}\sum_{m=0}^{n} H_{mn}Q_n^m(\xi_s)P_n^m(\eta_s)\left[P_n^{'m}(\xi_b)-B_{mn}Q_n^{'m}(\xi_b)\right]\int_{-1}^{1}\int_0^{2\pi} P_n^m(\eta)\cos(m\phi)d\phi d\eta = \iint_S \omega_{\xi_b}(\eta)dS \tag{81}$$

Since

$$\int_0^{2\pi} cos(m\phi)d\phi = 0, \qquad m \neq 0, \tag{82}$$

it must be that $m = 0$, yielding

$$\begin{aligned}
-\frac{a^2}{c^2}H_{00}Q_0(\xi_s)P_0(\eta_s)Q_0^{'}(\xi_b)B_{00}\left(\int_{-1}^{1}P_0(\eta)d\eta\right)\left(\int_0^{2\pi}(1)d\phi\right) &= 4\pi \\
\Rightarrow -\frac{a^2}{c^2}(1)Q_0(\xi_s)P_0(\eta_s)Q_0^{'}(\xi_b)B_{00}\left(\int_{-1}^{1}(1)d\eta\right)(2\pi) &= 4\pi \\
\Rightarrow -\frac{a^2}{c^2}Q_0(\xi_s)P_0(\eta_s)Q_0^{'}(\xi_b)B_{00}(2)(2\pi) &= 4\pi \\
\Rightarrow B_{00} &= -\frac{c^2}{a^2Q_0(\xi_s)P_0(\eta_s)Q_0^{'}(\xi_b)} \\
\Rightarrow B_{00} &= -\frac{c^2}{a^2Q_0(\xi_s)(1)(-c^2/a^2)} \\
\Rightarrow B_{00} &= \frac{1}{Q_0(\xi_s)}
\end{aligned}$$

Now, multiplying by $C_i^j(\eta,\phi)$ and integrating over S we obtain the expansion coefficient:

$$B_{mn} = \frac{P_n^{'m}(\xi_b)}{Q_n^{'m}(\xi_b)}, \qquad n \geq 1 \tag{83}$$

Hence the exterior Neumann function becomes

$$N^e(\mathbf{r},\mathbf{r}_s) = -\frac{1}{4\pi \mid \mathbf{r}-\mathbf{r}_s \mid}+\frac{1}{4\pi c Q_0(\xi_s)}+\frac{1}{4\pi c}\sum_{n=1}^{\infty}\sum_{m=0}^{n} H_{mn}\frac{P_n^{'m}(\xi_b)}{Q_n^{'m}(\xi_b)}Q_n^m(\xi_s)P_n^m(\eta_s)Q_n^m(\xi)C_n^m(\eta,\phi) \tag{84}$$

4.2 Exterior Neumann problem: Boundary condition 2

Here we derive the exterior Neumann fucntion for a new non-constant boundary condition. The problem is

$$\triangle N^e(\mathbf{r}, \mathbf{r}_s) = \delta(\mathbf{r} - \mathbf{r}_s), \qquad \xi_s > \xi > \xi_b \tag{85}$$

$$\frac{\partial}{\partial n} N^e(\mathbf{r}, \mathbf{r}_s) = -\frac{\alpha\eta + \beta}{\sqrt{\xi_b^2 - \eta^2}}, \qquad \xi = \xi_b \tag{86}$$

$$N^e(\mathbf{r}, \mathbf{r}_s) = O\left(\frac{1}{\mid \mathbf{r} \mid^2}\right), \qquad \mathbf{r} \to \infty \tag{87}$$

where β is an arbitrary real constant; also. Here we will choose $\beta = 1/(4\pi ac)$, in agreement with our choice above. Applying the boundary condition, multiplying by $C_i^j(\eta, \phi)$ for any $i \geq 0$, and integrating over S, we have

$$\frac{a^2}{4\pi c^2}\gamma_{ji}(\eta) H_{ji} Q_i^j(\xi_s) P_i^j(\eta_s) \left[P_i^{'j}(\xi_b) - B_{ji} Q_i^{'j}(\xi_b)\right] = \iint_S C_i^j(\eta, \phi) \frac{\alpha\eta + \beta}{\sqrt{\xi_b^2 - \eta^2}} dS \tag{88}$$

The righthand integral of (88) is identical to that seen in the second interior boundary value problem. Hence it equals

$$\begin{cases} 2\pi ac \int_{-1}^{1} P_i(\eta)(\alpha\eta + \beta) d\eta, & j = 0, i < 2 \\ 0, & \text{otherwise} \end{cases}$$

For the following two conditions, refer back to the techniques and integration used in the second interior boundary value problem. For $i = 0$ we have, via the manner used before,

$$\frac{a^2}{c^2} H_{00} Q_0(\xi_s) P_0(\eta_s) Q_0^{'}(\xi_b) = 4\pi ac\beta \tag{89}$$

Which for $\beta = 1/(4\pi ac)$ gives us

$$B_{00} = \frac{1}{Q_0(\xi_s)} \tag{90}$$

Which agrees with the B_{00} term for the first exterior boundary value problem.

Now, for $i = 1$ we have

$$\frac{a^2}{c^2} Q_1(\xi_s) P_1(\eta_s)[1 - B_{01} Q_1^{'}(\xi_b)] = \frac{4\pi ac\alpha}{3} \tag{91}$$

Simple algebra then yields

$$B_{01} = \frac{1}{Q_1^{'}(\xi_b)} \left[1 - \frac{4\pi c^3 \alpha}{3a Q_1(\xi_s) P_1(\eta_s)} \right] \tag{92}$$

Again, when $\alpha = 0$, this yields the result from the first exterior boundary value problem.

Finally, since the righthand integral of (88) vanishes for $j > 0$, we have

$$A_{mn} = \frac{P_n^{'m}(\xi_b)}{Q_n^{'m}(\xi_b)}, \qquad n \geq 2, m = 0, 1, ... \quad \text{or} \quad n = 1, m = 1 \tag{93}$$

Which matches the first exterior solution given the above constraints on m and n.

Since the two exterior boundary value problems have the same boundary conditions as the interior problems, the physical interpretations will be similar, except for the placement of the source outside of the prolate spheroid. We now begin the development of the image system for the exterior Neumann function.

4.3 Image system for the exterior Neumann function

The development of the image system for the exterior Neumann function will proceed along similar lines as above, but instead of a line image extending to infinity, we shall choose a line image extending from the point image to a point on the line confocal with the prolate spheroid. Importantly, since the two boundary value problems in the exterior case have different solutions for B_{01}, their image systems may differ. Here we consider only a homogeneous version of the second problem, with $\alpha = \beta = 0$. This is allowed since the source is located outside of the prolate spheroid and hence the divergence theorem does not constrain the analysis as it did with the interior problem. This will grant more freedom in developing the image system, as we will see. As before, we build an image system whose potential equals the reflected part of the exterior Neumann function:

$$R^e(\mathbf{r}, \mathbf{r}_s) = \frac{B_{00}}{4\pi c} + \frac{1}{4\pi c} \sum_{n=1}^{\infty} \sum_{m=0}^{n} H_{mn} \frac{P_n'^m(\xi_b)}{Q_n'^m(\xi_b)} Q_n^m(\xi_s) P_n^m(\eta_s) Q_n^m(\xi) C_n^m(\eta, \phi) \tag{94}$$

Since the point source at $\mathbf{r}_s$ is located outside of the prolate spheroid given by ξ_b, the divergence theorem does not imply that β cannot be zero. Hence we shall build the image system for the exterior Neumann function with a homogeneous boundary condition. The reflected part of this function is thus

$$R^e(\mathbf{r}, \mathbf{r}_s) = \frac{1}{4\pi c} \sum_{n=1}^{\infty} \sum_{m=0}^{n} H_{mn} \frac{P_n'^m(\xi_b)}{Q_n'^m(\xi_b)} Q_n^m(\xi_s) P_n^m(\eta_s) Q_n^m(\xi) C_n^m(\eta, \phi) \tag{95}$$

Now, similarly to the interior problem, the image system potential is given by

$$I^i(\mathbf{r}, \mathbf{r}_s) = -\frac{Q}{4\pi \mid \mathbf{r} - \mathbf{r}_k \mid} - \frac{\alpha}{4\pi} \int_1^{\xi_k} \frac{q(\xi')}{\mid \mathbf{r} - \mathbf{t}' \mid} d(\xi') - \frac{1}{4\pi} \iint_{S_{\xi_k}} \frac{d(\mathbf{s}')}{\mid \mathbf{r} - \mathbf{s}' \mid} dS_{\xi_k}(\eta', \phi') \tag{96}$$

Here the integral is from 1 to ξ_k since at the confocal line we have $\xi = 1$. Using the multipole expansion and demanding $I^e(\mathbf{r}, \mathbf{r}_s) = R^e(\mathbf{r}, \mathbf{r}_s)$ we have

$$\frac{1}{4\pi c}\sum_{n=0}^{\infty}\sum_{m=0}^{n} H_{mn} Q_n^m(\xi) P_n^m(\eta_k) C_n^m(\eta, \phi)\left(Q P_n^m(\xi_k) + \alpha \int_1^{\xi_k} q(\xi') P_n^m(\xi') d(\xi')\right)$$
$$+\frac{1}{4\pi c}\sum_{n=2}^{\infty}\sum_{m=0}^{n} H_{mn}\gamma_{mn} d_{mn} P_n^m(\eta_k) Q_n^m(\xi) C_n^m(\eta, \phi)$$
$$= -\frac{1}{4\pi c}\sum_{n=1}^{\infty}\sum_{m=0}^{n} H_{mn}\frac{P_n'^m(\xi_b)}{Q_n'^m(\xi_b)} Q_n^m(\xi_s) P_n^m(\eta_s) Q_n^m(\xi) C_n^m(\eta, \phi)$$

Next, we compare the monopole ($n = 0$) terms to obtain

$$Q = -\alpha \int_1^{\xi_k} q(\xi') d\xi' \tag{97}$$

We then define

$$g_1^m(\xi_k) = \int_1^{\xi_k} q(\xi')[P_1^m(\xi') - P_1^m(\xi_k)]d\xi', \qquad m = 0, 1 \tag{98}$$

Comparing the dipole ($n = 1$) terms yields

$$\alpha g_1^m(\xi_k) P_1^m(\eta_k) = -\frac{P_1'^m(\xi_b)}{Q_1'^m(\xi_b)} Q_1^m(\xi_s) P_1^m(\eta_s) \tag{99}$$

Finally, comparing the $n \geq 2$ terms we have

$$d_{mn} = -\frac{1}{\gamma_{mn} P_n^m(\xi_k)}\left[\frac{P_n'^m(\xi_b)}{Q_n'^m(\xi_b)} Q_n^m(\xi_s) P_n^m(\eta_s) + \left(Q P_n^m(\xi_k) + \alpha \int_1^{\xi_k} q(\xi') P_n^m(\xi') d\xi'\right) P_n^m(\eta_k)\right] \tag{100}$$

As before, it is clear that we need to determine Q, $q(\xi)$, and ξ_k. First, since the integral above will exist as long as $q(\xi)$ is continuous, we simply define

$$q(\xi) = 1 \tag{101}$$

Then it follows that

$$Q = -\alpha \int_1^{\xi_k} d\xi' = \alpha(1 - \xi_k) \tag{102}$$

Now, using the fact that, for $\phi = 0$,

$$x = c\sqrt{\xi^2 - 1}\sqrt{1 - \eta^2} = -cP_1^1(\xi)P_1^1(\eta) \tag{103}$$

$$z = c\xi\eta = cP_1(\xi)P_1(\eta) \tag{104}$$

We can rewrite to determine x_k and z_k:

$$x_k = -\frac{P_1^1(\xi_k)}{\alpha g_1^1(\xi_k)} \frac{Q_1^1(\xi_s)}{P_1^1(\xi_s)} \frac{P_1^{'1}(\xi_b)}{Q_1^{'1}(\xi_b)} x_s \tag{105}$$

$$z_k = -\frac{P_1(\xi_k)}{\alpha g_1(\xi_k)} \frac{Q_1(\xi_s)}{P_1(\xi_s)} \frac{P_1^{'}(\xi_b)}{Q_1^{'}(\xi_b)} x_s \tag{106}$$

Since the point image is located on the prolate spheroid S_{ξ_k}, we again note that

$$\frac{x_k^2}{c^2(\xi_k^2 - 1)} + \frac{z_k^2}{c^2\xi_k^2} = 1 \tag{107}$$

Substituting, we obtain

$$\frac{x_s^2}{a^2} \frac{P_1^1(\xi_k)^2}{\alpha g_1^1(\xi_k)^2} \frac{Q_1^1(\xi_s)^2}{P_1^1(\xi_s)^2} \frac{P_1^{'1}(\xi_b)^2}{Q_1^{'1}(\xi_b)^2} + \frac{z_s^2}{b^2} \frac{P_1(\xi_k)^2}{\alpha g_1(\xi_k)^2} \frac{Q_1(\xi_s)^2}{P_1(\xi_s)^2} \frac{P_1^{'}(\xi_b)^2}{Q_1^{'}(\xi_b)^2} = 1 \tag{108}$$

As it stands, this equation does not necessarily have a solution. However, it will if we constrain α in way similar to the case for the interior problem:

$$\alpha^2 = \beta^2 \left[\frac{x_s^2}{a^2} \frac{P_1^1(\xi_b)^2}{g_1^1(\xi_b)^2} \frac{Q_1^1(\xi_s)^2}{P_1^1(\xi_s)^2} \frac{P_1^{'1}(\xi_b)^2}{Q_1^{'1}(\xi_b)^2} + \frac{z_s^2}{b^2} \frac{P_1(\xi_b)^2}{g_1(\xi_b)^2} \frac{Q_1(\xi_s)^2}{P_1(\xi_s)^2} \frac{P_1'(\xi_b)^2}{Q_1'(\xi_b)^2} \right] \tag{109}$$

Where again β is a constant satisfying $\beta^2 > 1$. Now we define a function $f : [1, \xi_b] \to \mathbb{R}$ by

$$f(\xi) = \frac{x_s^2}{a^2} \frac{P_1^1(\xi)^2}{\alpha^2 g_1^1(\xi)^2} \frac{Q_1^1(\xi_s)^2}{P_1^1(\xi_s)^2} \frac{P_1^{'1}(\xi_b)^2}{Q_1^{'1}(\xi_b)^2} + \frac{z_s^2}{b^2} \frac{P_1(\xi)^2}{\alpha^2 g_1(\xi)^2} \frac{Q_1(\xi_s)^2}{P_1(\xi_s)^2} \frac{P_1'(\xi_b)^2}{Q_1'(\xi_b)^2} - 1 \tag{110}$$

It is easy to show that f is continuous and that

$$f(\xi_b) = \frac{1}{\beta^2} - 1 < 0 \tag{111}$$

Hence, by the intermediate value theorem, we see that (108) must have a solution in $(1, \xi_b)$. This completes the image system.

Indeed, the image system for the exterior Neumann function has not been developed as fully as that of the interior Neumann function. This is due to the fact that the expansion constant B_{00} is not allowed to be arbitrary; hence a homogenous boundary condition was used. We hope to more completely develop the image system in the future.

5 Discussion

In the present paper we derived interior Neumann functions for two non-constant boundary conditions, and exterior Neumann functions for two non-constant boundary conditions (the latter of which we allowed to be homogeneous). Indeed, the physical utility of these solutions will be limited by the types of boundary conditions encountered in nature. However, two of the solutions corresponded to boundary conditions consisting of multiplication by an arbitrary scalar and so may provide a wider range of physical application. In addition, the physically interesting interpretations of the two types of boundary conditions analyzed here merit future study and as such it is hoped that applications from thermodynamics to biophysics might be found.

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